# An LT-BEM for an Unsteady Helmholtz Type Equation of Exponentially Variable Coefficients

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*Abstract*—**In this paper a BEM is used to solve a class of variable coefficient parabolic equations numerically. Some examples are considered to show the convergence, consistency, and accuracy of the numerical solutions.**

*Index Terms*—**anisotropic properties, exponentially varying coefficients, unsteady Helmholtz equation, Laplace transform, boundary element method**

## I. Introduction

We will consider initial boundary value problems governed by a Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_j}\right]+\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=\alpha(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial t} \tag{1}$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial\mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial\mu}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial\mu}{\partial x_1}\right)\\&+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial\mu}{\partial x_2}\right)+\beta^2\mu=\alpha\frac{\partial c}{\partial t}\end{aligned}$$

Equation (1) is usually used to model acoustic problems (see for examples [1]–[8]).

Authors commonly define an FGM as an inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. Therefore equation (1) is relevant for FGMs.

Recently Azis and co-workers had been working on steady state problems of *anisotropic inhomogeneous* media for several types of governing equations, for examples [9], [10] for the modified Helmholtz equation, [11]–[15] for the diffusion convection equation, [16]–[19] for the Laplace type equation, [20]–[26] for the diffusion convection reaction equation. Some other classes of inhomogeneity functions for FGMs that differ from the class of constant-plus-variable coefficients are reported from these papers. Azis et al. also had been working on *unsteady state* problems of *anisotropic inhomogeneous* media for several types of governing equations (see for example [27]–[31]).

This paper is intended to extend the recently published works in [3]–[8] for steady anisotropic Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_j}\right]+\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=0$$

to unsteady anisotropic Helmholtz type equation with spatially variable coefficients of the form (1).

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. It is necessary to place some constraints on the class of coefficients $\kappa_{ij}$ and $\beta^2$ for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

## II. The initial-boundary value problem

Referred to a Cartesian frame $Ox_1x_2$ solutions $\mu(\mathbf{x},t)$ and its derivatives to (1) are sought which are valid for time interval $t\geq 0$ and in a region $\Omega$ in $R^2$ with a boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu(\mathbf{x},t)$ $(\mathbf{x}=(x_1,x_2))$ is specified and on $\partial\Omega_2$

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu(\mathbf{x},0)=0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable $s$. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution $c$ and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropic media, the coefficients in (1) take the form $\kappa_{11}=\kappa_{22}$ and $\kappa_{12}=0$ and use of these equations in the following analysis immediately yields the corresponding results for isotropic media.

## III. The boundary integral equation

The coefficients $\kappa_{ij},\beta^2,\alpha$ are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij}g(\mathbf{x}) \tag{4}$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \tag{5}$$

$$\alpha(\mathbf{x}) = \overline{\alpha}g(\mathbf{x}) \tag{6}$$

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where the $\overline{\kappa}_{ij}, \overline{\beta}^2, \overline{\alpha}$ are constants and $g$ is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$, $\beta^2(\mathbf{x})$ and $\alpha(\mathbf{x})$ are exponentially graded by taking $g(\mathbf{x})$ as an exponential function

$$g(\mathbf{x}) = [\exp(c_0 + c_i x_i)]^2 \tag{7}$$

where $c_0$ and $c_i$ are constants. Therefore if

$$\overline{\kappa}_{ij} c_i c_j - \lambda = 0 \tag{8}$$

then (7) satisfies

$$\overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \lambda g^{1/2} = 0 \tag{9}$$

Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g \frac{\partial \mu}{\partial x_j}\right) + \overline{\beta}^2 g\mu = \overline{\alpha} g \frac{\partial \mu}{\partial t} \tag{10}$$

Let

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, \psi(\mathbf{x}, t) \tag{11}$$

therefore substitution of (4) and (11) into (2) gives

$$P(\mathbf{x}, t) = -P_g(\mathbf{x})\, \psi(\mathbf{x}, t) + g^{1/2}(\mathbf{x})\, P_\psi(\mathbf{x}, t) \tag{12}$$

where

$$P_g(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \qquad P_\psi(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial \psi}{\partial x_j} n_i$$

Also, (10) may be written in the form

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left[g \frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right] + \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g \frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

A simplification yields

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} + g\psi \frac{\partial g^{-1/2}}{\partial x_j}\right) + \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i} = -g^{-1} \frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} - \psi \frac{\partial g^{1/2}}{\partial x_j}\right) + \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Rearranging and neglecting zero terms give

$$g^{1/2} \overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \overline{\beta}^2 g^{1/2}\psi = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t} \tag{13}$$

