The Property and Structure of Left Clifford Bi-semirings

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Abstract—**Based on the existing conclusions of left bi-rings and left Clifford bi-semirings, we give the spined product structure of left Clifford bi-semirings. Then we discuss a special case of left Clifford bi-semirings, that is, strong distributive lattices of left bi-rings. We also study a special class of left Clifford bi-semirings.**

Index Terms—**distributive lattice congruence, band bi-semiring, left bi-ring, left Clifford bi-semiring.**

I. Introduction and Preliminaries

A bi-semiring $(S,+,\cdot,*)$ is an algebraic system containing three binary operations "+", "·" and "∗". As a more general concept than semirings and distributive lattice ordered semigroups, the study of bi-semirings can unify the respective research methods of semirings and lattice ordered semigroups. Like the study of semiring theory, the study of bi-semirings is also a very important content. References [10], [11], [12] and [13] conduct related studies on the Clifford hierarchy of semigroups and semirings. In this paper, we investigate the properties and structures of left Clifford bi-semirings and left normal Clifford bi-semirings based on the research methods of semigroup and semiring Clifford hierarchy.

Definition 1.1[1] $(X,\leq)$ is a distributive lattice $\Leftrightarrow$ The algebraic system $(X,\vee,\wedge)$ satisfies:

(1) Commutative laws: $a\vee b=b\vee a$, $a\wedge b=b\wedge a$;

(2) Associative laws: $(a\vee b)\vee c=a\vee(b\vee c)$, $(a\wedge b)\wedge c=a\wedge(b\wedge c)$;

(3) Idempotent laws: $a\vee a=a$, $a\wedge a=a$;

(4) Absorption laws: $a\vee(a\wedge b)=a$, $a\wedge(a\vee b)=a$;

(5) Distributive laws: $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$.

Definition 1.2[2] A bi-semiring $(S,+,\cdot,*)$ is called an idempotent bi-semiring if $\forall s\in S$, $s+s=s\cdot s=s*s=s$.

Definition 1.3[3] Let ρ be an equivalence relation on the bi-semiring S. Then ρ is called a congruence relation on the bi-semiring S if it is a congruence relation on $(S,+)$, $(S,\cdot)$ and $(S,*)$, respectively.

Theorem 1.1[4] For an idempotent semiring $(S,+,\cdot)$, the following are equivalent:

(1) S satisfies $\forall s,t\in S$, $s+st+s=s$, $s+ts+s=s$;

(2) S satisfies $\forall s,t\in S$, $s+st+s=s$, $st+ts+st=st$;

(3) $\overset{+}{\mathcal{D}}$ is the least (distributive) lattice congruence on S.

Definition 1.4[2] A bi-semiring $(S,+,\cdot,*)$ is called a bi-ring if the additive reduct $(S,+)$ of S is an Abel group.

Definition 1.5[5] Let R, S be bi-semirings. A map $\gamma:R\to S$ is called a homomorphism of bi-semirings if

(1) $\forall r,s\in R$, $\gamma(r+s)=\gamma(r)+\gamma(s)$;

(2) $\forall r,s\in R$, $\gamma(r\cdot s)=\gamma(r)\cdot\gamma(s)$;

(3) $\forall r,s\in R$, $\gamma(r*s)=\gamma(r)*\gamma(s)$.

If γ is also injective (surjective, bijective), then γ is called a (an) monomorphism (epimorphism, isomorphism).

Theorem 1.2[6] Let S be a left Clifford semigroup. Then the following conditions are equivalent:

(1) S is a left Clifford semigroup;

(2) S is regular, and $\mathcal{L}=\mathcal{D}$ is a semilattice congruence for the Green relations $\mathcal{L}$ and $\mathcal{D}$;

(3) S is a semilattice of left groups (a left group is a left simple semigroup and a right cancellative semigroup simultaneously).

Definition 1.6[7] A nonempty subset A of a semigroup S is left (respectively, right) unitary if for any $s,t\in S$, $s,st\in A$ (respectively, $s,ts\in A$) implies that $t\in A$. If A is both a left unitary subset and a right unitary subset, then A is called a unitary subset.

II. Characterizations and Structure

Definition 2.1[8] Let $(S,+,\cdot,*)$ be a bi-semiring. If there is a distributive lattice D and a family of pairwise disjoint bi-semirings $\{S_\alpha|\alpha\in D\}$ such that $S=\bigcup\limits_{\alpha\in D}S_\alpha$ and $(S,+)=((D,+),(S_\alpha,+))$, $(S,\cdot)=((D,\cdot),(S_\alpha,\cdot))$, $(S,*)=((D,\cdot),(S_\alpha,*))$, then S is called a distributive lattice of bi-semirings $\{S_\alpha|\alpha\in D\}$. Denote this system by $S=\langle D;S_\alpha\rangle$.

