Quarter-Sweep Successive Over-relaxation Approximation to the Solution of Porous Medium Equations

Jackel Vui Lung Chew, Elayaraja Aruchunan, Andang Sunarto, and Jumat Sulaiman

Abstract—**This paper investigated the use of a successive over-relaxation parameter in a quarter-sweep finite difference approximation scheme. The performance of the developed quarter-sweep successive over-relaxation method is examined by considering a nonlinear partial differential equation, namely the porous medium equation. The main contribution of this paper is to present the stability, convergence and efficiency of the proposed method. Several initial-boundary value problems of the porous medium equation are solved to illustrate the efficiency of the proposed method. The numerical results showed that the quarter-sweep successive over-relaxation method is more efficient in reducing iterations and computational time than the standard and the existing numerical methods. In addition, the accuracy of the quarter-sweep successive over-relaxation method is comparable to the tested numerical methods.**

Index Terms—**Finite difference method, Newton method, porous medium equation, quarter-sweep, successive over-relaxation.**

I. Introduction

IN numerical analysis, the finite difference method, also commonly known as FDM, is one of the famous discretization approaches to solving partial differential equations. FDM is easy to derive for solving many linear and nonlinear mathematical models. The FDM works by approximating and replacing the derivatives of the equation with finite difference operators. The restricted spatial domain (and time interval for time-dependent partial differential equations) is usually defined for the solution. The unknown values, located at the discrete points under a restricted spatial domain, can be represented by algebraic equations. When the system of equations contains a set of finite differences and a set of known values from closest points or predefined boundaries, the sizes of such systems are usually large and possess a high computational complexity. Solving a large and complex system of equations requires an efficient iterative method or algebraic matrix technique. Various iterative methods and matrix algebra techniques are available in [1], [2].

Manuscript received August 21, 2021; revised April 15, 2022. The work was supported by the Research Management Center, Universiti Malaysia Sabah.

J. V. L. Chew is a Senior Lecturer of Faculty of Computing and Informatics, Universiti Malaysia Sabah Labuan International Campus, Labuan F.T., 87000, Malaysia, corresponding author, e-mail: jackelchew93@ums.edu.my.

E. Aruchunan is a Senior Lecturer of Institute of Mathematical Sciences, University of Malaya, Kuala Lumpur, 50603, Malaysia, e-mail: elayarajah@um.edu.my.

A. Sunarto is an Associate Professor of Tadris Matematika, Universitas Islam Negeri Fatmawati Sukarno, Bengkulu, 38211, Indonesia, e-mail: andang99@gmail.com.

J. Sulaiman is a Professor of Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, 88400, Malaysia, e-mail: jumat@ums.edu.my.

The FDM can be categorized into three basic schemes, namely explicit, implicit and mixed explicit-implicit. One of the well-known mixed explicit-implicit schemes is the Crank-Nicolson scheme. The above-mentioned finite difference schemes could be distinguished according to the stability, convergency, accuracy, and numerical intensity. The explicit scheme is conditionally and numerically stable, and it is convergent. The numerical errors from using the explicit scheme are linearly proportional to the time step and quadratically to the space step. The explicit scheme has the least numerical intensity because the solutions of the system of equations can be computed using a direct method such as Gaussian elimination. Next, the implicit scheme is unconditionally and numerically stable, and it is also convergent. The implicit scheme has a greater numerical intensity than the explicit scheme because the scheme requires solving a system of equations for each time step. The magnitude of numerical errors produced by the implicit scheme are comparable to the explicit scheme. The Crank-Nicolson scheme is the most accurate FDM scheme among the three main basic schemes. Although the scheme is unconditionally numerically stable and convergent, the numerical intensity is the highest as it solves a system of equations on each time step. The numerical errors produced by the Crank-Nicolson scheme are quadratically proportional to both the time and the space step [3].

From the basic schemes of FDM, many new schemes have been proposed to enrich the literature. For instance, [4], [5] constructed a nonstandard finite difference scheme to solve a nonlinear partial differential equation. The scheme focuses on the generalization of the discrete-time derivative. Then, [6] introduced a new high-order weighted essentially non-oscillatory finite difference scheme for solving several nonlinear degenerate parabolic equations. They extended the works from [7], [8] to obtain sixth-order accuracy in smooth regions. Another finite difference scheme has been proposed with the given name as a generalized FDM or GFDM [9]. This scheme is a meshless method and uses irregular clouds of nodes to model several nonlinear parabolic equations. Furthermore, [10] constructed invariant compact finite-difference schemes that preserve Lie symmetries for solving linear and nonlinear partial differential equations. Then, [11] studied the numerical solution of coupled Burgers' equation using the combination of FDM and sinc collocation.

Among the effective finite difference schemes, the quarter-sweep finite difference scheme emerges as a unique computational complexity reduction to solving complex mathematical equations. The quarter-sweep scheme was introduced by [12] for solving a two-dimensional Poisson equation. Since then, many researchers have studied the scheme and applied it to solve different mathematical equations. For instance, multidimensional Black-Scholes equation [13], [14], two-dimensional Helmholtz equation [15], first and second-order linear Fredholm integrodifferential equations [16]–[18], multidimensional fuzzy diffusion equation [19], fractional diffusion equation [20], [21] and nonlinear diffusion equation [22]. Besides that, the quarter-sweep scheme was used to simulate a steady-state problem [23] and design a quarter-sweep alternating decomposition explicit iterative method for solving linear and nonlinear two-point boundary value problems containing singularity [24]. Another article [25] investigated the performance of a nonlocal discretization scheme with a quarter-sweep iteration for solving nonlinear two-point boundary value problems. The quarter-sweep finite difference scheme has successfully overcome the computational complexity issue in solving systems of equations generated by many mathematical models and hence becomes the motivation of this paper.

This paper aims to investigate the impact of adding a successive over-relaxation (SOR) parameter to constructing an efficient quarter-sweep iteration scheme. A numerical method called the QSSOR method is introduced to solve nonlinear partial differential equations. One of the nonlinear partial differential equations called the porous medium equation (PME) is used as the test problem. PME is chosen for the numerical experiment because of its importance to describe a nonlinear process involving heat transfer, fluid flow and gas diffusion. PME has been applied to model the fire extinguishing process in an aircraft engine nacelle [26], the counter-current imbibition in a heterogeneous porous medium [27], the hydrodynamic for the slow reservoirs [28], and the instability phenomenon from the process of extracting crude oil from the field [29]. Besides that, a particular form of PME was used to model the moisture transport of rice and [30] applied explicit and implicit FDM to solve the mathematical model. The proposed numerical method in this paper can be a useful numerical tool for PME modelling and simulation. Moreover, this paper extends the work presented by [22] and contributes the stability and convergence theorems to the existing quarter-sweep iteration scheme for nonlinear partial differential equations. The stability and convergence theorems can be used to support the theoretical stability and convergence of the quarter-sweep finite difference approximation to nonlinear partial differential equations.

