A Non-Dimensional Mathematical Model of Shoreline Evolution with a Groin Structure Using an Unconditionally Stable Explicit Finite Difference Technique

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Abstract—Coastal erosion is a natural phenomenon that occurs when sediment transport away from the coast is not counterbalanced by fresh material growth along the shoreline. A sea wall and a groin were created to prevent coastal erosion and floods. The future topography of the beach is being investigated using shoreline evolution analysis. Erosion, accretion, and sea level changes are basic stages that have a significant impact on the coastal structure. A qualitative analysis of the model coastal behavior in relation to the controlling process is required to research beach erosion and beach deposition. When stated in terms of non-dimensional variables, all are mathematically equivalent. In general, the models do not have to be dimensionally different. Those might just be modifications of the same problems. One can solve a wide range of models with a single solution to the related non-dimensional equation. In this research, we provide a governing equation when a groin is introduced to a one-dimensional shoreline growth model. A non-dimensional shoreline evolution model with a groin structure model is provided. The model now has the ability to manipulate physical parameters. When groin structural effects are present, the initial condition setting method and boundary condition approaches are also given. To approximate the incremental model in each year, the forward time-centered space technique and the unconditionally stable Saulyev finite difference methods are used. The Saulyev finite difference approach can handle numerical solutions in almost any scenario since the stability requirements are not restricted. The Saulyev finite difference technique can be very useful for computing a practical conceptual design of shoreline evolution since the number of grids has increased. The numerical models offered provide a viable simulation for evaluating long-term coastal development. The proposed modeling may be used to forecast the effectiveness of constructing a groin system on a local beach.

Index Terms—shoreline evolution, groin structure, non-dimensional, mathematical model, finite difference method

INTRODUCTION

Coastal erosion is a natural process that occurs when sediment movement away from the coast is not counterbalanced by fresh material growth along the shoreline. This is certainly a problem that is related to coastal erosion. A sea wall and groin were built to prevent coastal erosion and floods. The beach's future topography is analyzed using shoreline evolution research. Basic phrases that have a substantial influence on the coastal structure are erosion, accretion, and sea-level variations.

The partial differential equation represents several disciplines, including mass, heat, energy, velocity, and vorticity (see for example [1-2]). In [3-8] these papers, the diffusion equation has been used to solve a variety of engineering problems, including pollutant and salinity transport in rivers and streams, and groundwater and contaminant dispersion in shallow lakes. In [9-11], they presented a case study of water level forecasting, a water quality evaluation based on probabilistic echo state networks, and the fluid dynamics of nonaqueous phase pollutants in groundwater.

Understanding the ideal shorelines' responses to the governing processes is important in the study of beach actions. A model for describing realistic situations involving general shoreline configuration settings and time-varying waves in more detail is proposed. As a result, numerical methods of shoreline evolution are preferred to analytical methods. In 1966, this paper [12] introduces modern logical design guidelines for groin structures. They are organized into three basic categories: Coastal processes, Functional design and Structural design. In [13] this paper expands on both the theoretical and practical concepts of mathematical models related to coastal behaviors: Theoretically, the influence of diffraction behind the groin is used to determine computer programming; Practically, the coastal constant in the theoretical model of the coast is expressed in terms of wave height and SVASEK's theoretical wave direction. In [14] this paper describes the development of the governing equations in general form and describes the assumptions and techniques used to obtain more than 25 analytical solutions. Solution for shoreline evolution with and without the influence of coastal structures. It covers situations involving beach filling of initial shape, sand mining, river discharge, groin and jetty and breakwater etc. The wind wave-driven longshore sediment transport rate and shoreline change are evaluated using a numerical model based on one-line theory in [15].
The model transforms waves from deep water into the surf zone and calculates their breaking characteristics. The model [16] provides complete, time-dependent simulations of shoreline evolution for coastlines driven by structures and a variety of boundary conditions that are both practical and reliable. This research looks at two case studies: New Jersey's Sea Isle City Beach and Egypt's Nile Delta Coast.