Substitution of (9) into (13) implies

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \left(\overline{\beta}^2 - \lambda\right) \psi = \overline{\alpha} \frac{\partial \psi}{\partial t} \tag{14}$$

Taking the Laplace transform of (11), (12), (14) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x}, s) = g^{1/2}(\mathbf{x})\, \mu^*(\mathbf{x}, s) \tag{15}$$

$$P_{\psi^*}(\mathbf{x}, s) = [P^*(\mathbf{x}, s) + P_g(\mathbf{x})\, \psi^*(\mathbf{x}, s)]\, g^{-1/2}(\mathbf{x}) \tag{16}$$

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi^*}{\partial x_i \partial x_j} + \left(\overline{\beta}^2 - \lambda - s\overline{\alpha}\right) \psi^* = 0 \tag{17}$$

where $s$ is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (17) is given in the form

$$\begin{aligned} \eta(\mathbf{x}_0)\, \psi^*(\mathbf{x}_0, s) = \int_{\partial\Omega} & [\Gamma(\mathbf{x}, \mathbf{x}_0)\, \psi^*(\mathbf{x}, s) \\ & - \Phi(\mathbf{x}, \mathbf{x}_0)\, P_{\psi^*}(\mathbf{x}, s)]\, dS(\mathbf{x}) \end{aligned} \tag{18}$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(a, b) \in \Omega$, $\eta = \frac{1}{2}$ if $(a, b) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $(a, b)$. The so called fundamental solution $\Phi$ in (18) is any solution of the equation

$$\overline{\kappa}_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \left(\overline{\beta}^2 - s\overline{\alpha} - \lambda\right) \Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the $\Gamma$ is given by

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \overline{\kappa}_{ij} \frac{\partial \Phi(\mathbf{x}, \mathbf{x}_0)}{\partial x_j} n_i$$

where $\delta$ is the Dirac delta function. For two-dimensional problems $\Phi$ and $\Gamma$ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \ln R & \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda > 0 \\ \frac{-K}{2\pi} K_0(\omega R) & \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda < 0 \end{cases}$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \frac{1}{R} \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{-\imath K\omega}{4} H_1^{(2)}(\omega R)\, \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{K\omega}{2\pi} K_1(\omega R)\, \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \end{cases}$$

$$\begin{cases} \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda = 0 \\ \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda > 0 \\ \text{if } \overline{\beta}^2 - s\overline{\alpha} - \lambda < 0 \end{cases} \tag{19}$$

where

$$\begin{aligned} K &= \ddot{\tau}/D \\ \omega &= \sqrt{|\overline{\beta}^2 - s\overline{\alpha} - \lambda|/D} \\ D &= \left[\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\dot{\tau} + \overline{\kappa}_{22}\left(\dot{\tau}^2 + \ddot{\tau}^2\right)\right]/2 \\ R &= \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \\ \dot{x}_1 &= x_1 + \dot{\tau} x_2 \\ \dot{a} &= a + \dot{\tau} b \\ \dot{x}_2 &= \ddot{\tau} x_2 \\ \dot{b} &= \ddot{\tau} b \end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root $\tau$ of the quadratic

$$\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\tau + \overline{\kappa}_{22}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. $K_0$, $K_1$ denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the $\Gamma$ in (19) are given by

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1 - \dot{a}) \\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1 - \dot{a})\right] + \ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right] \end{aligned}$$

TABLE I
VALUES OF $V_m$ OF THE STEHFEST FORMULA

| $V_m$ | $N=6$ | $N=8$ | $N=10$ | $N=12$ |
|---|---|---|---|---|
| $V_1$ | 1 | $-1/3$ | $1/12$ | $-1/60$ |
| $V_2$ | $-49$ | $145/3$ | $-385/12$ | $961/60$ |
| $V_3$ | 366 | $-906$ | 1279 | $-1247$ |
| $V_4$ | $-858$ | $16394/3$ | $-46871/3$ | $82663/3$ |
| $V_5$ | 810 | $-43130/3$ | $505465/6$ | $-1579685/6$ |
| $V_6$ | $-270$ | 18730 | $-236957.5$ | 1324138.7 |
| $V_7$ | | $-35840/3$ | $1127735/3$ | $-58375583/15$ |
| $V_8$ | | $8960/3$ | $-1020215/3$ | $21159859/3$ |
| $V_9$ | | | 164062.5 | $-8005336.5$ |
| $V_{10}$ | | | $-32812.5$ | 5552830.5 |
| $V_{11}$ | | | | $-2155507.2$ |
| $V_{12}$ | | | | 359251.2 |

Use of (15) and (16) in (18) yields

$$\eta g^{1/2}\mu^* = \int_{\partial\Omega}\left[\left(g^{1/2}\Gamma - P_g\Phi\right)\mu^* - \left(g^{-1/2}\Phi\right)P^*\right]dS \tag{20}$$

This equation provides a boundary integral equation for determining $\mu^*$ and its derivatives at all points of $\Omega$.