Definition 2.2[8] Let ρ be a congruence on the bi-semiring $(S,+,\cdot,*)$. If $(S/\rho,+,\cdot)$ is a distributive lattice and $S=\langle(S/\rho,+,\cdot);\rho\text{-class}\rangle$, then ρ is called a distributive lattice congruence on S.

Definition 2.3[8] Let $(S,+,\cdot,*)$ be an idempotent bi-semiring. Then $(S,+,\cdot,*)$ is called a band bi-semiring if $\forall s,t\in S$,

(1) $s+st+s=s$, $s+ts+s=s$;

(2) $s+s*t+s=s$, $s+t*s+s=s$;

(3) $st\overset{+}{\mathcal{D}}s*t$.

Lemma 2.1 An idempotent bi-semiring $(S,+,\cdot,*)$ is a band bi-semiring if and only if $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on S.

Proof: Let $(S,+,\cdot,*)$ be a band bi-semiring, then $(S,+,\cdot)$ and $(S,+,*)$ are obviously band semirings. So from Theorem 1.1 we know that $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on $(S,+,\cdot)$ and $(S,+,*)$. Thus $(S/\overset{+}{\mathcal{D}},+,\cdot)$ is a distributive lattice, each $\overset{+}{\mathcal{D}}$-class is a bi-semiring and

$$(S,+)=((S/\overset{+}{\mathcal{D}},+),(\overset{+}{\mathcal{D}}\text{-class},+)),$$

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$$(S,\cdot)=((S/\overset{+}{\mathcal{D}},\cdot),(\overset{+}{\mathcal{D}}\text{-class},\cdot)).$$

Since $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on $(S,+,*)$, we have

$$(S,*)=((S/\overset{+}{\mathcal{D}},*),(\overset{+}{\mathcal{D}}\text{-class},*)).$$

From Definition 2.3, we know that $\forall s,t\in S$, $st\,\overset{+}{\mathcal{D}}\,s*t$, that is, $\forall s,t\in S$, $s\overset{+}{\mathcal{D}}t\overset{+}{\mathcal{D}}=s\overset{+}{\mathcal{D}}*t\overset{+}{\mathcal{D}}$. Then $(S/\overset{+}{\mathcal{D}},\cdot)=(S/\overset{+}{\mathcal{D}},*)$, and so

$$(S,*)=((S/\overset{+}{\mathcal{D}},\cdot),(\overset{+}{\mathcal{D}}\text{-class},*)).$$

Thus $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on the bi-semiring S. Let δ be any distributive lattice congruence on the bi-semiring S, then δ is a semilattice congruence on $(S,+)$. And because $\overset{+}{\mathcal{D}}$ is the least semilattice congruence on $(S,+)$, $\overset{+}{\mathcal{D}}\subseteq\delta$. Thus $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on S.

Conversely, suppose that $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on S. Then $(S/\overset{+}{\mathcal{D}},+,\cdot)$ is a distributive lattice and

$$(S,*)=((S/\overset{+}{\mathcal{D}},\cdot),(\overset{+}{\mathcal{D}}\text{-class},*)).$$

It is easy to prove that $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on $(S,+,\cdot)$, and hence $\forall s,t\in S$, $s+st+s=s$, $s+ts+s=s$. Since $\forall s,t\in S$, $s*t\in s\overset{+}{\mathcal{D}}t\overset{+}{\mathcal{D}}=(st)\overset{+}{\mathcal{D}}$, we have $\forall s,t\in S$, $st\,\overset{+}{\mathcal{D}}\,s*t$, that is, $\forall s,t\in S$, $s\overset{+}{\mathcal{D}}t\overset{+}{\mathcal{D}}=s\overset{+}{\mathcal{D}}*t\overset{+}{\mathcal{D}}$. It follows that $(S/\overset{+}{\mathcal{D}},\cdot)=(S/\overset{+}{\mathcal{D}},*)$ and so $(S/\overset{+}{\mathcal{D}},+,*)$ is also a distributive lattice. It is also easy to prove that $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on $(S,+,*)$, and so $\forall s,t\in S$, $s+s*t+s=s$, $s+t*s+s=s$. Thus S is a band bi-semiring. ■

Definition 2.4[8] A band bi-semiring $(S,+,\cdot,*)$ is called a T band bi-semiring if the additive reduct $(S,+)$ of S is a T band, where T band means "left (right) zero", "left regular" and "left normal" bands, etc.

Definition 2.5[8] A bi-semiring $(S,+,\cdot,*)$ is called a left bi-ring if S can be decomposed as a direct product of a left zero band bi-semiring L and a bi-ring R.

