The Property and Structure of Left Clifford Bi-semirings

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Abstract—Based on the existing conclusions of left bi-rings and left Clifford bi-semirings, we give the spined product structure of left Clifford bi-semirings. Then we discuss a special case of left Clifford bi-semirings, that is, strong distributive lattices of left bi-rings. We also study a special class of left Clifford bi-semirings.

Index Terms—distributive lattice congruence, band bi-semiring, left bi-ring, left Clifford bi-semiring.

I. INTRODUCTION AND PRELIMINARIES

A bi-semiring \((S, +, \cdot, *)\) is an algebraic system containing three binary operations \(\cdot\), \(\cdot\) and \(\cdot\). As a more general concept than semirings and distributive lattice ordered semigroups, the study of bi-semirings can unify the respective research methods of semirings and lattice ordered semigroups. Like the study of semiring theory, the study of bi-semirings is also a very important content. References [10], [11], [12] and [13] conduct related studies on the Clifford hierarchy of semigroups and semirings. In this paper, we investigate the properties and structures of left Clifford bi-semirings and left normal Clifford bi-semirings based on the research methods of semigroup and semiring Clifford hierarchy.

Definition 1.1\(^{[1]}\) \((X, \leq)\) is a distributive lattice \(\Leftrightarrow\) The algebraic system \((X, \lor, \land)\) satisfies:

1. \textit{Commutative laws}: \(a \lor b = b \lor a, a \land b = b \land a\);
2. \textit{Associative laws}: \((a \lor b) \lor c = a \lor (b \lor c), (a \land b) \land c = a \land (b \land c)\);
3. \textit{Idempotent laws}: \(a \lor a = a, a \land a = a\);
4. \textit{Absorption laws}: \((a \lor (a \land b)) = a, (a \land (a \lor b)) = b\);
5. \textit{Distributive laws}: \((a \lor (b \land c)) = (a \lor b) \land (a \lor c), a \land (b \lor c) = (a \land b) \lor (a \land c)\).

Definition 1.2\(^{[2]}\) A bi-semiring \((S, +, \cdot, *)\) is called an idempotent bi-semiring if \(\forall s \in S, s + s = s \cdot s = s = s\).

Definition 1.3\(^{[3]}\) Let \(\rho\) be an equivalence relation on the bi-semiring \(S\). Then \(\rho\) is called a congruence relation on the bi-semiring \(S\) if it is a congruence relation on \((S, +), (S, \cdot)\) and \((S, *)\), respectively.

Theorem 1.1\(^{[4]}\) For an idempotent semiring \((S, +, \cdot)\), the following are equivalent:

1. \(S\) satisfies \(\forall s, t \in S, s + t + s = s, s + t s = s;\)
2. \(S\) satisfies \(\forall s, t \in S, s + t s + s = s, s + t s s = s;\)
3. \(\bar{D}\) is the least (distributive) lattice congruence on \(S\).

Definition 1.4\(^{[2]}\) A bi-semiring \((S, +, \cdot, *)\) is called a band if the additive reduct \((S, +)\) of \(S\) is an Abel group.

Definition 1.5\(^{[5]}\) Let \(R, S\) be bi-semirings. A map \(\gamma : R \to S\) is called a homomorphism of bi-semirings if

1. \(\forall r, s \in R, \gamma(r + s) = \gamma(r) + \gamma(s);\)
2. \(\forall r, s \in R, \gamma(r \cdot s) = \gamma(r) \cdot \gamma(s);\)
3. \(\forall r, s \in R, \gamma(r + s) = \gamma(r) + \gamma(s).\)

If \(\gamma\) is also injective (surjective, bijective), then \(\gamma\) is called a (an) monomorphism (epimorphism, isomorphism).

Theorem 1.2\(^{[6]}\) Let \(S\) be a left Clifford semigroup. Then the following conditions are equivalent:

1. \(S\) is a left Clifford semigroup;
2. \(S\) is regular, and \(\mathcal{L} = \mathcal{D}\) is a semilattice congruence for the Green relations \(\mathcal{L}\) and \(\mathcal{D}\);
3. \(S\) is a semilattice of left groups (a left group is a left simple semigroup and a right cancellative semigroup simultaneously).

Definition 1.6\(^{[7]}\) A nonempty subset \(A\) of a semigroup \(S\) is left (respectively, right) unitary if for any \(s, t \in S, s, t = A\) (respectively, \(s, t s \in A\)) implies that \(t \in A\). If \(A\) is both a left unitary subset and a right unitary subset, then \(A\) is called a unitary subset.