Knowing the solutions $\mu^*(\mathbf{x},s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (20), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x},t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned}
\mu(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m\mu^*(\mathbf{x},s_m)\\
\frac{\partial\mu(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_1}\\
\frac{\partial\mu(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_2}
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
s_m &= \frac{\ln 2}{t}m\\
V_m &= (-1)^{\frac{N}{2}+m}\times \sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}
\end{aligned}$$

A simple script is developed and embedded into the main FORTRAN code to calculate the values of the coefficients $V_m, m=1,2,\ldots,N$ for any number $N$. Table (I) shows the values of $V_m$ for $N=6,8,10,12$.

## IV. NUMERICAL EXAMPLES

In order to justify the analysis derived in the previous sections, we will consider several problems either as test examples of analytical solutions or problems without simple analytical solutions.

We assume each problem belongs to a system which is valid in given spatial and time domains and governed by equation (1) and satisfying the initial condition (3) and some boundary conditions as mentioned in Section II. The characteristics of the system which are represented by the coefficients $\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \alpha(\mathbf{x})$ in equation (1) are assumed to be of the form (4), (5) and (6) in which $g(\mathbf{x})$ is an exponential function of the form (7). The coefficients $\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \alpha(\mathbf{x})$ may represent respectively the diffusivity or conductivity, the wave number and the change rate of the unknown function $\mu(\mathbf{x},t)$.

Standard BEM with constant elements is employed to obtain numerical results. For a simplicity, a unit square (depicted in Figure 1) will be taken as the geometrical domain for all problems. A number of 320 elements of equal length, namely 80 elements on each side of the unit square, are used. And the time interval is chosen to be $0\le t\le 5$. A FORTRAN script is developed to compute the solutions and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results.

We try to use $N=6,8,10,12$ for the Stehfest formula and find out the convergence of the error when $N$ changes from $N=6$ to $N=10$ and $N=10$ is the best value of $N$ that makes the error stable and optimized. Increasing $N$ from $N=10$ to $N=12$ gives worse results. According to Hassanzadeh and Pooladi-Darvish [32] these worse results are induced by round-off errors. This justifies to choose $N=10$ in the Stehfest formula (21) for all problems.

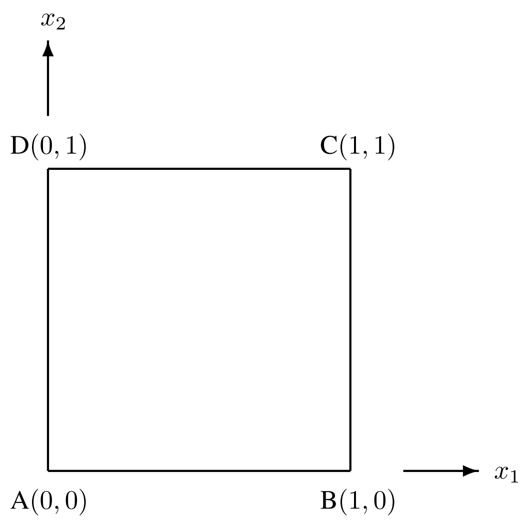


Fig. 1. The domain $\Omega$

For all problems the inhomogeneity function is taken to be

$$g^{1/2}(\mathbf{x}) = \exp[-0.75+0.25x_1+0.5x_2]$$

and the constant anisotropy coefficient $\overline{\kappa}_{ij}$

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.1\\ 0.1 & 0.85\end{bmatrix}$$

so that (8) implies

$$\lambda = 0.3$$

We set the constant coefficient $\overline{\beta}^2$

$$\overline{\beta}^2 = 1$$

*A. Examples with analytical solutions*

*Problem 1:*

Other aspects that will be justified are the accuracy and consistency (between the scattering and flow) of the numerical solutions. The analytical solutions are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where $h(\mathbf{x}), f(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

$P$ is given on side AB
$P$ is given on side BC
$\mu$ is given on side CD
$P$ is given on side AD

*Case 1:* We take

$$\begin{aligned} h(\mathbf{x}) &= 1 - 0.4x_1 - 0.6x_2 \\ f(t) &= 1 - \exp(-1.75t) \end{aligned}$$

Thus for $h(\mathbf{x})$ to satisfy (17)

$$\overline{\alpha} = 0.7/s$$

*Case 2:* For the analytical solution we take

$$\begin{aligned} h(\mathbf{x}) &= \cos(1 - 0.4x_1 - 0.6x_2) \\ f(t) &= t/5 \end{aligned}$$