The following sections of the paper can be outlined as follows: Section 2 presents the quarter-sweep discretization procedure of the one-dimensional PME model. Section 3 and 4 show the theorems and proofs of stability and convergence of the quarter-sweep finite difference scheme, respectively. Section 5 discusses the derivation of the QSSOR method, followed by the numerical experiment and discussion in Section 6. Section 7 concludes the paper by highlighting the findings and future direction of the research.

II. Quarter-sweep discretization procedure

In this section, the quarter-sweep finite difference scheme and its discretization procedure is described. The construction of the quarter-sweep finite difference scheme is similar to the standard implicit scheme of FDM which both are

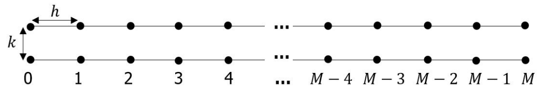

Fig. 1. Implicit finite difference scheme framework

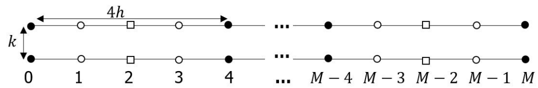

Fig. 2. Quarter-sweep finite difference scheme framework

based on Taylor series expansion. However, for the quarter-sweep finite difference scheme, the distance between two consecutive grid points to be calculated is quadrupled after a skipped three grid points. Moreover, the quarter-sweep finite difference approximation uses a mixed iterative-direct computation approach to solve a system of equations efficiently. Figure 1 and 2 show the comparison between the quarter-sweep and the implicit finite difference schemes.

To illustrate the quarter-sweep finite difference discretization procedure, let a one-dimensional PME be defined as [31], [32]:

$$u_t = \rho(u^m u_x)_x, \tag{1}$$

and can be further derived using the method of calculus into

$$u_t = \rho(u^m u_{xx} + m u^{m-1}(u_x)^2), \tag{2}$$

where ρ and m are real numbers with $\rho > 0$. $u(x,t)$ may express the temperature distribution or the velocity of fluid flow or gas-particle diffusion in a porous medium. Then, x represents the coordinate at a specific time t. Meanwhile, u^m is a nonlinear term that determines the type of diffusion process.

The quarter-sweep scheme to approximate Equation 2 depends on the transformation of a differential equation into an algebraic system of equations in which the finite domain is discretized into several discrete grid points of $U_{p,n}$, for $p = 0, 1, 2, \dots, M, n = 0, 1, 2, \dots, N$. The grid point $U_{p,n}$ is used to approximate the exact values by the expression $u(x_p, t_n) = U_{p,n} + \epsilon_{p,n}, p = 0, 1, 2, ..., M, n = 0, 1, 2, ..., N$, where $u(x_p, t_n)$ and $\epsilon_{p,n}$ represent the exact solution and the approximation error, respectively. The quarter-sweep scheme uses a common rectangular finite domain where the space $0 \leq x \leq L$ is partitioned uniformly by $ph, h = L/M$ with M is a positive integer. Since PME is a time-dependent partial differential equation, the time level is set equally by $t_n = nk, k = T/N$ where T and N are the total time and the number of time steps respectively. The quarter-sweep operators used to approximate Equation 2 are defined as [22]

$$u_t \approx \frac{U_{p,n} - U_{p,n-1}}{k}, \tag{3}$$

$$u_x \approx \frac{U_{p+4,n} - U_{p-4,n}}{8h}, \tag{4}$$

and

$$u_{xx} \approx \frac{U_{p+4,n} - 2U_{p,n} + U_{p-4,n}}{16h^2}, \tag{5}$$

where $p = 4, 8, ..., M-4, n = 1, 2, ..., N$. Substituting the operators shown by Equation 3, 4, and 5 into Equation 2 gives a quarter-sweep finite difference approximation to PME in the form of

$$\begin{aligned}U_{p,n} - \alpha_1 U_{p,n}^m (U_{p+4,n} - 2U_{p,n} + U_{p-4,n})\\ -\alpha_2 U_{p,n}^{m-1}(U_{p+4,n} - U_{p-4,n})^2 = U_{p,n-1},\end{aligned} \tag{6}$$

where $\alpha_1 = \rho k/16h^2$ and $\alpha_2 = \rho mk/64h^2$. The next two sections will discuss the stability and convergence analysis of the approximation equation shown by Equation 6.

III. Stability analysis

In this section, the stability analysis of the quarter-sweep finite difference approximation is discussed.

Proposition 1: $\|E_n\|_\infty \leq \|E_0\|_\infty, n = 1, 2, 3, \ldots$.

Theorem 1: The quarter-sweep finite difference approximation defined by Equation 6 is unconditionally stable.

Proof: Suppose that $\widetilde{U}_{p,n}$ is the approximate solution and $\epsilon_{p,n}$ is the error such that $\epsilon_{p,n} = \widetilde{U}_{p,n} - U_{p,n}$ satisfies

$$\begin{aligned}\epsilon_{p,n} - \alpha_1 \epsilon_{p,n}^m (\epsilon_{p+4,n} - 2\epsilon_{p,n} + \epsilon_{p-4,n})\\ -\alpha_2 \epsilon_{p,n}^{m-1} (\epsilon_{p+4,n} - \epsilon_{p-4,n})^2 = \epsilon_{p,n-1},\end{aligned} \tag{7}$$

where $p = 4, 8, ..., M - 4, n = 1, 2, ..., N$. Also, let $E_n = (\epsilon_{4,n}, \epsilon_{8,n}, ..., \epsilon_{M-4,n})^T$.