II. CHARACTERIZATIONS AND STRUCTURE

Definition 2.1\(^{[8]}\) Let \((S, +, \cdot, *)\) be a bi-semiring. If there is a distributive lattice \(D\) and a family of pairwise disjoint bi-semirings \(\{S_\alpha \mid \alpha \in D\}\) such that \(S = \bigcup \{S_\alpha \mid \alpha \in D\}\) and \(S_+ = ((D_+, S_0, +)), (S, \cdot) = ((D, \cdot, S_0, \cdot)), (S, *) = ((D, *, (S_0, *))\), then \(S\) is called a distributive lattice of bi-semirings \(\{S_\alpha \mid \alpha \in D\}\). Denote this system by \(S = (D; S_0)\).

Definition 2.2\(^{[8]}\) Let \(\rho\) be a congruence on the bi-semiring \((S, +, \cdot, *)\). If \((S/\rho, +, \cdot, *)\) is a distributive lattice and \(S = ((S/\rho, +, \cdot, *); \rho\text{-class})\), then \(\rho\) is called a distributive lattice congruence on \(S\).

Definition 2.3\(^{[8]}\) Let \((S, +, \cdot, *)\) be an idempotent bi-semiring. Then \((S, +, \cdot, *)\) is called a band bi-semiring if \(\forall s, t \in S, (s + t s + s = s; s + t s s + s = s; s + s t + s = s; s + t s s = s)\).

Lemma 2.1 An idempotent bi-semiring \((S, +, \cdot, *)\) is a band bi-semiring if and only if \(\bar{D}\) is the least distributive lattice congruence on \(S\).

Proof: Let \((S, +, \cdot, *)\) be a band bi-semiring, then \((S, +, \cdot)\) and \((S, +, *)\) are obviously band semirings. So from Theorem 1.1 we know that \(\bar{D}\) is the least distributive lattice congruence on \((S, +, \cdot)\) and \((S, +, *)\). Thus \((S/\bar{D}, +, \cdot)\) is a distributive lattice, each \(\bar{D}\text{-class}\) is a bi-semiring and \((S, +) = ((S/\bar{D}, +), (\bar{D}\text{-class}, +))\).

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\[(S, \cdot) = ((S/\bar{D}, \cdot), (\bar{D}-\text{-class, } \cdot)).\]

Since \(\bar{D}\) is a distributive lattice congruence on \((S, +, \cdot)\), we have

\[(S, +, \cdot) = ((S/\bar{D}, +, \cdot), (\bar{D}-\text{-class, } +, \cdot)).\]

From Definition 2.3, we know that \(\forall s, t \in S, s \bar{D} t \iff s + t = s\).

Thus \(\bar{D}\) is a distributive lattice congruence on the bi-semiring \(S\). Let \(\delta\) be any distributive lattice congruence on the bi-semiring \(S\), then \(\delta\) is a semilattice congruence on \((S, +)\).

And because \(\bar{D}\) is the least semilattice congruence on \((S, +)\), \(\bar{D} \subseteq \delta\). Thus \(\bar{D}\) is the least distributive lattice congruence on \(S\).

Conversely, suppose that \(\bar{D}\) is the least distributive lattice congruence on \(S\). Then \((S/\bar{D}, +, \cdot)\) is a distributive lattice and

\[(S, +, \cdot) = ((S/\bar{D}, +, \cdot), (\bar{D}-\text{-class, } +, \cdot)).\]

It is easy to prove that \(\bar{D}\) is the least distributive lattice congruence on \((S, +, \cdot)\), and hence \(\forall s, t \in S, s + t + s = s\), \(s + t s + s = s\). Since \(\forall s, t \in S, s + t \in s \bar{D} t \bar{D} = (st) \bar{D}\), we have \(\forall s, t \in S, st \bar{D} s + t\), that is, \(\forall s, t \in S, s \bar{D} t = s \bar{D} t \bar{D} = s + t s = s\).

It follows that \((S/\bar{D}, +, \cdot) = (S/\bar{D}, +, \cdot)\), and so \((S/\bar{D}, +, \cdot)\) is also a distributive lattice. It is also easy to prove that \(\bar{D}\) is the least distributive lattice congruence on \((S, +, \cdot)\), and so \(\forall s, t \in S, s s + t = s s t + s, s s + t s + s = s\).

