Sensitivity Analysis of Variance Gamma Parameters for Interest Rate Derivatives

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Abstract—**Interest rate derivatives being financial instruments whose values are affected by movements in interest rates experience jumps due to many unforeseen circumstances, and thus, require adequate modelling and sensitivity analysis that consider such scenarios in order to minimize risks. This paper derives expressions for the greeks from parameters of a variance gamma process required when computing the sensitivities of the parameters of an interest rate derivative called *zero-coupon bond* driven by the variance gamma process.**

Index Terms—**Interest Rates, Vasicek model, Greeks, Malliavin derivative, Zero-coupon bond.**

I. Introduction

A GOOD investor or risk manager should be able to understand how changes in the parameters of a financial derivative affect its price in order to minimize risk. Variance gamma (VG) process was introduced by Madan and Seneta [1] as a Lévy process that provides a better model which captures spikes and jumps that occur in financial markets. It has been applied in different fields such as finance and engineering (Bayazit & Nolder [2], Udoye & Ekhaguere [3], Bavouzet & Messaoud [4], Salem et al [5]). The process takes care of the fact that trading activities do not occur in a uniform way, but display fluctuations of peak and less activity periods (Aguilar [6]).

This paper is an extension of the work of Udoye & Ekhaguere [3] who derived an extended Vasicek model driven by a VG process, used the extended Vasicek model to derive an expression for the price of an interest rate derivative called a *zero-coupon bond* and obtained the greeks *delta* and *gamma* of the derived price. The VG process is a type of Lévy process that captures jumps which occur in financial markets and other fields due to certain phenomena such as natural disaster, presence of abrupt information, pandemic, changes in government policies, etc. Lévy noise is a non-Gaussian noise that has found attention in different fields such as engineering, society, etc (Wei [7]).

The paper is also an extension of the work of Bayazit & Nolder [2] on sensitivity analysis in a stock market driven by an exponential Lévy process. We apply Malliavin calculus in the sensitivity analysis of the interest rate derivative with respect to the parameters of the VG process. The differentiability tools of Malliavin calculus seen in Bavouzet & Messaoud [4], Bavouzet et al [8] and Bayazit & Nolder [2] are adopted in deriving expressions for the greeks based on the parameters of the VG process. Greeks describe the sensitivity of a bond option price to changes in certain parameters and enable investors to hedge their risks.

Other part of the paper is structured as follows: Section II discusses some important tools and theorems needed in obtaining the results while Section III derives the greeks with respect to the parameters of the VG process, and then concludes the work.

II. Foundational Concept

In this section, some definitions and results from Udoye and Ekhaguere [3] needed for the success of this paper are highlighted.

Definition II.1. The *Vasicek model* [9] of interest rate is given by the following stochastic differential equation:

$$dr_t = \eta(b - r_t)dt + \sigma dX_t$$

where $\eta, b, \sigma \neq 0$ and X_t represents speed of mean-reversion, its long-term mean rate, volatility of the interest rate and a Lévy process, respectively.

Definition II.2. *Arithmetic Brownian motion* is a Lévy process given by

$$X_t = \theta t + \widehat{\sigma} W_t$$

where θ and $\widehat{\sigma} \neq 0$ denote drift and volatility of the arithmetic Brownian motion, respectively. W_t represents Wiener process. The VG process is obtained by time-changing arithmetic Brownian motion with a gamma process.

Theorem II.1. *The price $P(t,T)$ of a zero-coupon bond at time t with maturity time T driven by a VG process under extended Vasicek model is given by*

$$\begin{aligned} P(t,T) = \exp\Bigg(- \Bigg(\Bigg[&- \frac{r_0}{\eta}\big(e^{-\eta T} - e^{-\eta t}\big) + b\big(T - t \\ &+ \frac{1}{\eta}(e^{-\eta T} - e^{-\eta t})\big) + \frac{\sigma \widetilde{w}}{\eta}\Bigg[T - t \\ &+ \frac{1}{\eta}(e^{-\eta T} - e^{-\eta t})\Bigg] + \sigma \sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\theta\Delta G(s) \\ &\cdot e^{-\eta(u-s)} + \widehat{\sigma}\Delta\sqrt{G(s)}e^{-\eta(u-s)}Z\big)\Bigg] + \widetilde{w}\sigma[T-t] \\ &+ \sigma \sum_{t\leq u\leq T}(\theta\Delta G(u) + \widehat{\sigma}\Delta\sqrt{G(u)}Z) \\ &- \frac{\sigma^2}{2}\Bigg(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widehat{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg)\Bigg)\Bigg), \end{aligned} \tag{1}$$

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where

$$\widetilde{w} = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu);$$

η, b, σ and r_0 represent mean-reversion speed, long-term mean rate, volatility of the Vasicek model and initial interest rate, respectively; while θ and $\widehat{\sigma}$, respectively, represent the drift and volatility of the arithmetic Brownian motion time-changed to obtain the VG process; ν is the variance of the gamma process used as subordinator; $\Delta G(t) = G(t_+) - G(t_-)$; whereas G and Z represent gamma and Gaussian random variables, respectively.