The purpose of [17] is to quantify the changes in the shoreline along with the sand reclamation for the Sultan Mahmud Airport runway in Kuala Terengganu. Littoral Processes and Coastline Kinetics (LITPACK) numerical model is the numerical device employed to solve the shoreline problem. This research [18] describes a numerical modelling framework called GENESIS that is used to simulate long-term shoreline change caused by spatial and temporal variations in longshore sand transport at coastal engineering projects. The modelling system is managed via an organized and user-friendly interface, which reduces the need for the operator to get concerned with computer code specifics. The modelling-system application method is described from the viewpoint of engineers and planners involved in the evaluation of shore-protection projects [19-21].

Then, if a groin is added to a one-dimensional shoreline evolution model, we propose a governing equation for this research. It is provided with a non-dimensional shoreline evolution model with a groin construction model. The model can now be used to manipulate physical parameters.

I. GOVERNING EQUATION

A. Shoreline Evolution Model

The beach profile is supposed to travel landward and seaward while maintaining the same form in the one-line model, meaning that all bottom contours are parallel. As a result, specifying the horizontal location of the profile with respect to the baseline is sufficient under this assumption, and one contour line may be used to represent changes in the beach plan shape and volume as the beach reduces and accretes. Sand is carried alongshore between two well-defined limiting heights on the profile, according to the model’s main assumption. If there is a variation in the alongshore sand transport rate at the lateral sides of the section and the related sand continuity, it contributes to the volume change.

At all times, the laws of mass conservation must be applied to the system. The following differential equation for shoreline evolution is produced using the given definitions:

\[ \frac{\partial y}{\partial t} = \frac{1}{D_b + D_c} \left( -\frac{\partial Q}{\partial x} \right), \]  

(1)

where \( x \) is the alongshore coordinate (m), \( y \) is the shoreline positions (m) and perpendicular to x-axis, \( t \) is time (day), \( Q \) is the long-shore sand transport rate (m³/day), \( D_b \) is the average berm height (m) and \( D_c \) is the average closure depth (m).

In order to solve Eq.(1), an equation for the longshore sand transport rate \( Q \) must be specified. This quantity is thought to be created by a wave that strikes the coastline obliquely. [22] provided a general expression for the long-shore sand transport rate,

\[ Q = Q_0 \sin (2\alpha_s), \]  

(2)

where \( Q_0 \) is the amplitude of the long-shore sand transport rate. The empirical predictive formula for the amplitude of the long-shore sand transport rate is [23]:

\[ Q_0 = \frac{\rho}{16} \left( \frac{H^2 c_w}{\sigma_s} \right) \frac{K}{(\sigma_s - \rho)(1-n)}, \]  

(3)

where the subscript \( b \) represent the value at the point breaking, \( \rho \) is the density of sea water (kg/m³), \( \rho_s \) is the density of the sediment (kg/m³), \( n \) is the porosity, \( K \) is the dimensionless coefficient which is a function of particle size, \( H \) is the wave height and \( c_s \) is the wave group velocity.

The quantity \( \alpha_s \) the impact angle between breaking wave crests angle with local shoreline, and may be written as,

\[ \alpha_s = \alpha_0 - \tan^{-1} \left( \frac{\partial y}{\partial x} \right), \]  

(4)

where \( \alpha_0 \) is the angle between breaking wave crests and the x-axis. For beaches with a slight slope, the breaking wave angle to the coastline is likely to be minimal. Assuming that,

\[ \sin(2\alpha_s) \approx 2\alpha_s, \]

and

\[ \tan^{-1} \left( \frac{\partial y}{\partial x} \right) \approx \frac{\partial y}{\partial x}. \]

Substituting Eq.(4) into Eq.(2), and assuming the beach with mild slope yields,

\[ Q = Q_0 \left( 2\alpha_0 - 2 \frac{\partial y}{\partial x} \right), \]  

(5)

Substituting Eq.(5) into Eq.(1), and neglecting the sources or sinks along the coast gives,

\[ \frac{\partial^2 y}{\partial t^2} = D \frac{\partial^2 y}{\partial x^2}, \]  

(6)

for all \( (x,t) \in [0,L] \times [0,T] \), where \( D = \frac{2Q_0}{D_b + D_c} \).

B. The Initial and Boundary Conditions of the One-Dimensional Model

a) The Initial Condition

Groin system that is impermeable and straight. The shoreline of the initials is considered to be parallel to the x-axis.