Thus \(S\) is a band bi-semiring.

**Definition 2.4** A band bi-semiring \((S, +, \cdot)\) is called a \(T\) band bi-semiring if the additive reduct \((S, +)\) is a \(T\) band, where \(T\) band means "left (right) zero", "left regular" and "left band" elements, etc.

**Definition 2.5** A bi-semiring \((S, +, \cdot)\) is called a left bi-ring if \(S\) can be decomposed as a direct product of a left zero band bi-semiring \(L\) and a bi-ring \(R\).

**Theorem 2.1** A bi-semiring \((S, +, \cdot)\) is a left bi-ring if and only if:

1. The additive reduct \((S, +)\) of \(S\) is a left commutative-group, that is, it is a direct product of a left zero band and a commutative group, and
2. \(E^+(S) \subseteq E(S), E^+(S) \subseteq E^+(S)\), where \(E^+(S)\) \((E(S), E^+(S))\) is the set of idempotents of \((S, +, \cdot)\) \((S, +)\).

**Definition 2.6** A bi-semiring \((S, +, \cdot)\) is called a left Clifford bi-semiring if \(S\) is a distributive lattice of left bi-rings.

**Theorem 2.2** A bi-semiring \((S, +, \cdot)\) is a left Clifford bi-semiring if and only if \(\bar{D}\) is a distributive lattice congruence on \(S\) and each \(\bar{D}^\perp\) is a left bi-ring.

Let \(S\) be a left Clifford bi-semiring. \(\forall s \in S\), we denote the group inverse of \(s\) by \(-s\), that is, \([-s] = V^+(s) \cap \bar{H}_s\). It can be known from Theorem 2.2 that the additive reduct \((S, +)\) of \(S\) is a left regular orthogroup. By using the distributive laws, the following lemma is obvious:

**Lemma 2.2** Let \((S, +, \cdot)\) be a left Clifford bi-semiring.

1. \(s \cdot (t - s) = (s - t)(s - t)\).
2. \(s \cdot t = s \cdot t - s \cdot t = (s - s) \cdot t\).

**Theorem 2.3** Let \((S, +, \cdot)\) be a left Clifford bi-semiring. Then \(\bar{H}\) is a congruence on \(S\).

**Corollary 2.1** A bi-semiring \((S, +, \cdot, *)\) is a left Clifford bi-semiring if and only if \(\bar{L}\) is a distributive lattice congruence on \(S\) and each \(\bar{L}^\perp\) is a left bi-ring.

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**Theorem 2.3** Let \((S, +, \cdot, *)\) be a left Clifford bi-semiring. Then \(\bar{H}\) is a congruence on \(S\).

**Proof:** It is clear that \(\bar{H}\) is a congruence on \((S, +)\) and \((S, +)\). Since \((S, +)\) is a left Clifford semigroup, \(\bar{H} = \bar{R}\) is a left congruence on \((S, +)\). So just need to prove that \(\bar{H}\) is a right congruence on \((S, +)\).

**Definition 2.7** A bi-semiring \((S, +, \cdot)\) is called a left bi-semiring if and only if \(S\) is a distributive lattice of left bi-rings.

**Lemma 2.3** A bi-semiring \((S, +, \cdot)\) is a left regular band bi-semiring if and only if \(S\) is a distributive lattice of left zero band bi-semirings.

**Proof:** Let \((S, +, \cdot)\) is a left regular band bi-semiring.

Then \(\bar{D} = \bar{J}\) is the least semilattice congruence on \((S, +)\) and \((S, +)\) is a semilattice \(S/\bar{D}\) of left zero bands \(\bar{D}\)-class. For any \(\bar{D}\)-class \(\bar{D}_\alpha, \bar{D}_\alpha\) is clearly a band bi-semiring. Thus each \(\bar{D}\)-class is a left zero band bi-semiring.

By Lemma 2.1, \(\bar{D}\) is a distributive lattice congruence on \(S\), and so the least regular band bi-semiring \(S\) is a distributive lattice \(S/\bar{D}\) of left zero band bi-semirings \(\bar{D}\)-class.
Conversely, let the bi-semiring $S$ is a distributive lattice $D$ of left zero band bi-semirings $L_{\alpha}(\alpha \in D)$. Obviously, $S$ is an idempotent bi-semiring and $(S,+) = \bigcup_{\alpha \in D}(L_{\alpha},+)$ is a left regular band. We now prove that $S$ is a band bi-semiring, that is, $\bar{D}$ is the least distributive lattice congruence on $S$.