Definition II.3. The call option price, with P as the underlying is given by

$$\mathbb{V} = e^{-r_0 T}\mathbb{E}[\Phi(P)],$$

where $\Phi(P) = \max(P-K, 0)$ represents the payoff function and K denotes the strike price.

Remark II.1. $\mathbb{V}$ is sensitive to changes in a number of parameters.
The following important greeks will be computed:
(i) Drift := $\mathcal{D} = \dfrac{\partial\mathbb{V}}{\partial\theta}$; (ii) Vega$_\nu$:= $\dfrac{\partial\mathbb{V}}{\partial\nu}$;
(iii) Vega$_{\widehat{\sigma}}$:= $\dfrac{\partial\mathbb{V}}{\partial\widehat{\sigma}}$.

Drift $\mathcal{D}$ measures the sensitivity of the bond option price to changes in the drift of the VG process. In other words, it determines the effect of changes in the skewness parameter to the value of the option price. Vega$_\nu$ determines the sensitivity of the bond option price with respect to changes in the variance of the gamma process, whereas Vega$_{\widehat{\sigma}}$ measures the sensitivity of the bond option price with respect to changes in the volatility of the arithmetic Brownian motion.

Theorem II.2. *(Malliavin integration by part theorem [2]) Let $Q_\psi = \dfrac{\partial P}{\partial\psi}$ where ψ denotes some parameters of the zero-coupon bond. Let D be the Malliavin derivative operator, then $\mathbb{M}(P) = \langle DP, DP\rangle$ is the Malliavin covariance matrix, with inverse $\mathbb{M}(P)^{-1} = \dfrac{1}{\mathbb{M}(P)}$ where $DP \neq 0$, and L is the Ornstein-Uhlenbeck operator. For a smooth function $\Phi : \mathbb{R} \to \mathbb{R}$, the following equation holds:*

$$\mathbb{E}[\partial\Phi(P)Q] = \mathbb{E}[\Phi(P)H(P,Q)]$$

where $H(P,Q)$ is the Malliavin weight given by

$$\begin{aligned} H(P,Q) &= Q\mathbb{M}(P)^{-1}LP - \mathbb{M}(P)^{-1}\langle DP, DQ\rangle \\ &\quad - Q\langle DP, D\mathbb{M}(P)^{-1}\rangle \end{aligned}$$

with $\mathbb{E}[H(P,Q)] < \infty$.

The following theorems whose proofs are in Udoye and Ekhaguere [3] will be needed for easier derivation of the greeks.

Theorem II.3. *The Malliavian derivative on a zero-coupon bond price P driven by a VG process is given by*

$$\begin{aligned} DP = -\Big[&\sigma\widehat{\sigma}\Big(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+ \sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Big) - \sigma^2\widehat{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+ \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P. \end{aligned} \tag{2}$$

Theorem II.4. *The action of the Ornstein-Uhlenbeck operator L ([3], [10]) on the price P of a zero-coupon bond driven by a VG process is given by*

$$\begin{aligned} LP = -\Big[&\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2 \\ &+ \Big(\sigma\widehat{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+ \sigma\widehat{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widehat{\sigma}(\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+ \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)^2 + Z\Big(\sigma\widehat{\sigma} \\ &\cdot(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+ \sum_{t\le u\le T}\Delta\sqrt{G(u)}) - \sigma^2\widehat{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+ \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big)\Big]P. \end{aligned} \tag{3}$$

Theorem II.5. *The inverse Malliavin covariance matrix of the zero-coupon bond price P driven by a VG process is given by*

$$\begin{aligned} \mathbb{M}(P)^{-1} = \Big(\Big[&\sigma\widehat{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+ \sigma\widehat{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widehat{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+ \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P\Big)^{-2}. \end{aligned} \tag{4}$$

while the Malliavin derivative
$D\mathbb{M}(P)^{-1}$

$$\begin{aligned} &= 2\Big[\Big(\sigma\widehat{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+ \sigma\widehat{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widehat{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) \\ &\quad + \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)^{-3}\Big]P^{-2} \\ &\quad \times\Big[\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2 \\ &\quad + \Big[\sigma\widehat{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &\quad + \sigma\widehat{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widehat{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) \\ &\quad + \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2\Big]. \end{aligned} \tag{5}$$

III. SENSITIVITY ANALYSIS WITH RESPECT TO CERTAIN PARAMETERS OF ZERO-COUPON BOND PRICE UNDER VG-DRIVEN LÉVY MARKET

In this section, the greeks of the zero-coupon bond price associated with the parameters of the VG process are derived.