So that in order for $h(\mathbf{x})$ to satisfy (17)

$$\overline{\alpha} = 0.186/s$$

*Case 3:* We take

$$\begin{aligned} h(\mathbf{x}) &= \exp(-1 + 0.4x_1 + 0.6x_2) \\ f(t) &= 0.16t(5 - t) \end{aligned}$$

Therefore (17) gives

$$\overline{\alpha} = 1.214/s$$

Figure 2 shows the accuracy of the numerical solutions $\mu$ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ solutions in the domain. For the Case 1 the errors occur in the third decimal place, whereas for the Cases 2 and 3 the errors occur in the fourth decimal place. Figures 3, 4, 5 show the consistency between the scattering and the flow solutions which indicates that the solutions for the derivatives had been computed correctly. Figure 6 shows a variation of the $\mu$ solution values at some interior points as the time increases from $t = 0.0005$ to $t = 5$. As expected, the variation follows the way the associated function $f(t)$ changes. Specifically for the Case 1 of associated function $f(t) = 1 - \exp(-1.75t)$ the $\mu$ solution will tend to approach a steady state solution. This is also expected, as the function $f(t) = 1 - \exp(-1.75t)$ will converge to 1 as $t$ gets bigger.

The elapsed CPU time for the computation of the numerical solutions at $19 \times 19$ spatial positions and 11 time steps is 6709.890625 seconds for the Case 1, 7697.125 seconds for the Case 2, and 3026.5 seconds for the Case 3. The longer computation time for the Cases 1 and 2 is produced by the iterative calculation of the polynomial approximation of the Hankel and Bessel functions in the fundamental solutions (19).

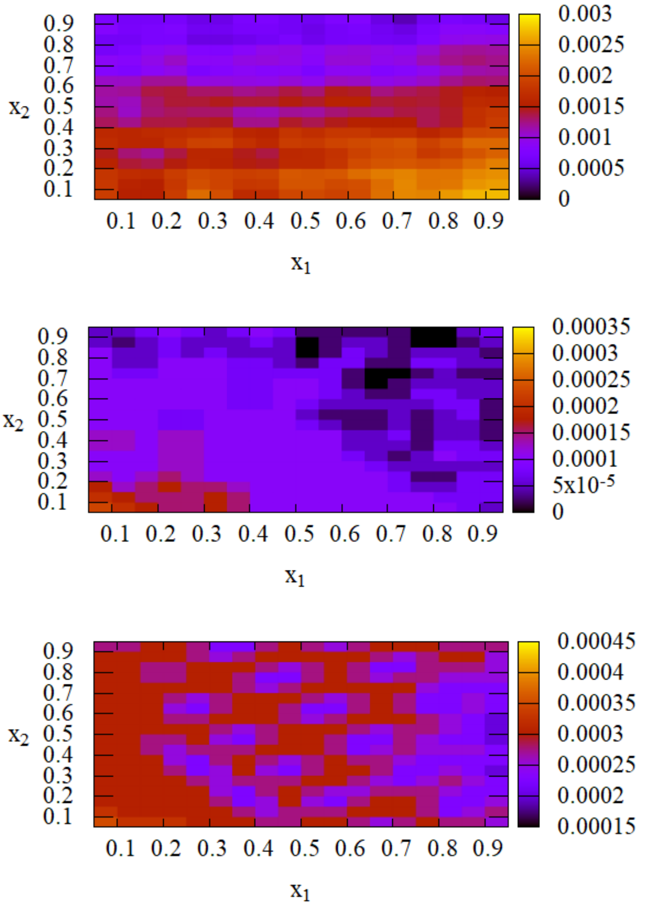


Fig. 2. The errors of interior solution $\mu$ at $t = 2.5$ for the Case 1 (top), Case 2 (center), Case 3 (bottom) of Problem 1

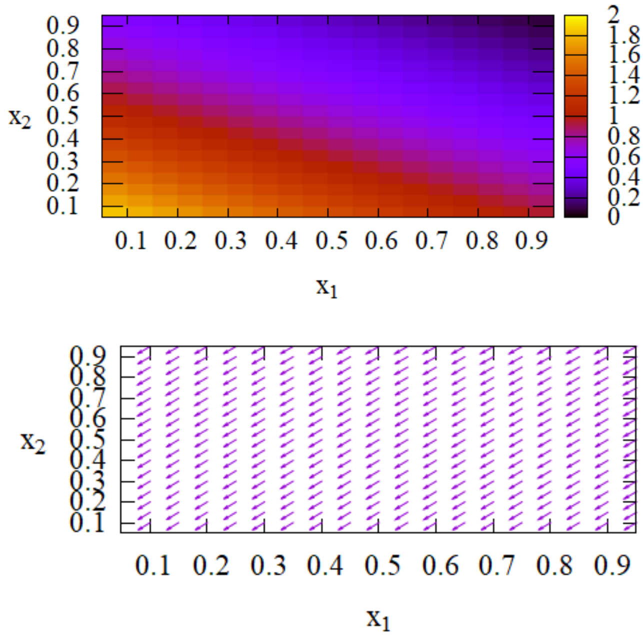


Fig. 3. Consistency of the scattering $\mu$ and the flow vector $(\partial\mu/\partial x_1, \partial\mu/\partial x_2)$ solutions at $t = 2.5$ for the Case 1 of Problem 1