Theorem 2.1[8] A bi-semiring $(S,+,\cdot,*)$ is a left bi-ring if and only if:

(1) The additive reduct $(S,+)$ of S is a left commutative-group, that is, it is a direct product of a left zero band and a commutative group, and

(2) $E^+(S)\subseteq E^{\cdot}(S)$, $E^+(S)\subseteq E^*(S)$, where $E^+(S)$ ($E^{\cdot}(S)$, $E^*(S)$) is the set of idempotents of $(S,+)$ ($(S,\cdot)$, $(S,*)$).

Definition 2.6[8] A bi-semiring $(S,+,\cdot,*)$ is called a left Clifford bi-semiring if S is a distributive lattice of left bi-rings.

Theorem 2.2[8] A bi-semiring $(S,+,\cdot,*)$ is a left Clifford bi-semiring if and only if the additive reduct $(S,+)$ of S is a left Clifford semigroup (left regular orthogroup[7]) in which each maximal subgroup is abelian, $E^+(S)\subseteq E^{\cdot}(S)$, $E^+(S)\subseteq E^*(S)$ and S satisfies the following conditions:

(1) $\forall s\in S$, $V^+(s)+s\supseteq s(V^+(s)+s)$;

(2) $\forall s,t\in S$, $V^+(st)+st\supseteq(V^+(t)+t)s$;

(3) $\forall s,t\in S$, $V^+(s)+s\supseteq V^+(st)+(V^+(s)+s)+st$;

(4) $\forall s,t\in S$, $st\,\overset{+}{\mathcal{L}}\,s*t$.

Corollary 2.1[8] A bi-semiring $(S,+,\cdot,*)$ is a left Clifford bi-semiring if and only if $\overset{+}{\mathcal{L}}$ is a distributive lattice congruence on S and each $\overset{+}{\mathcal{L}}$-class is a left bi-ring.

Let S be a left Clifford bi-semiring. $\forall s\in S$, we denote the group inverse of s by $-s$, that is, $\{-s\}=V^+(s)\cap\overset{+}{H}_s$. It can be known from Theorem 2.2 that the additive reduct $(S,+)$ of S is a left regular orthogroup. By using the distributive laws, the following lemma is obvious:

Lemma 2.2 Let $(S,+,\cdot,*)$ be a left Clifford bi-semiring. Let $\forall s,t\in S$, $e,f\in E^+(S)$ and $e\,\overset{+}{\mathcal{H}}\,s$, $f\,\overset{+}{\mathcal{H}}\,t$. Then

(1) $s(-t)=-st=(-s)t$, $s*(-t)=-s*t=(-s)*t$;

(2) $sf=et=ef\in E^+(S)\cap\overset{+}{H}_{st}$, $s*f=e*t=e*f\in E^+(S)\cap\overset{+}{H}_{s*t}$.

Theorem 2.3 Let $(S,+,\cdot,*)$ be a left Clifford bi-semiring. Then $\overset{+}{\mathcal{H}}$ is a congruence on S.

Proof: It is clear that $\overset{+}{\mathcal{H}}$ is a congruence on $(S,\cdot)$ and $(S,*)$. Since $(S,+)$ is a left Clifford semigroup, $\overset{+}{\mathcal{H}}=\overset{+}{\mathcal{R}}$ is a left congruence on $(S,+)$. So just need to prove that $\overset{+}{\mathcal{H}}$ is a right congruence on $(S,+)$. Let $s,t\in S$, $e\in E^+(S)$ and $e\,\overset{+}{\mathcal{H}}\,s$. We now prove that $(e+t)\,\overset{+}{\mathcal{H}}(s+t)$. Let $f_1,f_2\in E^+(S)$ such that $f_1\,\overset{+}{\mathcal{H}}(e+t)$, $f_2\,\overset{+}{\mathcal{H}}(s+t)$. Because $\overset{+}{\mathcal{H}}$ is a left congruence on $(S,+)$, we have

$$(e+f_1)\,\overset{+}{\mathcal{H}}(e+t),(e+f_2)\,\overset{+}{\mathcal{H}}(s+t),$$

$$(-s+f_2)\,\overset{+}{\mathcal{H}}(e+t),(s+f_1)\,\overset{+}{\mathcal{H}}(s+t).$$