A. Derivation of the greek drift *for a zero-coupon bond price driven by a VG process*

In this subsection, an expression for the greek *drift* for a zero-coupon bond price driven by a VG process is derived. The greek *drift* $\mathcal{D}$ for a VG-driven zero-coupon bond price is given by

$$\mathcal{D}=\frac{\partial}{\partial\theta}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big].$$

Recall that by equation (1),

$$\widetilde{w}=\frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)\ \Rightarrow\ \frac{\partial\widetilde{w}}{\partial\theta}=-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}.$$

Lemma III.1. *Let P be the zero-coupon bond price driven by a VG process. Then,*

$$\begin{aligned}Q_\theta=-\Big[&\frac{-\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}\\&-\frac{\widetilde{w}\sigma[T-t]}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}+\sigma\sum_{t\leq u\leq T}\Delta G(u)\\&-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]P.\end{aligned}\tag{6}$$

Furthermore, the Malliavin derivative

$$\begin{aligned}DQ_\theta=\Big[&\sigma^2\widehat{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Delta G(u)\\&+\Big[\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\Big(-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}\\&-\frac{\widetilde{w}\sigma[T-t]}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}+\sigma\sum_{t\leq u\leq T}\Delta G(u)-\sigma^2\\&\cdot\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]\mathcal{K}\Big]P,\end{aligned}\tag{7}$$

where

$$\begin{aligned}\mathcal{K}=\sigma\widehat{\sigma}&\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-\eta(u-s)}\\&+\sigma\widehat{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})-\sigma^2\widehat{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)\\&+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}.\end{aligned}\tag{8}$$

Proof: By equation (1), it follows that

$$\begin{aligned}Q_\theta=\frac{\partial P}{\partial\theta}=-\Big[&\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\frac{\partial\widetilde{w}}{\partial\theta}\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}+\widetilde{w}\sigma[T-t]\frac{\partial\widetilde{w}}{\partial\theta}\\&+\sigma\sum_{t\leq u\leq T}\Delta G(u)-\frac{\sigma^2}{2}\Big(2\sum_{t\leq u\leq T}(\theta\Delta G(u)\\&+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]P.\end{aligned}$$

Substituting the value of $\frac{\partial\widetilde{w}}{\partial\theta}$ into Q_θ gives equation (6). Hence, the Malliavin derivative

$$\begin{aligned}DQ_\theta=&-\Big[-\sigma^2\sum_{t\leq u\leq T}(\widehat{\sigma}\Delta\sqrt{G(u)})\Delta G(u)\Big]P+\\&-\Big[\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\Big(-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}+\widetilde{w}\sigma[T-t]\\&\cdot\Big(-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)+\sigma\sum_{t\leq u\leq T}\Delta G(u)-\frac{\sigma^2}{2}\\&\cdot\Big(2\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]DP\\=&\Big[\sigma^2\widehat{\sigma}\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Delta G(u)\big)P\\&+\Big[\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\Big(-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}+\widetilde{w}\sigma[T-t]\\&\cdot\Big(-\frac{1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)+\sigma\sum_{t\leq u\leq T}\Delta G(u)\\&-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]\mathcal{K}P\end{aligned}$$

which gives equation (7). ■

Lemma III.2. *Let P be the zero-coupon bond price driven by a VG process. Then,*
$Q_\theta\mathbb{M}(P)^{-1}LP$

$$=\mathcal{L}\sigma^2\widehat{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+\mathcal{L}+\frac{Z\mathcal{L}}{\mathcal{K}},\tag{9}$$

where

$$\begin{aligned}\mathcal{L}=&\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\Big(\frac{-1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)\\&+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\eta(u-s)}+\widetilde{w}\sigma[T-t]\\&\cdot\Big(\frac{-1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\Big)+\sigma\sum_{t\leq u\leq T}\Delta G(u)\\&-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big).\end{aligned}\tag{10}$$

Proof: The result follows by substituting equation (6) for Q_θ, equation (4) for $\mathbb{M}(P)^{-1}$ and equation (3) for LP, and simplifying. ■

Lemma III.3. *Let P be the zero-coupon bond price driven by a VG process, then*
$\mathbb{M}(P)^{-1}\langle DP,DQ_\theta\rangle$

$$=-\sigma^2\widehat{\sigma}\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Delta G(u)\big)\mathcal{K}^{-1}-\mathcal{L}\tag{11}$$

where $\mathcal{K}$ and $\mathcal{L}$ are given by equations (8) and (10), respectively.