Assume \( \alpha_0 \) is the braking wave angle to the beach as show in Fig. 3. As a result, the rate of sand transport along the beach is homogeneous. As seen in Fig. 1, the groin is
inserted instantly at \( x = 0 \). As a result, the initial conditions become,
\[
y(x,0) = 0 ,
\]
for all \( x \in [0,L] \).

*b) The Left Boundary Condition*

The left boundary condition is defined by the interpolation function of the measured evolutionary data. It is shoreline evolution with the left-hand side groin system. The boundary conditions are assumed to be as follows,
\[
y(0,t) = g(t) ,
\]
for all \( t \in [0,T] \), where \( g(t) \) is a given interpolation function of the measured evolutionary data at the left-hand side groin system.

*c) The Right Boundary Condition*

The right boundary condition is defined by the interpolation function of the measured evolutionary data. It is shoreline evolution with the right-hand side groin system. The boundary conditions are assumed to be as follows,
\[
y(L,t) = h(t) ,
\]
for all \( t \in [0,T] \), where \( h(t) \) is a given interpolation function of the measured evolutionary data at the right-hand side groin system.

\[ Q_0 \] is the amplitude of the long-shore sand transport rate \((m^3/day)\).  
\[ D_a \] is the average berm height \((m)\).  
\[ D_c \] is the average closure depth \((m)\).  
\( L \) is the length of alongshore \((m)\).  
\( \tau \) is time of simulation \((day)\).  

**Fig. 3. Breaking wave crests impact angle.**

**Fig. 4. Beach profile and shoreline physical parameters.**

**II. A NON-DIMENSIONAL SHORELINE EVOLUTION MODEL**

**A. A Non-Dimensional Shoreline Evolution Model**

Taking non-dimensional technique [24] into Eq.(6), we obtain the following,
\[
\frac{\partial Y}{\partial T} = \frac{\partial^2 Y}{\partial X^2} ,
\]
for all \( (X,T) \in [0,1] \times [0,\Gamma] \), where \( L \) is the length of alongshore, \( Y \) is the expected shoreline evolution,
\[
Y = \frac{x}{Y} , \quad X = \frac{x}{L} , \quad \text{and} \quad T = \frac{Dt}{L^2} .
\]

Equation (10) is similar to the one-dimensional heat equation which has a thermal conductivity coefficient of 1, so in order to solve a problem it is necessary to define the initial conditions and the boundary conditions.

**B. The Initial and Boundary Conditions of the Non-Dimensional Model**

The initial and boundary conditions for a non-dimensional model can be simply defined under the known initial and boundary conditions according to Eqs.(7)-(9) for a one-dimensional model, so we can define the necessary conditions as the follows. These means that the initial condition becomes,
\[
y(X,0) = F(X) \quad \text{for all} \quad X \in [0,1] ,
\]
Boundary conditions are also defined by:
\[ Y(0, T) = G(T) \quad \text{for all } T \in [0, \Gamma], \]
(12)
and
\[ Y(1, T) = H(T) \quad \text{for all } T \in [0, \Gamma], \]
(13)
where \( F(X) = 0 \), \( G(T) = \frac{g(t)}{Y} \) and \( H(T) = \frac{h(t)}{Y} \).

III. NUMERICAL TECHNIQUES

A. Grid Spacing

We now discretize the domain of Eq.(10) by dividing the interval \([0, 1]\) into \( M \) subintervals such that \( M\Delta X = 1 \) and the time interval \([0, \Gamma]\) into \( N \) subintervals such that \( N\Delta T = \Gamma \). We then approximate \( Y(X, T) \) by \( Y^n \), at the point \( X = m\Delta X \) and \( T = n\Delta T \), where \( 0 \leq m \leq M \) and \( 0 \leq n \leq N \) in which \( M \) and \( N \) are positive integers.

![Grid spacing](image)

Fig. 5. Grid spacing.