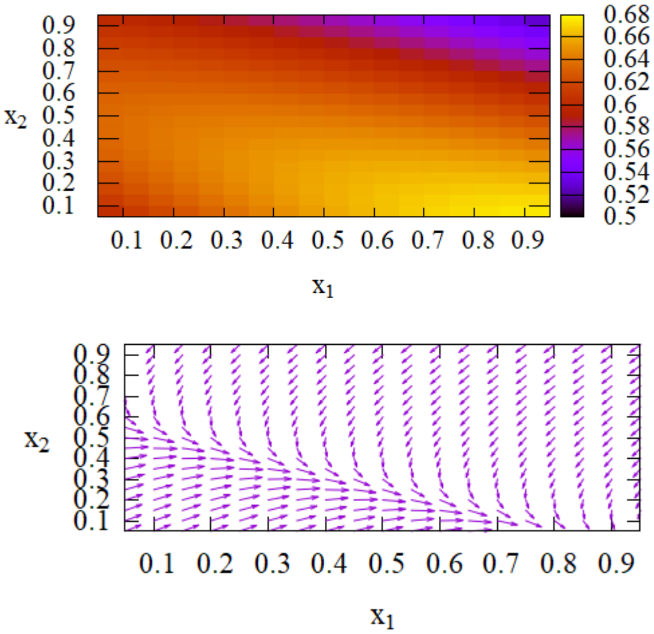


Fig. 4. Consistency of the scattering $\mu$ and the flow vector $(\partial\mu/\partial x_1, \partial\mu/\partial x_2)$ solutions at $t = 2.5$ for the Case 2 of Problem 1

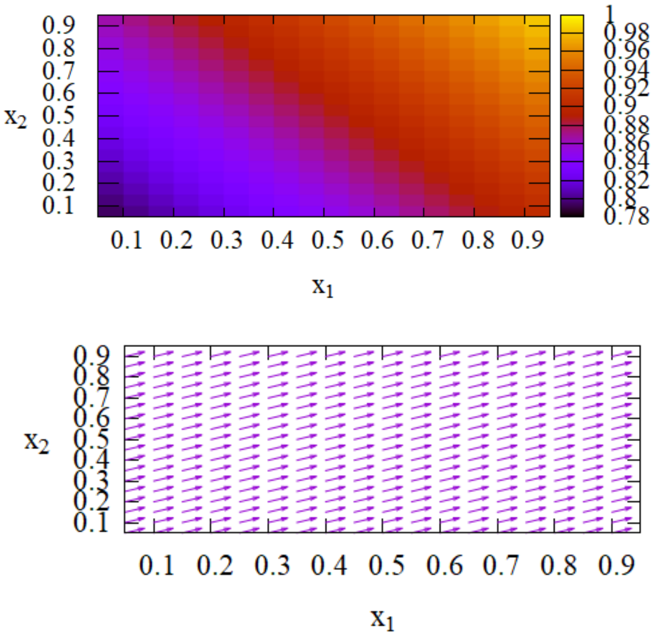


Fig. 5. Consistency of the scattering $\mu$ and the flow vector $(\partial\mu/\partial x_1, \partial\mu/\partial x_2)$ solutions at $t = 2.5$ for the Case 3 of Problem 1