Proof: The result follows by substituting equation (4) for $\mathbb{M}(P)^{-1}$, equation (2) for DP and equation (7) for DQ_θ, and simplifying. ■

Lemma III.4. *Let P be a zero-coupon bond price driven by a VG process, then*
$Q_\theta\langle DP, D\mathbb{M}(P)^{-1}\rangle$

$$= 2\mathcal{L}\Big[\sigma^2\widehat{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\cdot\mathcal{K}^{-2}+1\Big] \tag{12}$$

where $\mathcal{K}$ and $\mathcal{L}$ are given by equations (8) and (10), respectively.

Proof: Substituting equations (6), (2) and (5) for Q_θ, DP and $D\mathbb{M}(P)^{-1}\rangle$, respectively into $Q_\theta\langle DP, D\mathbb{M}(P)^{-1}\rangle$ yields the result. ■

Theorem III.5. *Let P be the zero-coupon bond price driven by a VG process. Then, the sensitivity* drift *is given by*

$$\mathcal{D} = e^{-r_0T}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p(t,T,g,z))H\Big(p,\frac{\partial p}{\partial\theta}\Big)(2\pi)^{-\frac{1}{2}}$$
$$\cdot\, e^{-\frac{1}{2}z^2}\bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}g^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}g}\bigg)dzdg\bigg)$$

where

$$H\Big(p,\frac{\partial p}{\partial\theta}\Big) = \frac{\mathcal{L}^*z}{\bar{\mathcal{K}}} - \mathcal{L}^*\sigma^2\widehat{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{g(u)})^2\big)\bar{\mathcal{K}}^{-2}$$
$$+\,\sigma^2\widehat{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{g(u)})\Delta g(u)\big)\bar{\mathcal{K}}^{-1},$$

$\Phi(p(t,T,g,z)) = \max(p(t,T,g,z)-K,0)$ *is from the payoff function with K as the strike price. $\bar{\mathcal{K}}$ and $\mathcal{L}^*$ are given by*

$$\bar{\mathcal{K}} = \sigma\widehat{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{g(s)}e^{-\eta(u-s)} + \sigma\widehat{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{g(u)})$$
$$-\,\sigma^2\widehat{\sigma}\sum_{t\le u\le T}(\theta\Delta g(u)+\widehat{\sigma}\Delta\sqrt{g(u)}z)\Delta\sqrt{g(u)}, \tag{13}$$

and

$$\mathcal{L}^* = \frac{\sigma}{\eta}\Big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\Big]\bigg(\frac{-1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\bigg)$$
$$+\,\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta g(s)e^{-\eta(u-s)}$$
$$+\,\widetilde{w}\sigma[T-t]\bigg(\frac{-1}{1-\theta\nu-\frac{\widehat{\sigma}^2}{2}\nu}\bigg)+\sigma\sum_{t\le u\le T}\Delta g(u)$$
$$-\,\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta g(u)+\widehat{\sigma}\Delta\sqrt{g(u)}z)\Delta g(u)\bigg). \tag{14}$$

Proof: It follows that

$$\frac{\partial\mathbb{V}}{\partial\theta} = \mathcal{D} = \frac{\partial}{\partial\theta}e^{-r_0T}\mathbb{E}[\Phi(P)]$$
$$= e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big].$$

Also, by substituting equations (9), (11) and (12) into

$$H(P,Q_\theta) = Q_\theta\mathbb{M}(P)^{-1}LP - \mathbb{M}(P)^{-1}\langle DP, DQ_\theta\rangle$$
$$-\,Q_\theta\langle DP, D\mathbb{M}(P)^{-1}\rangle$$

and simplifying, we obtain the Malliavin weight

$$H(P,Q_\theta) = \frac{\mathcal{L}Z}{\mathcal{K}} - \mathcal{L}\sigma^2\widehat{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}$$
$$+\,\sigma^2\widehat{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)\big)\mathcal{K}^{-1}.$$

Furthermore,

$$\mathcal{D} = e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big]$$
$$= e^{-r_0T}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p(t,T,g,z))H\Big(p,\frac{\partial p}{\partial\theta}\Big)f_{\mathcal{N}}(z;0,1)$$
$$\cdot\quad f_G(g;t\nu^{-1},\nu^{-1})dzdg\bigg)$$
$$= e^{-r_0T}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p(t,T,g,z))H\Big(p,\frac{\partial p}{\partial\theta}\Big)(2\pi)^{-\frac{1}{2}}$$
$$\cdot\, e^{-\frac{1}{2}z^2}\bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}g^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}g}\bigg)dzdg\bigg)$$

where $f_{\mathcal{N}}(z;0,1)$ and $f_G(g;t\nu^{-1},\nu^{-1})$ denote the probability density functions of the Gaussian random variable and the gamma process, respectively.