Since $(S, +)$ is a completely regular semigroup, $(\bar{D} = \bar{\bar{D}})$ is the least semilattice congruence on $(S, +)$. It is easy to show that $\bar{D}$ is a congruence on $(S, \cdot)$ and $(S, +)$ and that $S/\bar{D} \cong D$. Thus $\bar{D}$ is a distributive lattice congruence on $S$. Let $\eta$ be any distributive lattice congruence on $S$, then $\eta$ is a semilattice congruence on $(S, +)$. Also since $\bar{D}$ is the least semilattice congruence, $\bar{D} \subseteq \eta$. Thus $\bar{D}$ is the least distributive lattice congruence on $S$.

**Theorem 2.4** The spined product $L_{\times, D}R$ of a left regular band bi-semiring $L = (D; L_{\alpha})$ and a Clifford bi-semiring $R = (D; R_{\alpha})$ with respect to the distributive lattice $D$ is a left Clifford bi-semiring. Conversely, each left Clifford bi-semiring can be decomposed as a spined product of a left regular band bi-semiring and a Clifford bi-semiring.

**Proof:** The spined product $L_{\times, D}R$ is clearly a distributive lattice $D$ of left bi-rings $L_{\alpha} \times R_{\alpha}$, thus $L_{\times, D}R$ is a left Clifford bi-semiring.

Conversely, let $S$ is a left Clifford bi-semiring. Then $S$ is a distributive lattice $D$ of left bi-rings $L_{\alpha} \times R_{\alpha}$, where $L_{\alpha}(\alpha \in D)$ is a left zero band bi-semiring, $R_{\alpha}(\alpha \in D)$ is a bi-ring. By Theorem 2.3, we have $\bar{H}$ is a congruence on $(S, +)$. Thus, by a conclusion in [6], we can get that $(S, +)$ is a spined product of a left regular band $(L, +)$ and a Clifford semigroup $(R, +)$, where $(L, +) = \bigcup_{\alpha \in D}(L_{\alpha}, +)$ is an upper semilattice $D^{+}$ of left zero bands $(L_{\alpha}, +)$, $(R, +) = \bigcup_{\alpha \in D}(R_{\alpha}, +)$ is an upper semilattice $D^{+}$ of commutative groups $(R_{\alpha}, +)$. If $(i, r) \in L_{\alpha} \times R_{\alpha}$, $(j, s) \in L_{\beta} \times R_{\beta}$, then

$$(i, r) + (j, s) = (i + j, r + s) \in L_{\alpha + \beta} \times R_{\alpha + \beta},$$

where $i + j$ and $(r + s)$ is the sum of $i$ and $j$ $(r$ and $s)$ in $(L, +)$ ($(R, +)$). Next, we will discuss $(i, r)(j, s)$ and $(i, r) * (j, s)$.

Let

$$(i, r)(j, s) = (k, t) \in L_{\alpha \beta} \times R_{\alpha \beta},$$

$$(i, r) * (j, s) = (l, u) \in L_{\alpha \beta} \times R_{\alpha \beta}.$$
III. SPECIAL CASE

**Definition 3.1** Let $D$ be a distributive lattice, $\{S_\alpha | \alpha \in D \}$ be a family of pairwise disjoint bi-semirings. For any $\alpha, \beta \in D$ with $\alpha \leq \beta$, if there is a map $\varphi_{\alpha, \beta} : S_\alpha \rightarrow S_\beta$, where $\varphi_{\alpha, \beta}$ is a monomorphism of bi-semirings and satisfies:

\[
\forall \alpha, \beta, \gamma \in D,
\]

(1) $\varphi_{\alpha, \alpha} = 1_{S_\alpha}$;

(2) $\varphi_{\beta, \alpha} \varphi_{\gamma, \beta} = \varphi_{\gamma, \alpha}$, if $\alpha \leq \beta \leq \gamma$;

(3) $(S_\alpha \varphi_{\beta, \gamma}) \subseteq S_{\beta \varphi_{\alpha, \gamma}}$, if $\alpha \leq \beta \leq \gamma$;

(4) $(S_\alpha \varphi_{\beta, \gamma}) \subseteq S_{\beta \varphi_{\alpha, \gamma}}$, if $\alpha \leq \beta \leq \gamma$.