Assume that $0 < \ldots \leq \epsilon_{p,n}^{m+1} \leq \epsilon_{p,n}^m \leq \epsilon_{p,n}^{m-1} \leq \epsilon_{p,n}$, where m is a positive integer. Using a mathematical induction method: For $n = 1$,

$$\begin{aligned}\epsilon_{p,1} - \alpha_1 \epsilon_{p,1}^m (\epsilon_{p+4,1} - 2\epsilon_{p,1} + \epsilon_{p-4,1})\\ -\alpha_2 \epsilon_{p,1}^{m-1} (\epsilon_{p+4,1} - \epsilon_{p-4,1})^2 = \epsilon_{p,0}.\end{aligned} \tag{8}$$

Let $|\epsilon_{L,1}| = \max_{4\leq p\leq M-4} |\epsilon_{p,1}|$. Then, we have

$$\begin{aligned}|\epsilon_{L,1}| &\leq |\epsilon_{L,1}| - \alpha_1 |\epsilon_{L,1}^m||\epsilon_{L+4,1}| + 2\alpha_1 |\epsilon_{L,1}^{m+1}|\\ &\quad -\alpha_1 |\epsilon_{L,1}^m||\epsilon_{L-4,1}| - \alpha_2 |\epsilon_{L,1}^{m-1}||\epsilon_{L+4,1}^2|\\ &\quad +2\alpha_2 |\epsilon_{L,1}^{m-1}||\epsilon_{L+4,1}||\epsilon_{L-4,1}| - \alpha_2 |\epsilon_{L,1}^{m-1}||\epsilon_{L-4,1}^2|\\ &\leq |\epsilon_{L,1} - \alpha_1 \epsilon_{L,1}^m \epsilon_{L+4,1} + 2\alpha_1 \epsilon_{L,1}^{m+1} - \alpha_1 \epsilon_{L,1}^m \epsilon_{L-4,1}\\ &\quad -\alpha_2 \epsilon_{L,1}^{m-1} \epsilon_{L+4,1}^2 + 2\alpha_2 \epsilon_{L,1}^{m-1} \epsilon_{L+4,1} \epsilon_{L-4,1}\\ &\quad -\alpha_2 \epsilon_{L,1}^{m-1} \epsilon_{L-4,1}^2| = |\epsilon_{L,0}| \leq \|E_0\|_\infty .\end{aligned} \tag{9}$$

Thus, $\|E_1\|_\infty \leq \|E_0\|_\infty$.

Suppose that $\|E_j\|_\infty \leq \|E_0\|_\infty, j = 1, 2, ..., n-1$ and let $|\epsilon_{L,n}| = \max_{4\leq p\leq M-4} |\epsilon_{p,n}|$. Then, we have

$$\begin{aligned}|\epsilon_{L,n}| &\leq |\epsilon_{L,n}| - \alpha_1 |\epsilon_{L,n}^m||\epsilon_{L+4,n}| + 2\alpha_1 |\epsilon_{L,n}^{m+1}|\\ &\quad -\alpha_1 |\epsilon_{L,n}^m||\epsilon_{L-4,n}| - \alpha_2 |\epsilon_{L,n}^{m-1}||\epsilon_{L+4,n}^2|\\ &\quad +2\alpha_2 |\epsilon_{L,n}^{m-1}||\epsilon_{L+4,n}||\epsilon_{L-4,n}| - \alpha_2 |\epsilon_{L,n}^{m-1}||\epsilon_{L-4,n}^2|\\ &\leq |\epsilon_{L,n} - \alpha_1 \epsilon_{L,n}^m \epsilon_{L+4,n} + 2\alpha_1 \epsilon_{L,n}^{m+1} - \alpha_1 \epsilon_{L,n}^m \epsilon_{L-4,n}\\ &\quad -\alpha_2 \epsilon_{L,n}^{m-1} \epsilon_{L+4,n}^2 + 2\alpha_2 \epsilon_{L,n}^{m-1} \epsilon_{L+4,n} \epsilon_{L-4,n}\\ &\quad -\alpha_2 \epsilon_{L,n}^{m-1} \epsilon_{L-4,n}^2| = |\epsilon_{L,n-1}| \leq \|E_0\|_\infty .\end{aligned} \tag{10}$$

Thus, $\|E_n\|_\infty \leq \|E_0\|_\infty$. ■

Hence, Theorem 1 is proved.

IV. Convergence analysis

In this section, the convergence analysis of the quarter-sweep finite difference approximation is discussed.

Proposition 2: $\|e_n\|_\infty \leq c(k+h^2), n = 1, 2, 3, ...,$ where $\|e_n\|_\infty = \max_{4\leq p\leq M-4} |e_{p,n}|$.

Theorem 2: Let $U_{p,n}$ be the approximation to $u(x_p, t_n)$ by the use of the quarter-sweep scheme. Then, there is a positive constant c such that

$$|U_{p,n} - u(x_p, t_n)| \leq c(k + h^2). \tag{11}$$

Proof: Let $u(x_p, t_n), p = 4, 8, ..., M-4, n = 1, 2, ..., N$ be the exact solution at grid point (x_p, t_n). Define $e_{p,n} = u(x_p, t_n) - U_{p,n}, p = 4, 8, ..., M - 4, n = 1, 2, ..., N$ and $e_n = (e_{4,n}, e_{8,n}, ..., e_{M-4,n})^T$. Then, we have

$$\begin{aligned}e_{p,n} - \alpha_1 e_{p,n}^m (e_{p+4,n} - 2e_{p,n} + e_{p-4,n})\\ -\alpha_2 e_{p,n}^{m-1} (e_{p+4,n} - e_{p-4,n})^2 = e_{p,n-1} + R_{p,n},\end{aligned} \tag{12}$$

where

$$\begin{aligned}R_{p,n} = u(x_p, t_n) - u(x_p, t_{n-1}) - \alpha_1 u^m(x_p, t_n)\\ (u(x_{p+4}, t_n) - 2u(x_p, t_n) + u(x_{p-4}, t_n))\\ -\alpha_2 u^{m-1}(x_p, t_n)(u(x_{p+4}, t_n) - u(x_{p-4}, t_n))^2.\end{aligned} \tag{13}$$

From the quarter-sweep scheme, we have

$$\frac{u(x_p, t_n) - u(x_p, t_{n-1})}{k} = \frac{\delta u(x_p, t_n)}{\delta t} + \tilde{c_1} k, \tag{14}$$

$$\begin{aligned}&\frac{u(x_{p+4}, t_n) - 2u(x_p, t_n) + u(x_{p-4}, t_n)}{16h^2}\\ &\qquad = \frac{\delta^2 u(x_p, t_n)}{\delta x^2} + \tilde{c_2} h^2,\end{aligned} \tag{15}$$

and

$$\frac{u(x_{p+4}, t_n) - u(x_{p-4}, t_n)}{8h} = \frac{\delta u(x_p, t_n)}{\delta x} + \tilde{c_3} h. \tag{16}$$

Using Equation 14, 15 and 16, Equation 12 becomes

$$\begin{aligned}R_{p,n} = \frac{\delta u(x_p, t_n)}{\delta t} - \alpha u^m(x_p, t_n) \frac{\delta^2 u(x_p, t_n)}{\delta x^2}\\ -\alpha m u^{m-1}(x_p, t_n) \left(\frac{\delta u(x_p, t_n)}{\delta x}\right)^2 + \tilde{c_1} k\\ +(\tilde{c_2} + \tilde{c_3}) h^2.\end{aligned} \tag{17}$$

Also, $|R_{p,n}| \leq c(k + h^2)$ where c is a constant.