Hence, the result follows. ■

B. Derivation of the greek vega$_\nu$ *for a VG-driven zero-coupon bond price*

In this subsection, we derive an expression for the greek *vega* for a zero-coupon bond price driven by a VG process. From equation (1), $\widetilde{w} = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)$ implies that

$$\frac{\partial\widetilde{w}}{\partial\nu} = \frac{(-\theta-\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)} - \frac{\widetilde{w}}{\nu} = \frac{(-\theta-\frac{1}{2}\widehat{\sigma}^2)}{\nu e^{\nu\widetilde{w}}} - \frac{\widetilde{w}}{\nu}.$$

Lemma III.6. *Let P be the zero-coupon bond price under a VG process. Then,*

$$Q_\nu = \bigg(\frac{(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}+\frac{\widetilde{w}}{\nu}\bigg)$$
$$\cdot\bigg(\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\bigg)P \tag{15}$$

and

$$DQ_\nu = \bigg(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\bigg)$$
$$\cdot\bigg(\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\bigg)\mathcal{K}P, \tag{16}$$

where $\mathcal{K}$ is given by equation (8).

Proof: By equation (1), it follows that

$$Q_\nu = -\bigg[\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)$$
$$\cdot\bigg(\frac{(-\theta-\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\bigg)$$
$$+\,\sigma[T-t]\bigg(\frac{(-\theta-\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\bigg)\bigg]P.$$

Thus, the Malliavin derivative gives

$$\begin{aligned}
DQ_\nu &= -\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\\
&\quad\cdot\Big[\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]DP\\
&= -\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\\
&\quad\Big[\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\qquad\times-\Big[\sigma\widehat{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-\eta(u-s)}\\
&\quad+\sigma\widehat{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)\\
&\quad+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\widehat{\sigma}\Delta\sqrt{G(u)}\Big)\Big]P.
\end{aligned}$$

Hence, the result follows. ■

Lemma III.7. *Let P be the zero-coupon bond price driven by a VG process, then the following results hold:*
$Q_\nu\mathbb{M}(P)^{-1}LP$

$$\begin{aligned}
&= \Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\quad\times\Big(\frac{\sigma^2\widehat{\sigma}^2}{\mathcal{K}^2}\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)+1+\frac{Z}{\mathcal{K}}\Big).
\end{aligned}\tag{17}$$

$\mathbb{M}(P)^{-1}\langle DP, DQ_\nu\rangle$

$$\begin{aligned}
&= -\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\\
&\quad\cdot\Big(\frac{\sigma}{\eta}\big(T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big).
\end{aligned}\tag{18}$$

$Q_\nu\langle DP, D\mathbb{M}(P)^{-1}\rangle$

$$\begin{aligned}
&= 2\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\quad\cdot\Big(\sigma^2\widehat{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+1\Big).
\end{aligned}\tag{19}$$

Proof: Expression in equation (17) is obtained by substituting equation (15) for Q_ν, equation (4) for $\mathbb{M}(P)^{-1}$ and equation (3) for LP, and simplifying.

Expression in equation (18) holds by substituting equation (4) for $\mathbb{M}(P)^{-1}$, equation (2) for DP and equation (16) for DQ_ν, and simplification.

Expression in equation (19) is obtained by substituting equation (15) for Q_ν, equation (2) for DP and equation (5) for $D\mathbb{M}(P)^{-1}$, and simplifying. ■

Theorem III.8. *Let P be the zero-coupon bond price driven by a VG process, then the greek*

$$\begin{aligned}
\mathcal{V}_\nu &= e^{-r_0T}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p)H\Big(p,\frac{\partial p}{\partial\nu}\Big)(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}\\
&\quad\Big(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}g^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}g}\Big)dzdg+\mathbb{E}_{(\nu)}[\Phi(P)]\Big),
\end{aligned}$$

where $\Phi(p)=\max(p(t,T,g,z)-K,0)$,

$$\begin{aligned}
H\Big(p,\frac{\partial p}{\partial\nu}\Big) &= \Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\quad\cdot\Big(\frac{z}{\bar{\mathcal{K}}}-\frac{\sigma^2\widehat{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{g(u)})^2\big)}{\bar{\mathcal{K}}^2}\Big)
\end{aligned}$$

and $\bar{\mathcal{K}}$ is given by equation (13). $\mathbb{E}_{(\nu)}[\Phi(P)]$ is given in the Appendix.