Also since $e+f_1,e+f_2\in E^+(S)$, $e+f_1=f_1$, $e+f_2=f_2$. Applying Lemma 2.2, we have

$$\begin{aligned}f_1*f_2&=(e+f_1)*f_2=e*f_2+f_1*f_2\\&=s*f_2+f_1*f_2=(s+f_1)*f_2\\&=f_2*f_2=f_2,\end{aligned}$$

$$\begin{aligned}f_1*f_2&=f_1*(e+f_2)=f_1*e+f_1*f_2\\&=f_1*(-s)+f_1*f_2=f_1*(-s+f_2)\\&=f_1*f_1=f_1,\end{aligned}$$

and so $(e+t)\,\overset{+}{\mathcal{H}}\,f_1=f_2\,\overset{+}{\mathcal{H}}(s+t)$. It follows that $\forall w\,\overset{+}{\mathcal{H}}\,v,t\in S$, $(w+t)\,\overset{+}{\mathcal{H}}(v+t)$, that is, $\overset{+}{\mathcal{H}}$ is a right congruence on $(S,+)$. ■

Definition 2.7 A bi-semiring $(S,+,\cdot,*)$ is called a Clifford bi-semiring if S is a distributive lattice of bi-rings.

Lemma 2.3 A bi-semiring $(S,+,\cdot,*)$ is a left regular band bi-semiring if and only if S is a distributive lattice of left zero band bi-semirings.

Proof: Let $(S,+,\cdot,*)$ is a left regular band bi-semiring. Then $\overset{+}{\mathcal{D}}=\overset{+}{\mathcal{J}}$ is the least semilattice congruence on $(S,+)$ and $(S,+)$ is a semilattice $S/\overset{+}{\mathcal{D}}$ of left zero bands $\overset{+}{\mathcal{D}}$-class. For any $\overset{+}{\mathcal{D}}$-class $\overset{+}{D}_a$, $\overset{+}{D}_a$ is clearly a band bi-semiring. Thus each $\overset{+}{\mathcal{D}}$-class is a left zero band bi-semiring. By Lemma 2.1, $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on S, and so the left regular band bi-semiring S is a distributive lattice $S/\overset{+}{\mathcal{D}}$ of left zero band bi-semirings $\overset{+}{\mathcal{D}}$-class.

Conversely, let the bi-semiring S is a distributive lattice D of left zero band bi-semirings $L_\alpha(\alpha \in D)$. Obviously, S is an idempotent bi-semiring and $(S,+) = \underset{\alpha\in D}{\cup}(L_\alpha,+)$ is a left regular band. We now prove that S is a band bi-semiring, that is, $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on S. Since $(S,+)$ is a completely regular semigroup, $\overset{+}{\mathcal{D}} = \overset{+}{\mathcal{J}}$ is the least semilattice congruence on $(S,+)$. It is easy to show that $\overset{+}{\mathcal{D}}$ is a congruence on $(S,\cdot)$ and $(S,*)$ and that $S/\overset{+}{\mathcal{D}} \cong D$. Thus $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on S. Let η be any distributive lattice congruence on S, then η is a semilattice congruence on $(S,+)$. Also since $\overset{+}{\mathcal{D}}$ is the least semilattice congruence, $\overset{+}{\mathcal{D}} \subseteq \eta$. Thus $\overset{+}{\mathcal{D}}$ is the least distributive lattice congruence on S. ■

Theorem 2.4 The spined product $L\underline{\times}_D R$ of a left regular band bi-semiring $L = \langle D; L_\alpha\rangle$ and a Clifford bi-semiring $R = \langle D; R_\alpha\rangle$ with respect to the distributive lattice D is a left Clifford bi-semiring. Conversely, each left Clifford bi-semiring can be decomposed as a spined product of a left regular band bi-semiring and a Clifford bi-semiring.

Proof: The spined product $L\underline{\times}_D R$ is clearly a distributive lattice D of left bi-rings $L_\alpha \times R_\alpha$, thus $L\underline{\times}_D R$ is a left Clifford bi-semiring.

Conversely, let S is a left Clifford bi-semiring. Then S is a distributive lattice D of left bi-rings $L_\alpha \times R_\alpha$, where $L_\alpha(\alpha \in D)$ is a left zero band bi-semiring, $R_\alpha(\alpha \in D)$ is a bi-ring. By Theorem 2.3, we have $\overset{+}{\mathcal{H}}$ is a congruence on $(S,+)$. Thus, by a conclusion in [6], we can get that $(S,+)$ is a spined product of a left regular band $(L,+)$ and a Clifford semigroup $(R,+)$, where $(L,+) = \cup_{D^+}(L_\alpha,+)$ is an upper semilattice D^+ of left zero bands $(L_\alpha,+)$, $(R,+) = \cup_{D^+}(R_\alpha,+)$ is an upper semilattice D^+ of commutative groups $(R_\alpha,+)$. And if $(i,r) \in L_\alpha \times R_\alpha$, $(j,s) \in L_\beta \times R_\beta$, then