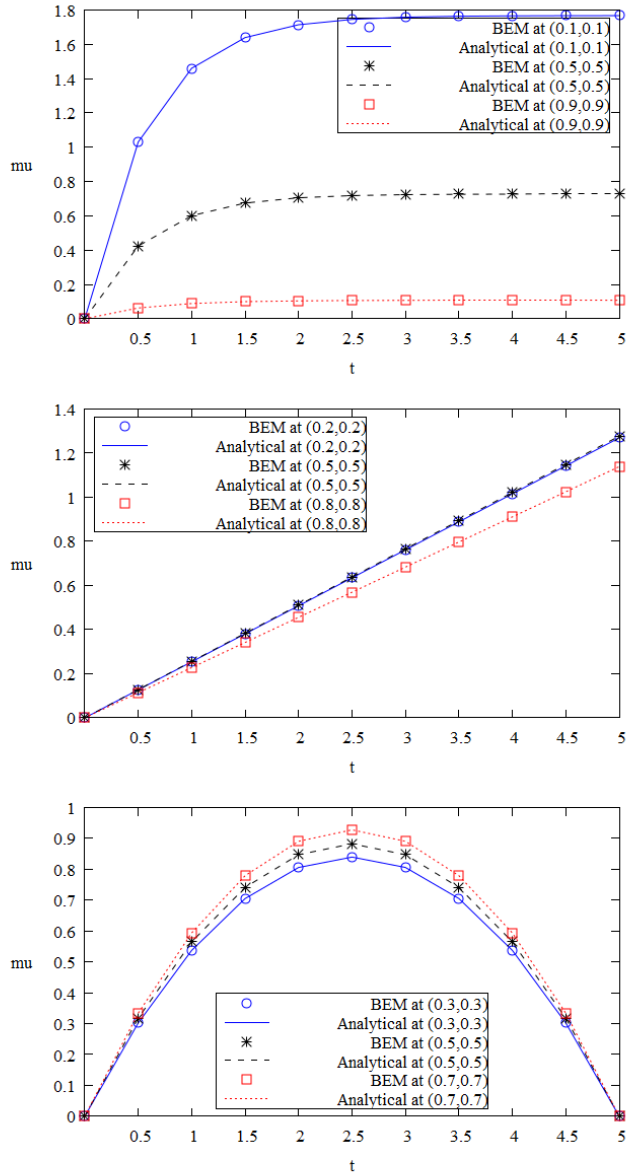


Fig. 6. Solutions $\mu$ at some interior points $(x_1, x_2)$ for the Case 1 (top), Case 2 (center) and Case 3 (bottom) of Problem 1

### *B. Examples without analytical solutions*

The aim is to show the effect of inhomogeneity and anisotropy of the considered material on the solution $\mu$.

*Problem 2:*

The material is supposed to be either inhomogeneous or homogeneous and either anisotropic or isotropic. For a homogeneous material we take

$$g(\mathbf{x}) = 1$$

and for an isotropic material we take

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases regarding the material, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. The corresponding value of $\lambda$ for each case is obtained from equation (8). We set $\overline{\alpha} = 1$ and the boundary conditions are (see Figure 1)

$$\begin{aligned} P &= f(t) \text{ on side AB} \\ P &= 0 \text{ on side BC} \\ \mu &= 0 \text{ on side CD} \\ P &= 0 \text{ on side AD} \end{aligned}$$

Four types of the function $f(t)$ defining the boundary condition on side AB will be considered, namely

$$\begin{aligned} f(t) = f_1(t) &= 1 \\ f(t) = f_2(t) &= 1 - \exp(-1.75t) \\ f(t) = f_3(t) &= t/5 \\ f(t) = f_4(t) &= 0.16t\,(5 - t) \end{aligned}$$

In fact, for the case of isotropic and homogeneous material the system is geometrically symmetric about the axis