Proof: It follows that
$\mathcal{V}_\nu=\frac{\partial}{\partial\nu}e^{-r_0T}\mathbb{E}[\Phi(P)]$

$$\begin{aligned}
&= e^{-r_0T}(\mathbb{E}[\Phi(P)H(P,Q_\nu)]+\mathbb{E}_{(\nu)}[\Phi(P)])\\
&= e^{-r_0T}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p)H\Big(p,\frac{\partial p}{\partial\nu}\Big)f_{\mathcal{N}}(z;0,1)\\
&\quad\cdot f_G(g;t\nu^{-1},\nu^{-1})dzdg+\mathbb{E}_{(\nu)}[\Phi(P)]\Big)
\end{aligned}$$

where $f_{\mathcal{N}}(z;0,1)$ and $f_G(g;t\nu^{-1},\nu^{-1})$ denote the density function of a Gaussian random variable and the density function of a gamma random variable, respectively.
Also, by substituting and simplifying equations (17), (18) and (19), the Malliavin weight becomes

$$\begin{aligned}
H(P,Q_\nu) &= Z\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\mathcal{K}^{-1}\\
&\quad-\Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\quad\cdot\sigma^2\widehat{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}\\
&= \Big(\frac{-(\theta+\frac{1}{2}\widehat{\sigma}^2)}{\nu(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu)}-\frac{\widetilde{w}}{\nu}\Big)\Big[\frac{\sigma}{\eta}\big(T-t\\
&\quad+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big)+\sigma[T-t]\Big]\\
&\quad\cdot\Big(\frac{Z}{\mathcal{K}}-\frac{\sigma^2\widehat{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\Big).
\end{aligned}$$

Since the computation of $E_{(\nu)}[\Phi(P)]$ is given in the Appendix, the result follows. ■

C. Derivation of the greek vega$_{\widehat{\sigma}}$ *for a VG-driven zero-coupon bond price*

In this subsection, we compute *vega*$_{\widehat{\sigma}}$ for a VG-driven interest rate derivative.

$$\mathcal{V}_{\widehat{\sigma}}=\frac{\partial}{\partial\widehat{\sigma}}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\widehat{\sigma}}\Big)\Big].$$

By equation (1),

$$\widetilde{w} = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu) \ \Rightarrow \frac{\partial\widetilde{w}}{\partial\widehat{\sigma}} = \frac{-\widehat{\sigma}}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}.$$

Lemma III.9. *Let P be the price of the zero-coupon bond driven by a VG process. Then,*

$$\begin{aligned} Q_{\widehat{\sigma}} = -\Bigg[&\Big(\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]+\sigma[T-t]\Big) \\ &\cdot\Big(\frac{-\widehat{\sigma}}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}\Big) \\ &+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-\eta(u-s)}Z\big) \\ &+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]P \end{aligned} \tag{20}$$

and

$$\begin{aligned} DQ_{\widehat{\sigma}} = \Bigg(-\Bigg[&\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}-\sigma^2\Bigg[\sum_{t\le u\le T}\Bigg((\theta\Delta G(u) \\ &+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)} \\ &+\widehat{\sigma}(\Delta\sqrt{G(u)})^2Z\Bigg)\Bigg]\Bigg]\Bigg)P+\widetilde{L}\mathcal{K}P, \end{aligned} \tag{21}$$

where $\mathcal{K}$ is given by equation (8), and

$$\begin{aligned} \widetilde{L} = \Big(&\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]+\sigma[T-t]\Big) \\ &\cdot\big(\frac{-\widehat{\sigma}}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)} \\ &\cdot e^{-\eta(u-s)}Z\big)+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z) \\ &-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big). \end{aligned} \tag{22}$$

Proof: By equation (1), applying partial derivative with respect to $\widehat{\sigma}$ gives

$$\begin{aligned} Q_{\widehat{\sigma}} = -\Bigg[&\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]\frac{\partial\widetilde{w}}{\partial\widehat{\sigma}}+\sigma[T-t]\frac{\partial\widetilde{w}}{\partial\widehat{\sigma}} \\ &+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-\eta(u-s)}Z\big) \\ &+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u) \\ &+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]P \end{aligned}$$

which gives equation (20).