By a mathematical induction: For $n = 1$, let $\|e_1\|_\infty = |e_{L,1}| = \max_{4\leq p\leq M-4} |e_{p,1}|$, we have

$$\begin{aligned}|e_{L,1}| &\leq |e_{L,1}| - \alpha_1 |e_{L,1}^m||e_{L+4,1}| + 2\alpha_1 |e_{L,1}^{m+1}|\\ &\quad -\alpha_1 |e_{L,1}^m||\epsilon_{L-4,1}| - \alpha_2 |e_{L,1}^{m-1}||e_{L+4,1}^2|\\ &\quad +2\alpha_2 |e_{L,1}^{m-1}||e_{L+4,1}||e_{L-4,1}| - \alpha_2 |e_{L,1}^{m-1}||e_{L-4,1}^2|\\ &\leq |e_{L,1} - \alpha_1 e_{L,1}^m e_{L+4,1} + 2\alpha_1 e_{L,1}^{m+1} - \alpha_1 e_{L,1}^m e_{L-4,1}\\ &\quad -\alpha_2 e_{L,1}^{m-1} e_{L+4,1}^2 + 2\alpha_2 e_{L,1}^{m-1} e_{L+4,1} e_{L-4,1}\\ &\quad -\alpha_2 e_{L,1}^{m-1} e_{L-4,1}^2| = |R_{p,1}| \leq c(k + h^2).\end{aligned} \tag{18}$$

Suppose that $\|e_j\|_\infty \leq c(k + h^2), j = 1, 2, ..., n - 1$ and $|e_{L,n}| = \max_{4\leq p\leq M-4} |e_{p,n}|$. We have

$$\begin{aligned}|e_{L,n}| &\leq |e_{L,n}| - \alpha_1 |e_{L,n}^m||e_{L+4,n}| + 2\alpha_1 |e_{L,n}^{m+1}|\\ &\quad -\alpha_1 |e_{L,n}^m||\epsilon_{L-4,n}| - \alpha_2 |e_{L,n}^{m-1}||e_{L+4,n}^2|\\ &\quad +2\alpha_2 |e_{L,n}^{m-1}||e_{L+4,n}||e_{L-4,n}| - \alpha_2 |e_{L,n}^{m-1}||e_{L-4,n}^2|\\ &\leq |e_{L,n} - \alpha_1 e_{L,n}^m e_{L+4,n} + 2\alpha_1 e_{L,n}^{m+1} - \alpha_1 e_{L,n}^m e_{L-4,n}\\ &\quad -\alpha_2 e_{L,n}^{m-1} e_{L+4,n}^2 + 2\alpha_2 e_{L,n}^{m-1} e_{L+4,n} e_{L-4,n}\\ &\quad -\alpha_2 e_{L,n}^{m-1} e_{L-4,n}^2| = |\tilde{c} e_{L,n-1} + R_{p,1}|\\ &\leq \tilde{c} |e_{L,n-1}| + c(k + h^2)\\ &\leq \tilde{c} \|e_{n-1}\|_\infty + c(k + h^2) = c(k + h^2).\end{aligned} \tag{19}$$

Hence, Theorem 2 is proved. ■

V. Formulation of the Quarter-Sweep Successive Over-Relaxation Method

In this section, the formulation of the QSSOR method to solve one-dimensional PME is presented. Let Equation 6 be written in the form of

$$\begin{aligned} F_{p,n} &= U_{p,n} - \alpha_1 U_{p,n}^m (U_{p+4,n} - 2U_{p,n} + U_{p-4,n}) \\ &-\alpha_2 U_{p,n}^{m-1}(U_{p+4,n} - U_{p-4,n})^2 - U_{p,n-1}, \end{aligned} \quad (20)$$

with $p = 4, 8, ..., M-4, n = 1, 2, ..., N$. The corresponding system of nonlinear equations to Equation 20 is

$$F(\widehat{U}) = 0, \quad (21)$$

where $F(.) = (F_{4,n}(.), F_{8,n}(.), ..., F_{M-4,n}(.))^T$ and $\widehat{U} = (U_{4,n}, U_{8,n}, ..., U_{M-4,n})$. A second order Newton method is applied to solve the nonlinear system shown by Equation 21. Then, we have a system of linear equations as follows:

$$J_F(\widehat{U}^{(iter)})\underline{\widehat{U}}^{(iter)} = -F(\widehat{U}^{(iter)}), \quad (22)$$

where

$$J_F(\widehat{U}^{(iter)}) = \begin{bmatrix} \frac{\delta F_{4,n}}{\delta U_{4,n}} & \frac{\delta F_{4,n}}{\delta U_{8,n}} & \cdots & \frac{\delta F_{4,n}}{\delta U_{M-4,n}} \\ \frac{\delta F_{8,n}}{\delta U_{4,n}} & \frac{\delta F_{8,n}}{\delta U_{8,n}} & \cdots & \frac{\delta F_{8,n}}{\delta U_{M-4,n}} \\ \vdots & \vdots & & \vdots \\ \frac{\delta F_{M-4,n}}{\delta U_{4,n}} & \frac{\delta F_{M-4,n}}{\delta U_{8,n}} & \cdots & \frac{\delta F_{M-4,n}}{\delta U_{M-4,n}} \end{bmatrix}, \quad (23)$$

and

$$\underline{\widehat{U}}^{(iter)} = \widehat{U}^{(iter)} - \widehat{U}^{(iter-1)}, iter = 1, 2, ... \quad (24)$$