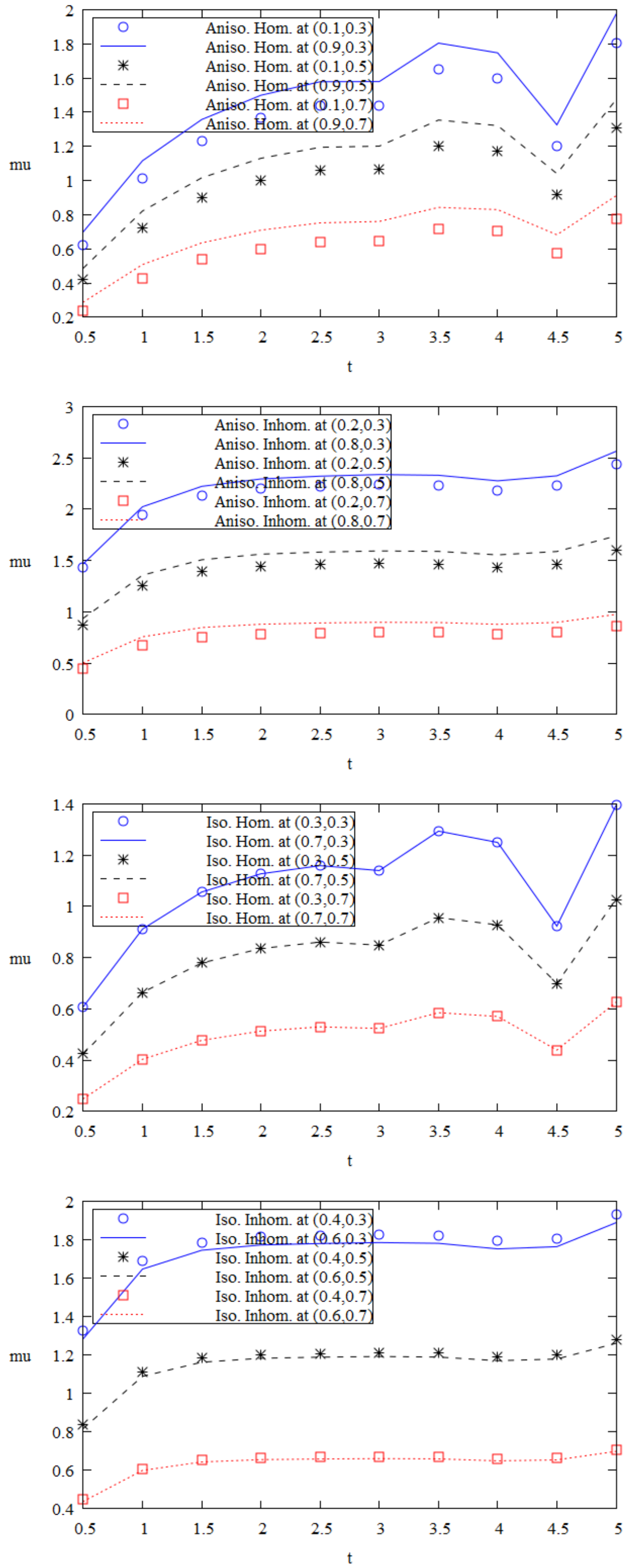


Fig. 7. Symmetry of solution $\mu$ when $f(t) = 1$ for Problem 2

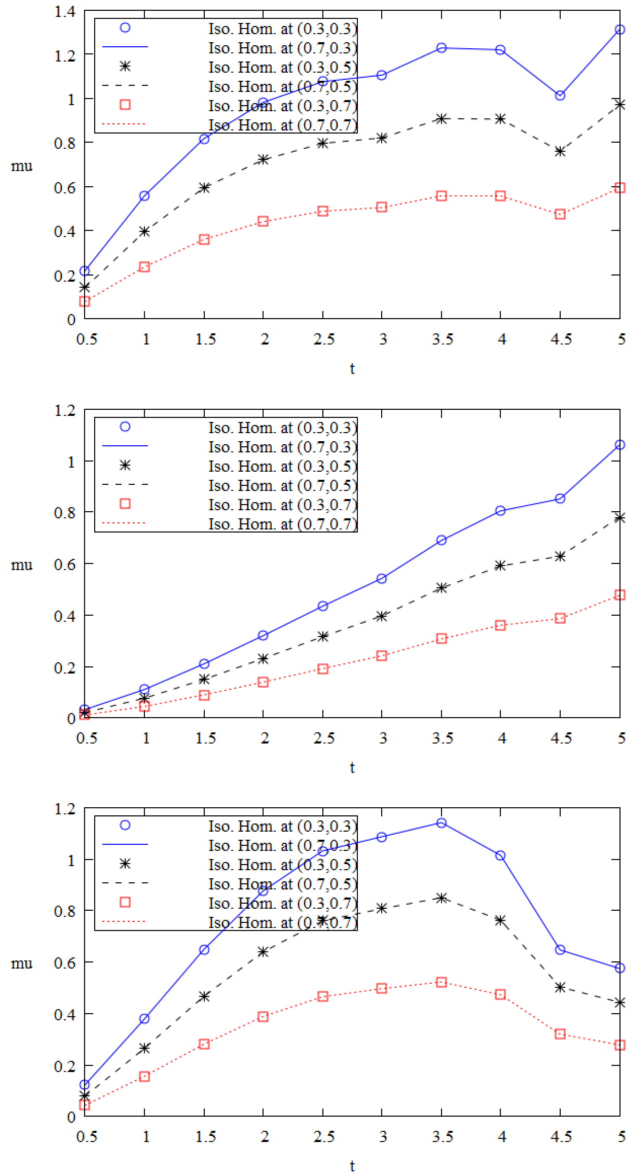


Fig. 8. Symmetry of solution $\mu$ when $f(t) = 1 - \exp(-1.75t)$ (top), $f(t) = t/5$ (center) and $f(t) = 0.16t(5-t)$ (bottom) for Problem 2

$x_1 = 0.5$. And this is verified by the results in Figures 7 and 8. In addition, Figure 7 also shows the effect of anisotropy and inhomogeneity on the asymmetry of the solution $\mu$. And Figure 8 indicates that the solution $\mu$ tends to follow the variation of the function $f(t)$ associated for the boundary condition on the side AB.