Thus, the Malliavin derivative

$$\begin{aligned} DQ_{\widehat{\sigma}} = \Bigg(-\Bigg[&\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}-\sigma^2\Bigg[\sum_{t\le u\le T}\Bigg((\theta\Delta G(u) \\ &+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widehat{\sigma}(\Delta\sqrt{G(u)})^2Z\Bigg)\Bigg]\Bigg]\Bigg)P \\ &+\Bigg[\Big(\frac{\sigma}{\eta}\big[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\big]+\sigma[T-t]\Big) \\ &\cdot\Big(\frac{-\widehat{\sigma}}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}\Big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)} \\ &\cdot e^{-\eta(u-s)}Z\big)+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)-\sigma^2 \\ &\cdot\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]\mathcal{K}P, \end{aligned}$$

where $\mathcal{K}$ is given by equation (8). ■

Lemma III.10. *Let P be the zero-coupon bond price driven by a VG process. Then,*
$Q_{\widehat{\sigma}}\mathbb{M}(P)^{-1}LP$

$$=\widetilde{L}\Big[\frac{(\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2)}{\mathcal{K}^2}+1+\frac{Z}{\mathcal{K}}\Big], \tag{23}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (8) and (22), respectively.

Proof: The result follows by substituting equation (20) for $Q_{\widehat{\sigma}}$, equation (4) for $\mathbb{M}(P)^{-1}$ and equation (3) for LP, and simplifying. ■

Lemma III.11. *Let P be the zero-coupon bond price driven by the VG process. Then,*
$\mathbb{M}(P)^{-1}\langle DP, DQ_{\widehat{\sigma}}\rangle$

$$\begin{aligned} =\frac{1}{\mathcal{K}}\Bigg(\Bigg[&\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)} \\ &+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}-\sigma^2\Bigg[\sum_{t\le u\le T}\Bigg((\theta\Delta G(u) \\ &+\widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widehat{\sigma}(\Delta\sqrt{G(u)})^2Z\Bigg)\Bigg]\Bigg]\Bigg)-\widetilde{L} \end{aligned} \tag{24}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (8) and (22), respectively.

Proof: The result follows by substituting equations (4), (2) and (21) for $\mathbb{M}(P)^{-1}$, DP and $DQ_{\widehat{\sigma}}$, respectively, and then simplifying. ■

Lemma III.12. *Let P be the zero-coupon bond price driven by a VG process. Then,*

$Q_{\widehat{\sigma}}\langle DP, D\mathbb{M}(P)^{-1}\rangle$

$$=2\widetilde{L}\mathcal{K}^{-2}\big(\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)+2\widetilde{L}, \tag{25}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (8) and (22), respectively.

Proof: The result follows by substituting the expression for $Q_{\widehat{\sigma}}$, DP and $D\mathbb{M}(P)^{-1}$ in equations (20), (2) and (5), respectively into $Q_{\widehat{\sigma}}\langle DP, D\mathbb{M}(P)^{-1}\rangle$, and simplifying. ■

Theorem III.13. *Let P be the zero-coupon bond price driven by a VG process. Then,*

$$\mathcal{V}_{\widehat{\sigma}} = e^{-r_0T}\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p)H\Big(p, \frac{\partial p}{\partial\widehat{\sigma}}\Big)(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2} \cdot \Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}g^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}g}\Bigg)dzdg\Bigg),$$

where $\Phi(p) = \max(p(t,T,g,z)-K,0)$, *and*

$$\begin{aligned}
&H\Big(p, \frac{\partial p}{\partial\widehat{\sigma}}\Big)\\
&= \frac{z\widetilde{L}^*}{\bar{\mathcal{K}}} - \frac{\widetilde{L}^*\Big(\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{g(u)})^2\Big)}{\bar{\mathcal{K}}^2}\\
&- \frac{1}{\bar{\mathcal{K}}}\Bigg(\Bigg[\sigma\Bigg(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{g(s)}e^{-\eta(u-s)}\Bigg)\\
&+ \sigma\Bigg(\sum_{t\le u\le T}\Delta\sqrt{g(u)}\Bigg) - \sigma^2\Bigg[\sum_{t\le u\le T}\Bigg((\theta\Delta g(u)\\
&+ \widehat{\sigma}\Delta\sqrt{g(u)}z)\Delta\sqrt{g(u)} + \widehat{\sigma}(\Delta\sqrt{g(u)})^2z\Bigg)\Bigg]\Bigg]\Bigg),
\end{aligned}$$

$\bar{\mathcal{K}}$ is given by equation (13) and $\widetilde{L}^*$ is given by

$$\begin{aligned}
\widetilde{L}^* = &\Bigg(\frac{\sigma}{\eta}\Bigg[T-t+\frac{1}{\eta}(e^{-\eta T}-e^{-\eta t})\Bigg]+\sigma[T-t]\Bigg)\\
&\cdot\Bigg(\frac{-\widehat{\sigma}}{1-\theta\nu-\frac{1}{2}\widehat{\sigma}^2\nu}\Bigg)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Bigg(\Delta\sqrt{g(s)}\\
&\cdot e^{-\eta(u-s)}z\Bigg)+\sigma\sum_{t\le u\le T}(\Delta\sqrt{g(u)}z)\\
&-\sigma^2\Bigg(\sum_{t\le u\le T}(\theta\Delta g(u)+\widehat{\sigma}\Delta\sqrt{g(u)}Z)\Delta\sqrt{g(u)}z\Bigg).
\end{aligned} \tag{26}$$