$$(i,r) + (j,s) = (i+j, r+s) \in L_{\alpha+\beta} \times R_{\alpha+\beta},$$

where $i+j$ $(r+s)$ is the sum of i and j (r and s) in $(L,+)$ ($(R,+)$). Next, we will discuss $(i,r)(j,s)$ and $(i,r)*(j,s)$. Let

$$(i,r)(j,s) = (k,t) \in L_{\alpha\beta} \times R_{\alpha\beta},$$

$$(i,r)*(j,s) = (l,u) \in L_{\alpha\beta} \times R_{\alpha\beta}.$$

We first prove that k and l (t and u) only depend on i and j (r and s). Note that $\overset{+}{\mathcal{H}}$ is a congruence on $(S,\cdot)$ and $(S,*)$, and

$$(i,r)\overset{+}{\mathcal{H}}(j,s) \Leftrightarrow i = j, (\forall (i,r),(j,s) \in S).$$

So it is easy to prove that k and l only depend on i and j. For any $(i',r) \in L_\alpha \times R_\alpha$, let $(i',r)(j,s) = (k',t')$. By the distributive laws of S, we have

$$\begin{aligned}(k,t) &= (i,r)(j,s)\\ &= [(i,0)+(i',r)](j,s)\\ &= (i,0)(j,s)+(i',r)(j,s)\\ &= (k,0)+(k',t')\\ &= (k,t'),\end{aligned}$$

then $t = t'$, and hence t is independent of i. Similarly, we can prove that t is independent of j. Let $(i',r)*(j,s) = (l',u')$. By the distributive laws of S, we have

$$\begin{aligned}(l,u) &= (i,r)*(j,s)\\ &= [(i,0)+(i',r)]*(j,s)\\ &= (i,0)*(j,s)+(i',r)*(j,s)\\ &= (l,0)+(l',u')\\ &= (l,u'),\end{aligned}$$

then $u = u'$, and hence u is independent of i. Similarly, we can prove that u is independent of j. Now we define "$\cdot$" and "$*$" on L (R) as follows: $\forall i \in L_\alpha, j \in L_\beta$ ($\forall r \in R_\alpha, s \in R_\beta$),

$$ij = k \Leftrightarrow (i,0)(j,0) = (k,0)$$
$$(rs = t \Leftrightarrow (i,r)(j,s) = (ij,t)),$$

$$i*j = l \Leftrightarrow (i,0)*(j,0) = (l,0)$$
$$(r*s = u \Leftrightarrow (i,r)*(j,s) = (i*j,u)).$$

It is easy to prove that $(L,+,\cdot,*) = \langle D; L_\alpha\rangle$ and $(R,+,\cdot,*) = \langle D; R_\alpha\rangle$, that is, $(L,+,\cdot,*)$ is a left regular band bi-semiring and $(R,+,\cdot,*)$ is a Clifford bi-semiring. Thus the left Clifford bi-semiring $S \cong L\underline{\times}_D R$ can be decomposed as a spined product of the left regular band bi-semiring $L = \underset{\alpha\in D}{\cup} L_\alpha$ and the Clifford bi-semiring $R = \underset{\alpha\in D}{\cup} R_\alpha$ with respect to distributive lattice D. ■

Example 2.1 Let $S = \{s_1, s_2, s_3\}$. Define "+", "$\cdot$" and "$*$" as below:

$+$	s_1	s_2	s_3
s_1	s_1	s_1	s_1
s_2	s_2	s_2	s_2
s_3	s_3	s_3	s_3

$\cdot$	s_1	s_2	s_3
s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_2
s_3	s_1	s_3	s_3

$*$	s_1	s_2	s_3
s_1	s_1	s_1	s_1
s_2	s_2	s_2	s_2
s_3	s_3	s_3	s_3

In fact, $(S,+)$ and $(S,*)$ are left zero band, $(S,\cdot)$ is the semigroup $(\{s_2,s_3\},\cdot)^0$ with zero s_1, where $(\{s_2,s_3\},\cdot)$ is a left zero band. It is very easy to prove that the two side distributive laws of "$\cdot$" over "+", "$\cdot$" over "$*$", "+" over "$*$" and "$*$" over "+" hold. Therefore, $(S,+,\cdot,*)$ is a bi-semiring. And since $\forall s,t \in S$, $s\overset{+}{\mathcal{L}}t$ and

$$s+st+s = s, s+ts+s = s,$$
$$s+s*t+s = s, s+t*s+s = s,$$

we have $(S,+,\cdot,*)$ is a band bi-semiring and obviously a left regular band bi-semiring. Let $T = \{e\}$ and define "+", "$\cdot$" and "$*$" as below:

$$e+e = ee = e*e = e.$$

Obviously, $(T,+,\cdot,*)$ is a Clifford bi-semiring. Thus $S \cong S \times T$ is a left Clifford bi-semiring.