From the previous work [22], using the separation of the matrix (Equation 23) into

$$J_F(\widehat{U}^{(iter)}) = D + L + V, \quad (25)$$

where D, L and V indicate the diagonal, the lower and upper part of the large and sparse matrix, a Gauss-Seidel iterative method via a quarter-sweep approximation can be derived as

$$\underline{\widehat{U}}^{(iter)} = (D+L)^{-1}(-V\underline{\widehat{U}}^{(iter-1)} - F(\widehat{U}^{(iter)}). \quad (26)$$

The derivation of the proposed QSSOR method is made by adding a relaxation parameter ω to Equation 26. The idea of adding a SOR parameter to the Gauss-Seidel is adopted from the concept of a SOR iterative method [33]. Therefore, the QSSOR method can be derived in the form of

$$\begin{aligned} \underline{\widehat{U}}^{(iter)} &= (1-\omega)\underline{\widehat{U}}^{(iter-1)} \\ &+\omega(D+L)^{-1}(-V\underline{\widehat{U}}^{(iter-1)} - F(\widehat{U}^{(iter)}), \end{aligned} \quad (27)$$

where $1 < \omega < 2$.

The algorithm of the QSSOR method is provided in Algorithm 1. The selection of the optimal value of $\omega(\pm 0.01)$ can be determined by running the Algorithm 1 program several times until the lowest number of total iterations is recorded. The theoretical optimal value of ω for solving a large system of linear equations can be referred to in [33]. In addition, the experiment sets the tolerance error $\epsilon = 10^{-10}$ to standardize the convergence criteria for the implementation of the QSSOR method with several testing methods.

Algorithm 1 QSSOR method

Define the value of SOR parameter ω, initial guess $\underline{\widehat{U}}^{(0)}$ and the tolerance error ϵ;
Define the initial and boundaries;
while $n \leq N$ **do**
  Set $iter = 0$;
  Construct $J_F(\widehat{U}^{(iter)})\underline{\widehat{U}}^{(iter)} = -F(\widehat{U}^{(iter)})$;
  while $\left|(\widehat{U}^{(iter)}) - (\widehat{U}^{(iter-1)})\right| > \epsilon$ **do**
    while $\left|\underline{\widehat{U}}^{(iter)} - \underline{\widehat{U}}^{(iter-1)}\right| > \epsilon$ **do**
      $\underline{\widehat{U}}^{(iter)} = (1-\omega)\underline{\widehat{U}}^{(iter-1)} + \omega(D+L)^{-1}(-V\underline{\widehat{U}}^{(iter-1)} - F(\widehat{U}^{(iter)}))$;
    end while
    $\widehat{U}^{(iter)} = \widehat{U}^{(iter-1)} + \underline{\widehat{U}}^{(iter)}$;
    $iter++$;
  end while
  $n++$;
end while
Display the numerical outputs

VI. Numerical Experiment and Results

This paper selected three different initial-boundary value problems of one-dimensional PME to test the proposed QSSOR method. The outputs such as the number of iterations $(iter_{max})$, the computation time $(sec.)$ and the absolute errors (e_{max}) are recorded for five different sizes of matrices, $M = 256, 512, 1024, 2048$ and 4096. Then, the results are compared to the standard SOR and QSGS [22], which both are implemented independently using the same sizes of matrices. Below are the following tested problems:

Problem 1. Consider a simple quadratic PME, which describes the unsteady flow of groundwater with the presence of a free surface [31],

$$u_t = (uu_x)_x, \quad (28)$$

subjects to the initial condition, $u(x,0) = x, 0 \leq x \leq 1$, and the two-sided boundaries, $u(0,t) = t, u(1,t) = 1+t, 0 \leq t \leq 1$. The exact solution is $u(x,t) = x+t$.

Problem 2. Consider a slow diffusion type PME [31],

$$u_t = (u^2 u_x)_x, \quad (29)$$

with the initial condition, $u(x,0) = (x+1)/4, 0 \leq x \leq 1$, and the Dirichlet boundary conditions, $u(0,t) = 1/(2(\sqrt{4-t})), u(1,t) = 1/\sqrt{4-t}, 0 \leq t \leq 1$. The exact solution is given by $u(x,t) = (x+1)/(2(\sqrt{4-t}))$.

Problem 3. Consider a fast diffusion type PME, which is also known as the Fujita–Storm equation [31], [34],

$$u_t = 0.5(u^{-2}u_x)_x, \quad (30)$$

The initial condition is set at $u(x,0) = 1/\sqrt{0.7x+1.35}$ for $0 \leq x \leq 1$, whereas the boundaries are $u(0,t) = 1/\sqrt{1.35-0.1225t}, u(1,t) = 1/\sqrt{0.7-0.1225t+1.35}$, for $0 \leq t \leq 1$. The exact solution is $u(x,t) = 1/\sqrt{0.7x-0.1225t+1.35}$.

From the independent numerical experiments using the three numerical methods (SOR, QSGS and QSSOR) to solve Problem 1, 2 and 3, the recorded outputs are tabulated in the following Tables I, II and III, respectively.

TABLE I
A NUMERICAL OUTPUT FROM TESTING PROBLEM 1.

M	Method	$iter_{max}$	$sec.$	$emax$
256	SOR ($\omega = 1.93$)	2328	1.68	3.19×10^{-10}
	QSGS	3835	1.91	2.75×10^{-8}
	QSSOR($\omega = 1.77$)	562	0.89	1.18×10^{-10}
512	SOR ($\omega = 1.97$)	4942	5.89	9.10×10^{-11}
	QSGS	13,678	7.78	1.22×10^{-7}
	QSSOR($\omega = 1.87$)	1142	0.86	2.09×10^{-10}
1024	SOR ($\omega = 1.98$)	9445	21.36	1.21×10^{-9}
	QSGS	48,395	44.79	5.33×10^{-7}
	QSSOR($\omega = 1.93$)	2328	4.44	3.19×10^{-10}
2048	SOR ($\omega = 1.99$)	18,234	76.94	1.22×10^{-9}
	QSGS	169,693	270.08	2.10×10^{-6}
	QSSOR($\omega = 1.97$)	4942	8.71	9.10×10^{-11}
4096	SOR ($\omega = 1.99$)	65,027	664.53	2.79×10^{-7}
	QSGS	587.031	2068.44	7.62×10^{-6}
	QSSOR($\omega = 1.98$)	9445	73.92	1.21×10^{-9}

TABLE II
A NUMERICAL OUTPUT FROM TESTING PROBLEM 2.