B. The Traditional Forward Time Centered Space (FTCS) Techniques

The forward time centered space (FTCS) Technique is employed. Consequently, the finite difference approximation \([19],[25]\) becomes,
\[ Y(X_n, T_n) \approx Y^n, \]
(14)
\[ \frac{\partial Y}{\partial T} \approx \frac{Y^n_{m+1} - Y^n_{m-1}}{2\Delta T}, \]
(15)
\[ \frac{\partial Y}{\partial X} \approx \frac{Y^n_{m+1} - Y^n_{m-1}}{2\Delta X}, \]
(16)
\[ \frac{\partial^2 Y}{\partial X^2} \approx \frac{Y^n_{m+1} - 2Y^n_{m} + Y^n_{m-1}}{(\Delta X)^2}. \]
(17)
Substituting Eqs.(14)-(17) into Eq.(10), we obtain,
\[ \frac{Y^n_{m} - Y^n}{\Delta T} \approx \frac{Y^n_{m+1} - 2Y^n_{m} + Y^n_{m-1}}{(\Delta X)^2}, \]
(18)
for \( 1 \leq m \leq M - 1 \) and \( 0 \leq n \leq N - 1 \). Equation (18) can be written in an explicit form of finite difference as follows,
\[ Y^n_{m+1} \approx \mu Y^n_{m+1} + (1 - 2\mu)Y^n_{m} + \mu Y^n_{m-1}, \]
(19)
for \( 1 \leq m \leq M - 1 \) and \( 0 \leq n \leq N - 1 \), where \( \mu = \frac{\Delta T}{(\Delta X)^2} \).

C. An Unconditionally Saulve Finite Difference Techniques

The Saulve finite difference technique will be also employed. We can obtain that the finite difference approximation \([19],[25]\) becomes,
\[ Y(X_n, T_n) \approx Y^n, \]
(20)
\[ \frac{\partial Y}{\partial T} \approx \frac{Y^n_{m+1} - Y^n_{m-1}}{2\Delta T}, \]
(21)
\[ \frac{\partial Y}{\partial X} \approx \frac{Y^n_{m+1} - Y^n_{m-1}}{2\Delta X}, \]
(22)
\[ \frac{\partial^2 Y}{\partial X^2} \approx \frac{Y^n_{m+1} - 2Y^n_{m} + Y^n_{m-1}}{(\Delta X)^2}. \]
(23)
Substituting Eqs.(20)-(23) into Eq.(10), we obtain,
\[ \frac{Y^n_{m} - Y^n}{\Delta T} \approx \frac{Y^n_{m+1} - 2Y^n_{m} + Y^n_{m-1}}{(\Delta X)^2}, \]
(24)
for \( 1 \leq m \leq M - 1 \) and \( 0 \leq n \leq N - 1 \). Equation (24) can be written in an explicit form of finite difference as follows,
\[ Y^n_{m+1} \approx (1 + \mu)^{-1} \left( \mu Y^n_{m+1} + (1 - \mu)Y^n_{m} + \mu Y^n_{m-1} \right), \]
(25)
for \( 1 \leq m \leq M - 1 \) and \( 0 \leq n \leq N - 1 \), where \( \mu = \frac{\Delta T}{(\Delta X)^2} \).

IV. ERROR MEASUREMENT

A simple measure used to measure the difference between actual and approximate values is the absolute error method. The absolute error formula be as follows,
\[ E^n_m = \left| \tilde{y}^n_m - y^n_m \right|, \]
(26)
where \( \tilde{y}^n_m \) is the analytical solution of shoreline evolution and \( y^n_m \) is the approximate solution of shoreline evolution.

V. NUMERICAL EXPERIMENT

In order to investigate the shoreline evolution in the long-term scale. Assuming that the length of considered shoreline \( L \) is 5000 m, the amplitude of the long-shore transport rate \( Q_L \) is 7500 m³/day, the averaged berm height \( D_b \) is 2 m, the averaged closure depth \( D_c \) is 28 m, the breaking wave impact angle \( \alpha \) is 0.02, and the expected shoreline
The simulation setting is illustrated in Fig. 6-7.

We will employ the traditional forward time centered space (FTCS) techniques Eq.(19), and the Saulyev finite difference techniques Eq.(25), to approximate the model solution.

The analytical solution of the simulation is [26]:

\[
\tilde{y}(x,t) = \tan \alpha \sqrt{\frac{4Dt}{\pi}} e^{-\frac{x^2}{4Dt}} \frac{x}{2\sqrt{Dt}} \text{erfc} \left( \frac{x}{2\sqrt{Dt}} \right),
\]  
(27)

Fig. 6. Initial shoreline.