III. Special Case

Definition 3.1[9] Let D be a distributive lattice, $\{S_\alpha|\alpha \in D\}$ be a family of pairwise disjoint bi-semirings. For any $\alpha, \beta \in D$ with $\alpha \leq \beta$, if there is a map $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$, where $\varphi_{\alpha,\beta}$ is a monomorphism of bi-semirings and satisfies: $\forall \alpha, \beta, \gamma \in D$,

(1) $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$;
(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$;
(3) $(S_\alpha\varphi_{\alpha,\gamma})(S_\beta\varphi_{\beta,\gamma}) \subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$, if $\alpha + \beta \leq \gamma$;
(4) $(S_\alpha\varphi_{\alpha,\gamma}) * (S_\beta\varphi_{\beta,\gamma}) \subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$, if $\alpha + \beta \leq \gamma$.

On $S = \bigcup_{\alpha\in D} S_\alpha$, "+", "·" and "∗" are defined as follows: $\forall s \in S_\alpha, t \in S_\beta$,

(5) $s + t = s\varphi_{\alpha,\alpha+\beta} + t\varphi_{\beta,\alpha+\beta}$,
$st = [(s\varphi_{\alpha,\alpha+\beta})(t\varphi_{\beta,\alpha+\beta})]\varphi^{-1}_{\alpha\beta,\alpha+\beta}$,
$s * t = [(s\varphi_{\alpha,\alpha+\beta}) * (t\varphi_{\beta,\alpha+\beta})]\varphi^{-1}_{\alpha\beta,\alpha+\beta}$.

Denote the system by $S = \langle D; S_\alpha; \varphi_{\alpha,\beta}\rangle$, and call it a strong distributive lattice of bi-semirings $\{S_\alpha|\alpha \in D\}$.

Theorem 3.1 A left Clifford bi-semiring $(S, +, \cdot, *)$ is a strong distributive lattice of left bi-rings if and only if $E^+(S)$ is a left normal band bi-semiring and $E^+(S)$ is a right unitary subset in $(S, +)$.

Proof: Suppose that the left Clifford bi-semiring $S = \langle D; S_\alpha; \varphi_{\alpha,\beta}\rangle$ is a strong distributive lattice of left bi-rings $S_\alpha = L_\alpha \times R_\alpha$, where L_α is a left zero band bi-semiring and R_α is a bi-ring. According to the definition of "+" of S, $E^+(S)$ is obviously a left normal band bi-semiring. We now prove that $E^+(S)$ is a right unitary subset in $(S, +)$. If $(i, r) \in S_\alpha$, $(j, 0) \in E^+(S_\beta)$ and $(k, 0) \in E^+(S_{\alpha+\beta})$ such that $(i, r) + (j, 0) = (k, 0)$, then

$$(i, r)\varphi_{\alpha,\alpha+\beta} + (j, 0)\varphi_{\beta,\alpha+\beta} = (k, 0).$$

Denote

$$(i, r)\varphi_{\alpha,\alpha+\beta} = (k', r') \in S_{\alpha+\beta},$$
$$(j, 0)\varphi_{\beta,\alpha+\beta} = (k'', 0) \in S_{\alpha+\beta}.$$

Then

$$(k', r') = (k', r') + (k'', 0) = (k, 0),$$

and so

$$(i, r)\varphi_{\alpha,\alpha+\beta} = (k, 0) \in E^+(S_{\alpha+\beta}).$$

Also since $\varphi_{\alpha,\alpha+\beta}$ is injective and $(i, 0)\varphi_{\alpha,\alpha+\beta} = (k, 0)$, $(i, r) = (i, 0) \in E^+(S_\alpha)$. Thus $E^+(S)$ is a right unitary subset in $(S, +)$.

Conversely, let S is a left Clifford bi-semiring. Then S is a distributive lattice D of left bi-rings $S_\alpha = L_\alpha \times R_\alpha (\alpha \in D)$, where L_α is a left zero band bi-semiring and R_α is a bi-ring. $\forall \alpha, \beta \in D$ with $\alpha \leq \beta$ and a fixed $(j, 0) \in E^+(S_\beta)$, define

$$\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$$
$$(i, r) \mapsto (i, r) + (j, 0) \quad (\forall (i, r) \in S_\alpha).$$