M	Method	$iter_{max}$	$sec.$	$emax$
256	SOR ($\omega = 1.92$)	1784	2.85	8.39×10^{-5}
	QSGS	1345	1.43	8.39×10^{-5}
	QSSOR($\omega = 1.73$)	462	0.78	8.39×10^{-5}
512	SOR ($\omega = 1.96$)	3490	4.73	8.39×10^{-5}
	QSGS	4824	4.86	8.39×10^{-5}
	QSSOR($\omega = 1.85$)	908	1.96	8.39×10^{-5}
1024	SOR ($\omega = 1.98$)	6758	21.23	8.39×10^{-5}
	QSGS	17,308	27.22	8.39×10^{-5}
	QSSOR($\omega = 1.92$)	1784	4.65	8.39×10^{-5}
2048	SOR ($\omega = 1.99$)	13,290	82.05	8.39×10^{-5}
	QSGS	61,658	192.19	8.40×10^{-5}
	QSSOR($\omega = 1.96$)	3490	14.79	8.39×10^{-5}
4096	SOR ($\omega = 1.99$)	27,821	330.78	8.39×10^{-5}
	QSGS	218,147	1194.31	8.43×10^{-5}
	QSSOR($\omega = 1.98$)	6758	57.54	8.39×10^{-5}

TABLE III
A NUMERICAL OUTPUT FROM TESTING PROBLEM 3.

M	Method	$iter_{max}$	$sec.$	$emax$
256	SOR ($\omega = 1.92$)	1706	2.91	2.97×10^{-6}
	QSGS	2015	3.72	2.88×10^{-6}
	QSSOR($\omega = 1.70$)	420	0.78	2.90×10^{-6}
512	SOR ($\omega = 1.96$)	3381	5.39	2.98×10^{-6}
	QSGS	7082	6.61	2.90×10^{-6}
	QSSOR($\omega = 1.84$)	837	2.27	2.96×10^{-6}
1024	SOR ($\omega = 1.98$)	6687	20.14	2.98×10^{-6}
	QSGS	24,325	51.35	2.71×10^{-6}
	QSSOR($\omega = 1.92$)	1706	4.15	2.97×10^{-6}
2048	SOR ($\omega = 1.99$)	13,158	77.29	2.98×10^{-6}
	QSGS	81,729	343.43	1.86×10^{-6}
	QSSOR($\omega = 1.96$)	3381	17.42	2.98×10^{-6}
4096	SOR ($\omega = 1.99$)	33,611	404.27	2.91×10^{-6}
	QSGS	265,698	1772.49	3.33×10^{-6}
	QSSOR($\omega = 1.98$)	6687	46.46	2.98×10^{-6}

Based on the tabulated numerical outputs shown in Tables I, II and III, we observe that the QSSOR method required the least iterations and the shortest computational time between the three numerical methods for solving all PME problems. The QSSOR method is more efficient in solving PME problems than the SOR and QSGS methods. The outputs showed the superiority of a quarter-sweep scheme with single SOR parameter in terms of computational efficiency. The quarter-sweep iteration scheme has successfully reduced the computational complexity of solving the system of linearized equations at each time step. Then, the optimal SOR parameter has successfully improved the convergence rate of the approximate values. Hence, the QSSOR method obtained the final approximate solutions of the selected PME problems in a shorter time compared to the SOR and QSGS methods.

Based on the absolute errors produced from solving PME using different size of M (see Tables I, II and III), we observe that the absolute errors produced by the QSSOR method are smaller than the SOR and QSGS methods. It can be said that the accuracy of the QSSOR method is better than the SOR and QSGS methods. The QSSOR method produced a significantly smaller absolute error than the SOR and QSGS methods in solving Problem 1. Then, there is no significant difference in the accuracy of the methods used for solving Problem 2. The deviation of the e_{max} values are very small in solving Problem 2 due to the first-order accurate in time and second-order accurate in space of the quarter-sweep implicit finite difference scheme. For Problem 3, the accuracy of the QSSOR method is better than the SOR method from $M = 256$ to 1024. However, the value of e_{max} by the QSSOR method starts to increase and becomes bigger than the e_{max} of the SOR method at $M = 2048, 4096$. On the other hand, the accuracy of the QSSOR method is better than its variant, QSGS, when the size of a matrix is sufficiently large, $M = 4096$. The study also found that when the value of M increases, the magnitude of e_{max} does not consistently decrease for some PME problems. The magnitude of e_{max} will fluctuate at some sizes of matrices. After a thorough investigation on this issue, we found that the accumulation of errors occurred because of the direct computation via average points for the remaining points in the quarter-sweep approximation.

Furthermore, a study of arithmetic operations per iteration is conducted to investigate the computational complexity of the QSSOR iteration. The result is presented in IV. Based on IV, it can be observed that the number of unknown grid points computed by iteration via the quarter-sweep iteration scheme (QSGS and QSSOR) is lesser than the implicit iteration scheme (SOR). Thus, the number of arithmetic operations per quarter-sweep iteration scheme is lesser than the SOR iteration. Furthermore, the quarter-sweep scheme can reduce the computational complexity for solving a large linear system because of fewer arithmetic operations. Besides that, since the QSSOR method uses single SOR parameter in the computation, it needs more arithmetic operations for Plus/Minus and Multiply/Divide compared to its variant, the QSGS method. However, it does not negatively influence the performance of the QSSOR iteration since the use of optimal SOR parameter significantly improves the solution's convergence rate.

Besides that, a numerical convergence test is conducted to verify the theoretical convergence of the quarter-sweep finite difference approximation as discussed in Section IV. The numerical convergence test uses different values of spatial and

TABLE IV
ARITHMETIC OPERATIONS PER ITERATION BY SOR, QSGS, AND QSSOR METHODS

Method	Plus/Minus	Multiply/Divide
SOR	$4(M-1)$	$5(M-1)$
QSGS	$2(M/4-1)$	$3(M/4-1)$
QSSOR	$4(M/4-1)$	$5(M/4-1)$

TABLE V
A NUMERICAL CONVERGENCE TEST USING PROBLEM 1.

$h\backslash k$	1/100	1/1000	1/10000
$1/256$	$2.74(10^{-8})$	$3.85(10^{-11})$	$1.11(10^{-15})$
$1/512$	$1.22(10^{-7})$	$6.93(10^{-10})$	$5.11(10^{-15})$
$1/1024$	$5.29(10^{-7})$	$6.68(10^{-9})$	$2.26(10^{-14})$
$1/2048$	$2.09(10^{-6})$	$2.91(10^{-8})$	$9.78(10^{-14})$
$1/4096$	$7.59(10^{-6})$	$1.08(10^{-7})$	$3.81(10^{-13})$

TABLE VI
A NUMERICAL CONVERGENCE TEST USING PROBLEM 2.