On $S = \bigcup_{\alpha \in D} S_\alpha$, “$\cdot$” and “$*$” are defined as follows:

\[
\forall s \in S, t \in S_\beta,
\]

(5) $s + t = s \varphi_{\alpha, \alpha} + t \varphi_{\beta, \beta}$,

where $s \varphi_{\alpha, \alpha}$ and $t \varphi_{\beta, \beta}$ are clearly left normal band bi-semirings. We denote the system by $\langle D; S; \varphi_{\alpha, \beta} \rangle$, and call it a strong distributive lattice of left bi-semirings $\{S_\alpha | \alpha \in D \}$.

**Theorem 3.1** A left Clifford bi-semiring $(S, +, \cdot, *, \cdot)$ is a strong distributive lattice of left bi-rings if and only if $E^+(S)$ is a left normal band bi-semiring and $E^+(S)$ is a right unitary subset in $(S, +)$.

**Proof:** Suppose that the left Clifford bi-semiring $S = \langle D; S; \varphi_{\alpha, \beta} \rangle$ is a strong distributive lattice of left bi-rings $S_\alpha = L_\alpha \times R_\alpha$, where $L_\alpha$ is a left zero band bi-semiring and $R_\alpha$ is a bi-ring. According to the definition of “$+$” of $S$, $E^+(S)$ is obviously a left normal band bi-semiring. We now prove that $E^+(S)$ is a right unitary subset in $(S, +)$. If $(i, r) \in S_\alpha$, $(j, 0) \in E^+(S_\beta)$ and $(k, 0) \in E^+(S_\alpha)$ such that $(i, r) + (j, 0) = (k, 0)$, then

\[
(i, r) \varphi_{\alpha, \alpha} + (j, 0) \varphi_{\beta, \beta} = (k, 0).
\]

Denote

\[
(i, r) \varphi_{\alpha, \alpha} = (k', r') \in S_{\beta},
\]

\[
(j, 0) \varphi_{\beta, \beta} = (k'', 0) \in S_{\alpha},
\]

Then

\[
(k', r') + (k'', 0) = (k, 0),
\]

and so

\[
(i, r) \varphi_{\alpha, \alpha} + (j, 0) \in E^+(S_\beta).
\]

Thus the definition of $\varphi_{\alpha, \beta}$ is independent of the choice of the element in $E^+(S_\beta)$, $(i, r) + (j', r') \in S_\alpha$.

If $(i, r) + (j, 0) = (j', s) \in S_\beta$, then

\[
[i(i, r) + (j', r')] \varphi_{\alpha, \beta} = [i(i, r) + (j', r')] + (j, 0)
\]

\[
= [i(i, r) + (j', 0)] + (j', r') + (j, 0)
\]

\[
= (i, r) \varphi_{\alpha, \beta} + (i', r') \varphi_{\alpha, \beta}.
\]

Also since $(E^+(S), +)$ is a left normal band and $\forall s \in S$, $sE^+(S) \cup + E^+(S) \subseteq E^+(S)$, $s \ast E^+(S) \subseteq E^+(S)$, we have

\[
[(i, r)(i', r')] \varphi_{\alpha, \beta}
\]

\[
= (i, r) + (j, 0) + (i', r') + (j, 0)
\]

\[
= [(i, r) + (j, 0)] + [(i', r') + (j, 0)]
\]

\[
= (i, r) \varphi_{\alpha, \beta} + (i', r') \varphi_{\alpha, \beta},
\]

this is because $(j, 0)(i', r') + (i)(r, j) = (0, j) \ast (i', r') + (i')(r, j) = (0, j) \ast (i, r) + (j, 0)$.

Thus $\varphi_{\alpha, \beta}$ is a homomorphism of bi-semirings. If $(i, r), (i', r') \in S_\alpha$ such that

\[
(i, r) \varphi_{\alpha, \beta} = (i', r') \varphi_{\alpha, \beta},
\]

then

\[
(i, r) + (j, 0) = (i', r') + (j, 0)
\]

\[
(i, r) + (j, 0) = (i, r) + (j, 0)
\]

\[
(i, r) + (j, 0) = (i, r) + (j, 0)
\]

\[
(i, r) + (j, 0) = (i, r) + (j, 0)
\]

\[
(i, r) + (j, 0) = (i, r) + (j, 0)
\]

Let $(k, 0) = (0, j)(i', r') = E^+(S_\beta) = E^+(S_\alpha)$, then

\[
(i', r') + (k, 0) = (i', r') + (k, 0)
\]

\[
(i', r') = i'.
\]
Similarly, we can prove that $i = ii'$, and thus $i = i'$. And we have $(i, r) = (i', r')$, thus $\varphi_{\alpha, \beta}$ is a monomorphism of bi-semirings. At the same time, it is easy to verify that $\varphi_{\alpha, \beta}$ satisfies the conditions (1) $\sim$ (5) in Definition 3.1 by using known conditions. Thus $S = (D; S_0; \varphi_{\alpha, \beta})$.