For any $(j', 0) \in E^+(S_\beta)$, since $E^+(S)$ is a left normal band bi-semiring and $(E^+(S_\beta), +)$ is a left zero band, then we have

$$\begin{aligned}(i, r) + (j, 0) &= [(i, r) + (i, 0)] + [(j, 0) + (j', 0)]\\ &= (i, r) + [(i, 0) + (j, 0) + (j', 0)]\\ &= (i, r) + [(i, 0) + (j', 0) + (j, 0)]\\ &= [(i, r) + (i, 0)] + [(j', 0) + (j, 0)]\\ &= (i, r) + (j', 0).\end{aligned}$$

Thus the definition of $\varphi_{\alpha,\beta}$ is independent of the choice of the element in $E^+(S_\beta)$. $\forall (i, r), (i', r') \in S_\alpha$,

$$[(i, r) + (i', r')]\varphi_{\alpha,\beta} = (i, r) + (i', r') + (j, 0).$$

If $(i', r') + (j, 0) = (j', s) \in S_\beta$, then

$$\begin{aligned}[(i, r) + (i', r')]\varphi_{\alpha,\beta} &= (i, r) + [(j', 0) + (j', s)]\\ &= [(i, r) + (j', 0)] + [(i', r') + (j, 0)]\\ &= (i, r)\varphi_{\alpha,\beta} + (i', r')\varphi_{\alpha,\beta}.\end{aligned}$$

Also since $(E^+(S), +)$ is a left normal band and $\forall s \in S$,

$$sE^+(S) \cup E^+(S)s \subseteq E^+(S),$$
$$s * E^+(S) \cup E^+(S) * s \subseteq E^+(S),$$

we have

$$\begin{aligned}&[(i, r)(i', r')]\varphi_{\alpha,\beta}\\ =&(i, r)(i', r') + (j, 0)\\ =&(i, r)(i', r') + (j, 0)(i', r') + (i, r)(j, 0) + (j, 0)\\ =&[(i, r) + (j, 0)][(i', r') + (j, 0)]\\ =&(i, r)\varphi_{\alpha,\beta}(i', r')\varphi_{\alpha,\beta},\end{aligned}$$

$$\begin{aligned}&[(i, r) * (i', r')]\varphi_{\alpha,\beta}\\ =&(i, r) * (i', r') + (j, 0)\\ =&(i, r) * (i', r') + (j, 0) * (i', r') + (i, r) * (j, 0) + (j, 0)\\ =&[(i, r) + (j, 0)] * [(i', r') + (j, 0)]\\ =&(i, r)\varphi_{\alpha,\beta} * (i', r')\varphi_{\alpha,\beta},\end{aligned}$$

this is because $(j, 0)(i', r') + (i, r)(j, 0) + (j, 0), (j, 0) * (i', r') + (i, r) * (j, 0) + (j, 0) \in E^+(S_\beta)$. Thus $\varphi_{\alpha,\beta}$ is a homomorphism of bi-semirings. If $(i, r), (i', r') \in S_\alpha$ such that

$$(i, r)\varphi_{\alpha,\beta} = (i', r')\varphi_{\alpha,\beta},$$

then

$$\begin{aligned}&(i, r) + (j, 0) = (i', r') + (j, 0)\\ \Rightarrow&(i, 0) + (i, r) + (j, 0) = (i, 0) + (i', r') + (j, 0)\\ &((i, 0) \in E^+(S_\alpha))\\ \Rightarrow&(i, r) + (j, 0) = (i, r') + (j, 0)\\ \Rightarrow&(i, -r') + (i, r) + (j, 0) = (i, -r') + (i, r') + (j, 0)\\ &((i, -r') \in S_\alpha)\\ \Rightarrow&(i, r - r') + (j, 0) = (i, 0) + (j, 0) \in E^+(S_\beta).\end{aligned}$$

Since $E^+(S)$ is a right unitary subset in $(S, +)$, $(i, r - r') \in E^+(S_\alpha)$, and hence $r = r'$. Then we can see

$$\begin{aligned}&(i, r) + (j, 0) = (i', r) + (j, 0)\\ \Rightarrow&[(i, r) + (j, 0)] + [(i, -r) + (j, 0)] = [(i', r) + (j, 0)]+\\ &[(i, -r) + (j, 0)]\\ \Rightarrow&(i, r)\varphi_{\alpha,\beta} + (i, -r)\varphi_{\alpha,\beta} = (i', r)\varphi_{\alpha,\beta} + (i, -r)\varphi_{\alpha,\beta}\\ \Rightarrow&[(i, r) + (i, -r)]\varphi_{\alpha,\beta} = [(i', r) + (i, -r)]\varphi_{\alpha,\beta}\\ \Rightarrow&(i, 0) + (j, 0) = (i', 0) + (j, 0)\\ \Rightarrow&[(i, 0) + (j, 0)](i', 0) = [(i', 0) + (j, 0)](i', 0)\\ \Rightarrow&(i, 0)(i', 0) + (j, 0)(i', 0) = (i', 0) + (j, 0)(i', 0).\end{aligned}$$