$h\backslash k$	1/100	1/1000	1/10000
$1/256$	$8.39(10^{-5})$	$9.74(10^{-7})$	$1.08(10^{-8})$
$1/512$	$8.39(10^{-5})$	$9.74(10^{-7})$	$1.08(10^{-8})$
$1/1024$	$8.39(10^{-5})$	$9.74(10^{-7})$	$1.08(10^{-8})$
$1/2048$	$8.40(10^{-5})$	$9.74(10^{-7})$	$1.08(10^{-8})$
$1/4096$	$8.43(10^{-5})$	$9.75(10^{-7})$	$1.09(10^{-8})$

TABLE VII
A NUMERICAL CONVERGENCE TEST USING PROBLEM 3.

$h\backslash k$	1/100	1/1000	1/10000
$1/256$	$2.88(10^{-6})$	$9.84(10^{-8})$	$3.58(10^{-9})$
$1/512$	$2.90(10^{-6})$	$1.22(10^{-7})$	$6.44(10^{-10})$
$1/1024$	$2.71(10^{-6})$	$1.17(10^{-7})$	$3.39(10^{-9})$
$1/2048$	$1.86(10^{-6})$	$1.51(10^{-7})$	$2.51(10^{-8})$
$1/4096$	$3.33(10^{-6})$	$6.24(10^{-7})$	$9.80(10^{-8})$

time steps to observe whether the maximum absolute errors decrease as both spatial and time steps approach to zero. Tables V, VI and VII illustrate the numerical convergence of the quarter-sweep finite difference approximation to the one-dimensional PME for Problem 1, 2 and 3, respectively.

Based on Tables V, VI and VII, it can be observed that the maximum absolute errors produced by the quarter-sweep approximation to the one-dimensional PME decrease as both spatial and time steps decrease. Hence, the quarter-sweep approximation scheme is numerically convergent. Therefore, there is an agreement between the numerical convergence of the quarter-sweep approximation and the theoretical convergence presented in Section IV. Finally, the graphical representations of numerical solutions obtained from the use of the QSSOR method on Problem 1, 2 and 3 are provided. Figure 3, 4 and 5 show the numerical solutions subject to the predefined initial-boundary conditions of Problem 1, 2 and 3, respectively.

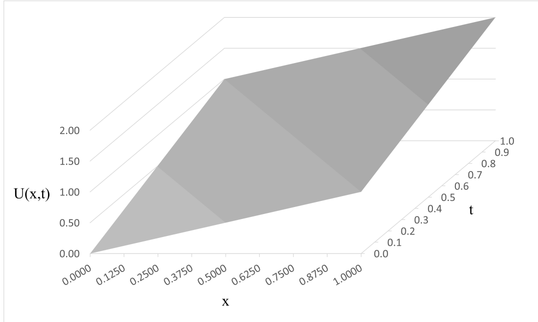

Fig. 3. Numerical solutions of Problem 1.

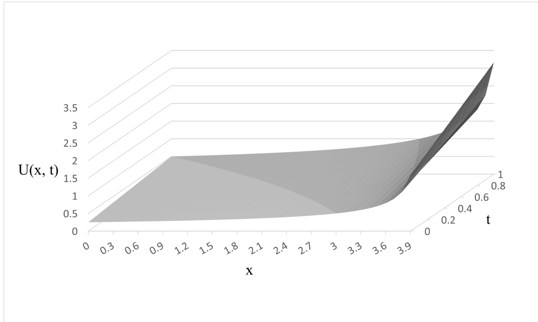

Fig. 4. Numerical solutions of Problem 2.

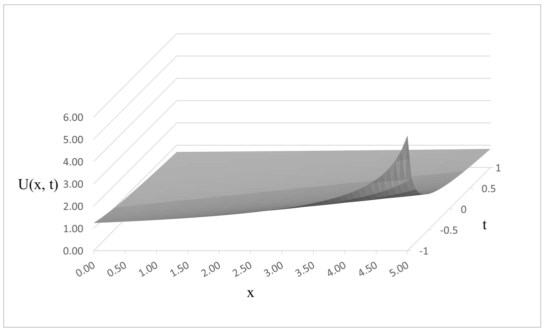

Fig. 5. Numerical solutions of Problem 3.

VII. CONCLUSION

This paper presented the FDM-based numerical method called the QSSOR method for solving the one-dimensional PME problems. The quarter-sweep approximation scheme is discussed together with its stability and convergence analysis. The stability and convergence theorems support the outputs obtained from the numerical experiment. The numerical outputs showed the superiority of the QSSOR method in terms of efficiency against the existing SOR and QSGS methods. The proposed QSSOR method required the least iterations with minimum computation time compared to the standard SOR and QSGS methods. Moreover, the QSSOR method obtains more accurate results when solving a simple quadratic PME and a fast diffusion type PME at a sufficiently large matrix. In future works, the applicability and efficiency of the QSSOR method will be further investigated to solve different cases of PME.

REFERENCES

[1] Y. Saad, "Iterative Methods for Sparse Linear Systems," 2^{nd} edition, *Society for Industrial and Applied Mathematics*, 2003.
[2] W. Hackbusch, "Iterative Solution of Large Sparse Systems of Equations," 2^{nd} edition, *Springer*, 2016.
[3] J. C. Strikwerda, "Finite Difference Schemes and Partial Differential Equations," 2^{nd} edition, *Society for Industrial and Applied Mathematics*, 2004.
[4] R. E. Mickens, "Construction of a novel finite-difference scheme for a nonlinear diffusion equation," *Numer. Methods Partial Differ. Equ.*, vol. 7, pp. 299-302, 1991.

[5] R. E. Mickens, "A nonstandard finite difference scheme for a Fisher PDE having nonlinear diffusion," *Comput. Math. Appl.*, vol. 45, pp. 429-436, 2003.

[6] R. Abedian, H. Adibi, and M. Dehghan, "A high-order weighted essentially non-oscillatory (WENO) finite difference scheme for nonlinear degenerate parabolic equations," *Comput. Phys. Commun.*, vol. 184, pp. 1874-1888, 2013.