A Clifford bi-semiring is clearly a left Clifford bi-semiring, thus the following corollary is obvious:

**Corollary 3.1** A Clifford bi-semiring $(S, +, \cdot, \ast)$ is a strong distributive lattice of left bi-rings if and only if $E^+(S)$ is unitary in $(S, +)$ of $S$.

**Definition 3.2** A bi-semiring $(S, +, \cdot, \ast)$ is called a left normal Clifford bi-semiring if $S$ is a distributive lattice of left bi-rings and $(E^+(S), +)$ is a left normal band.

Let $S$ be a left normal Clifford bi-semiring, then $S$ is a left Clifford bi-semiring and the set of all additive idempotents of $S$ is a left normal band. By applying Theorem 2.2 and Corollary 2.1, we can get Theorem 3.2 and Corollary 3.2.

**Theorem 3.2** A bi-semiring $(S, +, \cdot, \ast)$ is a left normal Clifford bi-semiring if and only if the additive reduct $(S, +)$ of $S$ is a left normal orthogroup in which each maximal subgroup is abelian, $E^+(S) \subseteq E(S)$, $E^+(S) \subseteq E^+(S)$ and $S$ satisfies the following conditions:

1. $\forall s \in S, V^+(s) + s \supseteq s(V^+(s) + s)$;
2. $\forall s, t \in S, V^+(st) + st \supseteq (V^+(t) + t)s$;
3. $\forall s, t \in S, V^+(s) + s \supseteq V^+(st) + (V^+(s) + s) + st$;
4. $\forall s, t \in S, st \in L$.

**Corollary 3.2** A bi-semiring $(S, +, \cdot, \ast)$ is a left normal Clifford bi-semiring if and only if $L$ is a distributive lattice of left bi-rings on $S$, each $L$-class is a left bi-ring and $(E^+(S), +)$ is a left normal band.

A left normal band bi-semiring is clearly a strong distributive lattice of left zero band bi-semirings by Theorem 3.1. Then we can verify the following theorem by the research methods of Theorem 2.4:

**Theorem 3.3** The spined product $L \lessdot D$ of left normal band bi-semiring $L = (D; L_\alpha; \varphi_{\alpha, \beta})$ and Clifford bi-semiring $R = (D; R_\alpha)$ with respect to the distributive lattice $D$ is a left normal Clifford bi-semiring. Conversely, each left normal Clifford bi-semiring can be decomposed as a spined product of a left normal band bi-semiring and a Clifford bi-semiring.

**Theorem 3.4** A left normal Clifford bi-semiring $S$ is a strong distributive lattice of left bi-rings if and only if $E^+(S)$ is a right unitary subset in $(S, +)$.

**Proof:** Let the left normal Clifford bi-semiring $S$ be a strong distributive lattice of left bi-rings. And because a left normal Clifford bi-semiring is a left Clifford bi-semiring, we have $E^+(S)$ is a right unitary subset in $(S, +)$.

Conversely, suppose that $S$ is a left normal Clifford bi-semiring, and let $E^+(S)$ is a right unitary subset in $(S, +)$. Then $S$ is a distributive lattice of left bi-rings $S_\alpha = L_\alpha \times R_\alpha$, where $L_\alpha$ is a left zero band bi-semiring and $R_\alpha$ is a bi-ring. Thus $E^+(S) = \bigcup_{\alpha \in D} (E^+(L_\alpha) \times E^+(R_\alpha))$, where $E^+(L_\alpha) \times E^+(R_\alpha) \cong L_\alpha$ is a left zero band bi-semiring. Obviously, $E^+(S)$ is a bi-semiring and is a left normal band bi-semiring. From Theorem 3.1, we know that $S$ is a strong distributive lattice of left bi-rings.

**References**