Let $(k, 0) = (j, 0)(i', 0) \in E^+(S_{\beta\alpha}) = E^+(S_\alpha)$, then

$$\begin{aligned}&(ii', 0) + (k, 0) = (i', 0) + (k, 0)\\ \Rightarrow&(ii', 0) = (i', 0)\\ \Rightarrow&i' = ii'.\end{aligned}$$

Similarly, we can prove that $i = ii'$, and thus $i = i'$. And we have $(i, r) = (i', r')$, thus $\varphi_{\alpha,\beta}$ is a monomorphism of bi-semirings. At the same time, it is easy to verify that $\varphi_{\alpha,\beta}$ satisfies the conditions $(1) \sim (5)$ in Definition 3.1 by using known conditions. Thus $S = \langle D; S_\alpha; \varphi_{\alpha,\beta} \rangle$. ■

A Clifford bi-semiring is clearly a left Clifford bi-semiring, thus the following corollary is obvious:

Corollary 3.1 A Clifford bi-semiring $(S, +, \cdot, *)$ is a strong distributive lattice of bi-rings if and only if $E^+(S)$ is unitary in $(S, +)$ of S.

Definition 3.2 A bi-semiring $(S, +, \cdot, *)$ is called a left normal Clifford bi-semiring if S is a distributive lattice of left bi-rings and $(E^+(S), +)$ is a left normal band.

Let S be a left normal Clifford bi-semiring, then S is a left Clifford bi-semiring and the set of all additive idempotents of S is a left normal band. By applying Theorem 2.2 and Corollary 2.1, we can get Theorem 3.2 and Corollary 3.2:

Theorem 3.2 A bi-semiring $(S, +, \cdot, *)$ is a left normal Clifford bi-semiring if and only if the additive reduct $(S, +)$ of S is a left normal orthogroup in which each maximal subgroup is abelian, $E^+(S) \subseteq E^\cdot(S)$, $E^+(S) \subseteq E^*(S)$ and S satisfies the following conditions:

(1) $\forall s \in S,\ V^+(s) + s \supseteq s(V^+(s) + s)$;

(2) $\forall s, t \in S,\ V^+(st) + st \supseteq (V^+(t) + t)s$;

(3) $\forall s, t \in S,\ V^+(s) + s \supseteq V^+(st) + (V^+(s) + s) + st$;

(4) $\forall s, t \in S,\ st \overset{+}{\mathcal{L}} s * t$.

Corollary 3.2 A bi-semiring $(S, +, \cdot, *)$ is a left normal Clifford bi-semiring if and only if $\overset{+}{\mathcal{L}}$ is a distributive lattice congruence on S, each $\overset{+}{\mathcal{L}}$-class is a left bi-ring and $(E^+(S), +)$ is a left normal band.

A left normal band bi-semiring is clearly a strong distributive lattice of left zero band bi-semirings by Theorem 3.1. Then we can verify the following theorem by the research methods of Theorem 2.4:

Theorem 3.3 The spined product $L \underline{\times}_D R$ of left normal band bi-semiring $L = \langle D; L_\alpha; \varphi_{\alpha,\beta} \rangle$ and Clifford bi-semiring $R = \langle D; R_\alpha \rangle$ with respect to the distributive lattice D is a left normal Clifford bi-semiring. Conversely, each left normal Clifford bi-semiring can be decomposed as a spined product of a left normal band bi-semiring and a Clifford bi-semiring.

Theorem 3.4 A left normal Clifford bi-semiring S is a strong distributive lattice of left bi-rings if and only if $E^+(S)$ is a right unitary subset in $(S, +)$.

Proof: Let the left normal Clifford bi-semiring S be a strong distributive lattice of left bi-rings. And because a left normal Clifford bi-semiring is a left Clifford bi-semiring, we have $E^+(S)$ is a right unitary subset in $(S, +)$.

Conversely, suppose that S is a left normal Clifford bi-semiring, and let $E^+(S)$ is a right unitary subset in $(S, +)$. Then S is a distributive lattice D of left bi-rings $S_\alpha = L_\alpha \times R_\alpha$, where L_α is a left zero band bi-semiring and R_α is a bi-ring. Thus $E^+(S) = \bigcup_{\alpha \in D} (E^+(L_\alpha) \times E^+(R_\alpha))$, where $E^+(L_\alpha) \times E^+(R_\alpha) \cong L_\alpha$ is a left zero band bi-semiring. Obviously, $E^+(S)$ is a bi-semiring and is a left normal band bi-semiring. From Theorem 3.1, we know that S is a strong distributive lattice of left bi-rings. ■

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