[7] Y. Liu, C. W. Shu, and M. Zhang, "High order finite difference WENO schemes for nonlinear degenerate parabolic equations," *SIAM J. Sci. Comput.*, vol. 33, no. 2, pp. 939–965, 2011.

[8] D. Levy, G. Puppo, and G. Russo, "Compact central WENO schemes for multidimensional conservation laws," *SIAM J. Sci. Comput.*, vol. 22, no. 2, pp. 656-672, 2000.

[9] F. Ureña, L. Gavete, A. García, J. J. Benito, and A. M. Vargas, "Solving second order nonlinear parabolic PDEs using generalized finite difference method (GFDM)," *J. Comput. Appl. Math.*, vol. 354, pp. 221-241, 2019.

[10] E. Ozbenli and P. Vedula, "Construction of invariant compact finite-difference schemes," *Phys. Rev. E*, vol. 101, Article ID 023303, 20 pages, 2020.

[11] L. Wang, H. Li, and Y. Meng, "Numerical solution of coupled Burgers' equation using finite difference and sinc collocation method," *Engineering Letters*, vol. 29, no. 2, pp. 668-674, 2021.

[12] M. Othman and A. R. Abdullah, "An efficient four points modified explicit group Poisson solver," *Int. J. Comput. Math.*, vol. 76, pp. 203-217, 2000.

[13] W. S. Koh, J. Sulaiman, and R. Mail, "Quarter-sweep improving modified Gauss-Seidel method for pricing European option," *Matematika*, vol. 26, no. 2, pp. 179-185, 2010.

[14] W. S. Koh, J. Sulaiman, and R. Mail, "Numerical solution for 2D European option pricing using quarter-sweep modified Gauss-Seidel method," *J. Math. Stat.*, vol. 8, no. 1, pp. 129-135, 2012.

[15] M. K. M. Akhir and J. Sulaiman, "The solution of triangle element approximation for 2D Helmholtz equations using QSGS method," *Int. J. Appl. Math.*, vol. 28, no. 6, pp. 703-714, 2015.

[16] E. Aruchunan and J. Sulaiman, "Quarter-sweep Gauss-Seidel method for solving first order linear Fredholm integro-differential equations," *Matematika*, vol. 27, no. 2, pp. 199-208, 2011.

[17] E. Aruchunan, M. S. Muthuvalu, J. Sulaiman, W. S. Koh, and K. M. Akhir, "An iterative solution for second order linear Fredholm integro-differential equations," *Malays. J. Math. Sci.*, vol. 8, no. 2, pp. 157-170, 2014.

[18] M. S. Muthuvalu and J. Sulaiman, "Solving second kind linear Fredholm integral equations via quarter-sweep SOR iterative method," *J. Fundam. Sci.*, vol. 6, no. 2, pp. 104-110, 2010.

[19] A. A. Dahalan and J. Sulaiman, "Approximate solution for 2 dimensional fuzzy parabolic equations in QSAGE iterative method," *Int. J. Math. Anal.*, vol. 9, no. 35, pp. 1733-1746, 2015.

[20] A. Sunarto, P. Agarwal, J. Sulaiman, J. V. L. Chew, and E. Aruchunan, "Iterative method for solving one-dimensional fractional mathematical physics model via quarter-sweep and PAOR," *Adv. Differ. Equ.*, vol. 2021, pp. 1-12, 2021.

[21] A. Sunarto, J. Sulaiman, and J. C. V. Lung, "Numerical solution of the time-fractional diffusion equations via quarter-sweep preconditioned Gauss-Seidel method," *Int. J. Appl. Math.*, vol. 34, no. 1, pp. 111-125, 2021.

[22] J. C. V. Lung and J. Sulaiman, "On quarter-sweep finite difference scheme for one-dimensional porous medium equations," *Int. J. Appl. Math.*, vol. 33, no. 3, pp. 439-450, 2020.

[23] Y. H. Ng and M. K. Hasan, "Investigation of steady state problems via quarter sweep schemes," *Sains Malaysiana*, vol. 42, no. 6, pp. 837-844, 2013.

[24] N. Jha, B. K. Sharma, and R. C. Chaudhary, "Design and analysis of quarter sweep ADEI algorithm for linear and nonlinear two point boundary value problems containing singularity: application to Burger's equation," *Asian J. Exp. Sci.*, vol. 23, no. 1, pp. 329-340, 2009.

[25] M. U. Alibubin, A. Sunarto, and J. Sulaiman, "Quarter-sweep nonlocal discretization scheme with QSSOR iteration for nonlinear two-point boundary value problems," *J. Phys.: Conf. Ser.*, vol. 710, Article ID 012023, 7 pages, 2016.

[26] J. L. Díaz, "Modeling of an aircraft fire extinguishing process with a porous medium equation," *SN Appl. Sci.*, vol. 2, Article ID 2108, 20 pages, 2020.

[27] J. Kesarwani and R. Meher, "Modeling of an imbibition phenomenon in a heterogeneous cracked porous medium on small inclination," *Spec. Top. Rev. Porous Media*, vol. 12, no. 1, pp. 27-52, 2021.

[28] L. Bonorino, R. de Paula, P. Gonçalves, and A. Neumann, "Hydrodynamics of porous medium model with slow reservoirs," *J. Stat. Phys.*, vol. 179, pp. 748-788, 2020.

[29] R. Borana, V. Pradhan, and M. Mehta, "Numerical solution of instability phenomenon arising in double phase flow through inclined homogeneous porous media," *Perspect. Sci.*, vol. 8, pp. 225-227, 2016.

[30] B. Jitsom, S. Sungnul, and E. Kunnawuttipreechachan, "Numerical solutions of the moisture transport in rough rice," *Engineering Letters*, vol. 28, no. 3, pp. 742-750, 2020.

[31] A. D. Polyanin and V. F. Zaitsev, "Handbook of Nonlinear Partial Differential Equations," 2^{nd} edition, *Chapman and Hall/CRC*, 2011.

[32] J. L. Vázquez, "The Porous Medium Equation: Mathematical Theory," 1^{st} edition, *Clarendon Press*, 2006.

[33] D. Young, "Iterative methods for solving partial difference equations of elliptic type," *Trans. Am. Math. Soc.*, vol. 76, no. 1, pp. 92-111, 1954.

[34] A. M. Wazwaz, "The variational iteration method: a powerful scheme for handling linear and nonlinear diffusion equations," *Comput. Math. Appl.*, vol. 54, pp. 933-939, 2007.