Proof:

$$\begin{aligned}
\mathcal{V}_{\widehat{\sigma}} = \frac{\partial\mathbb{V}}{\partial\widehat{\sigma}} &= e^{-r_0T}\mathbb{E}\big[\Phi(P)H\big(P, Q_{\widehat{\sigma}}\big)\big]\\
&= e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)H\Bigg(P, \frac{\partial P}{\partial\widehat{\sigma}}\Bigg)\Bigg].
\end{aligned}$$

For the Malliavin weight, substituting equations (21), (22) and (23) and simplifying gives

$$\begin{aligned}
&H\big(P, Q_{\widehat{\sigma}}\big)\\
&= \frac{Z\widetilde{L}}{\mathcal{K}} - \frac{\widetilde{L}\big(\sigma^2\widehat{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\\
&- \frac{1}{\mathcal{K}}\Bigg(\Bigg[\sigma\big(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-\eta(u-s)}\big)\\
&+ \sigma\big(\sum_{t\le u\le T}\Delta\sqrt{G(u)}\big) - \sigma^2\Bigg[\sum_{t\le u\le T}\Bigg((\theta\Delta G(u)\\
&+ \widehat{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)} + \widehat{\sigma}(\Delta\sqrt{G(u)})^2Z\Bigg)\Bigg]\Bigg]\Bigg).
\end{aligned}$$

Hence, the result follows. ■

IV. CONCLUSION

The derived greeks play a big role in hedging which is a process of reducing risk of interest rate derivatives. Each greek computation will give the rate at which change in the parameters of the model will affect the worth of the financial derivative, and thus, gives a guide to appropriate decision making.

APPENDIX A

EXPRESSION FOR $\mathbb{E}_{(\nu)}[\Phi(P)]$

The digamma function is given in Medina and Moll [11] by

$$\psi(a) = \frac{d}{da}\ln\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$$

where $\Gamma(a) = \displaystyle\int_0^\infty t^{a-1}e^{-t}dt.$

It follows that the digamma function can be written as

$$\psi\Big(\frac{t}{\nu}\Big) = \frac{d}{d\nu}\ln\Gamma\Big(\frac{t}{\nu}\Big).$$

Assume that $f_{\mathcal{N}}$ and f_g are the density functions for the Gaussian random variable and gamma random variable, respectively. Then, by Bayazit and Nolder [2],

$$\begin{aligned}
&\mathbb{E}_{(\nu)}[\Phi(P)] = \frac{\partial}{\partial\nu}\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\cdot f_{g(y;\frac{t}{\nu},\frac{1}{\nu})}dxdy\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\cdot\frac{\partial}{\partial\nu}\Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}y^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}y}\Bigg)dxdy\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\\
&\quad\cdot\frac{\partial}{\partial\nu}\exp\Bigg(\ln\Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}y^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}y}\Bigg)\Bigg)dxdy\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}f_{g(y;\frac{t}{\nu},\frac{1}{\nu})}\\
&\quad\cdot\frac{\partial}{\partial\nu}\ln\Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}y^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}y}\Bigg)dxdy,
\end{aligned}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\ln\Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}y^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}y}\Bigg)\\
&= \frac{\partial}{\partial\nu}\Bigg(-\frac{t}{\nu}\ln\nu-\ln\Gamma\big(\frac{t}{\nu}\big)+\big(\frac{t}{\nu}-1\big)\ln y-\frac{1}{\nu}y\Bigg)\\
&= \Bigg(\frac{t}{\nu^2}\ln\nu-\frac{t}{\nu^2}+\frac{t}{\nu^2}\frac{\Gamma'\big(\frac{t}{\nu}\big)}{\Gamma\big(\frac{t}{\nu}\big)}-\frac{t}{\nu^2}\ln y+\frac{1}{\nu^2}y\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&e^{-r_0T}\mathbb{E}_{(\nu)}[\Phi(P)]\\
&= e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)\sum_{t\le u\le T}\Bigg(\frac{t}{\nu^2}\ln\nu-\frac{t}{\nu^2}+\frac{t}{\nu^2}\frac{\Gamma'\big(\frac{t}{\nu}\big)}{\Gamma\big(\frac{t}{\nu}\big)}\\
&\quad-\frac{t}{\nu^2}\ln(\Delta G(u))+\frac{1}{\nu^2}\Delta G(u)\Bigg)\Bigg].
\end{aligned}$$

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