Figure 9 shows again the effect of anisotropy and inhomogeneity on the solution $\mu$ and the tendency of the solution $\mu$ to agree the variation of the corresponding function $f(t)$. In particular, for bigger $t$ the boundary conditions on the side AB with $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.75t)$ are identical. This is verified by the results in Figure 9, the two plots for the cases when $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.75t)$ will coincide as $t$ goes to infinity.

After all, the results suggest it is important to put the anisotropy and inhomogeneity into account in any practical application.

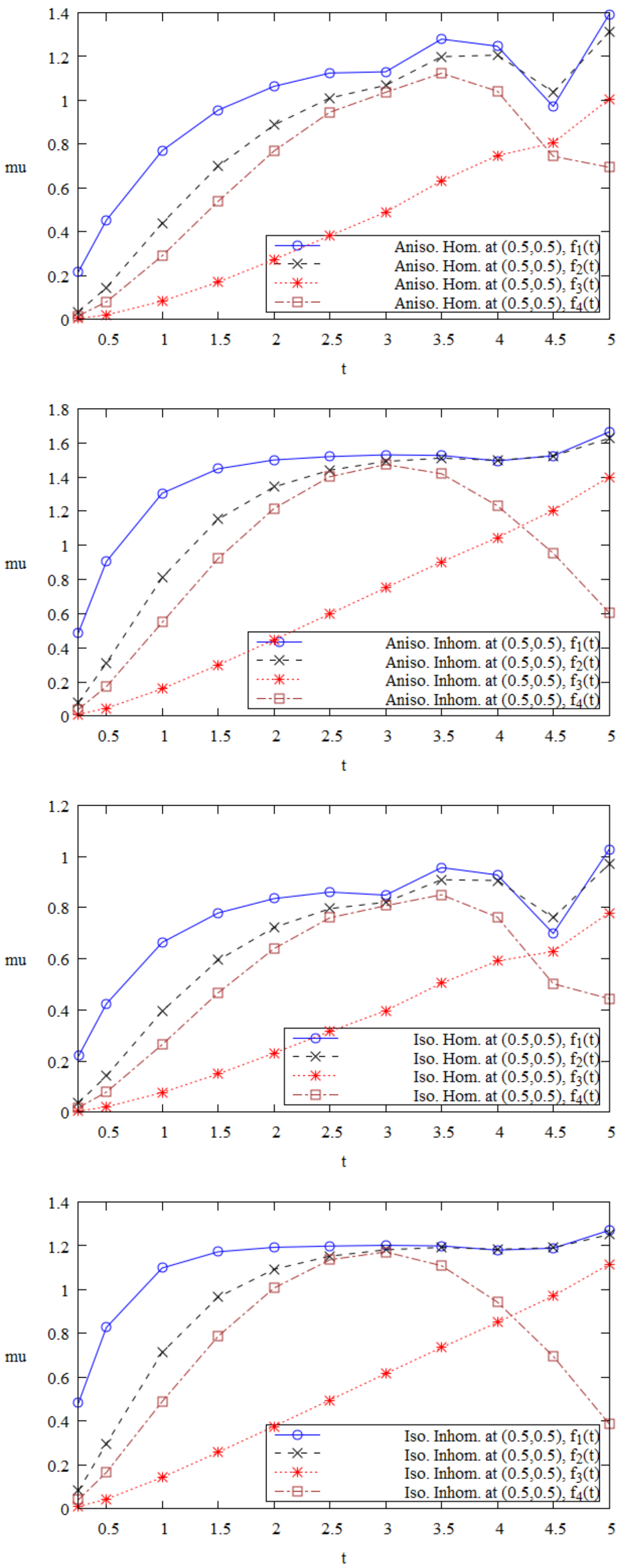


Fig. 9. Solutions $\mu$ at $(x_1, x_2) = (0.5, 0.5)$ for Problem 2

## V. Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic exponentially graded materials which are governed by the equation (1). It is easy to implement and accurate. It involves a time variable free fundamental solution so that it is accurate. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution usually has singular time points and the procedure may involve round-off error propagation.

It has been applied to a class of exponentially graded materials where the coefficients $\kappa_{ij}(\mathbf{x}), \beta(\mathbf{x}), \alpha(\mathbf{x})$ do depend on the spatial variable $\mathbf{x}$ only, taking the forms (4), (5) and (6) and on the same inhomogeneity or gradation function $g(\mathbf{x})$ of exponential form (7). Therefore, it is interesting to extend the study to the case when the coefficients depend on different gradation functions varying also with the time variable $t$.

In order to use the boundary integral equation (20), the values of the boundary conditions $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ as stated in Section (II) have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion.

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