Fig. 7. The evolution from initial shoreline.

Fig. 8. Approximated shoreline evolution under the non-dimensional equations in 20 years when the FTCS technique is used.

Fig. 9. Approximated shoreline evolution under the non-dimensional equations in 20 years when the Saulyev technique is used.

Fig. 10. Analytical shoreline evolution in 20 years.

Fig. 11. Approximated shoreline evolution under the one-dimensional equations in 20 years when the FTCS technique is used.
Fig. 12. Approximated shoreline evolution under the one-dimensional equations in 20 years when the Saulyev technique is used.

Fig. 13. Shoreline evolution in 1 year.

Fig. 14. Shoreline evolution in 5 years.

Fig. 15. Shoreline evolution in 10 years.

Fig. 16. Shoreline evolution in 15 years.

Fig. 17. Shoreline evolution in 20 years.
The analytical solution is illustrated in Table I and the approximate solutions of both the numerical techniques are illustrated in Tables II-III, respectively.

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<th>Time (Years)</th>
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In order to be able to analyze the computational efficiency, the absolute error and comparisons of stability are illustrated in Tables IV-V, respectively.

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In Fig. 18, it is clearly confirmed that both numerical techniques produce shoreline evolution solutions close to each other even after 1, 5, 10, 15, and 20 years.

In terms of accuracy, the traditional forward time centered space (FTCS) technique is more accurate than the Saulyev finite difference techniques as shown in Table IV and Fig. 19, but the solution cannot be handled in some cases when the time increment is increased for the traditional forward time centered space (FTCS) techniques. The Saulyev finite difference techniques, on the other hand, can still be used. The Saulyev finite difference technique, on the other hand, can handle numerical solutions in almost every scenario as shown in Table V, because the stability conditions are not constrained. As a result, the Saulyev finite difference technique can also be useful for computing a practical conceptual design of shoreline evolution when the number of grids is increased.

VII. CONCLUSION

The model includes a non-dimensional shoreline evolution model as well as a groin structure model. Physical parameters can now be manipulated by the model. The initial condition setting technique and boundary condition approaches are also presented where groin structural effects are involved. The traditional forward time centered space (FTCS) technique and the unconditionally stable Saulyev finite difference strategies are employed to approximate the incremental model every year. The numerical models available give a realistic simulation for predicting long-term coastal evolution. Because the stability requirements are not limited, the Saulyev finite difference Techniques can handle numerical solutions in practically any circumstance. Since the number of grids has increased, the Saulyev finite difference technique can be highly beneficial for computing a practical conceptual design of shoreline evolution. The simulation described here can be used to forecast the success of a groin system on a local beach.

REFERENCES


VI. DISCUSSION

Using the traditional forward time centered space (FTCS) techniques and the Saulyev finite difference techniques, the annual evolution of the shoreline can be determined.

In the calculations, when we compared the approximated shoreline evolution solutions under the non-dimensionless model which occurred between the traditional forward times centered space (FTCS) technique and the Saulyev finite difference technique, they were found to be close together as shown in Figs. 8-9. For this reason, the approximated shoreline evolution solutions obtained when converted back to the solutions under the one-dimensional model are also close as shown in Figs. 11-12. Therefore, we can see that the approximate solutions as shown in Figs. 11-12 are close to the analytical solution as shown in Fig. 10.

As demonstrated in Tables II, III, and Fig. 13, the distance from the furthest shoreline evolution after 1 year is 9.576 m. The smallest distance from the evolution of the shoreline is 0.000 m.

As demonstrated in Tables II, III, and Fig. 14, the distance from the furthest shoreline evolution after 5 years is 21.412 m. The smallest distance from the evolution of the shoreline is 0.001 m.

As demonstrated in Tables II, III, and Fig. 15, the distance from the furthest shoreline evolution after 10 years is 30.282 m. The smallest distance from the evolution of the shoreline is 0.099 m.

As demonstrated in Tables II, III, and Fig. 16, the distance from the furthest shoreline evolution after 15 years is 37.087 m. The smallest distance from the evolution of the shoreline is 0.521 m.

As demonstrated in Tables II, III, and Fig. 17, the distance from the furthest shoreline evolution after 20 years is 42.825 m. The smallest distance from the evolution of the shoreline is 1.304 m.
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