Improved Branch and Bound Global Optimization Algorithm for a Class of Sum of Linear Ratios Problems

Xianfeng Ding , Xinlei Liu , Hongyan Li

***Abstract*—Sum of linear ratios problem is a particularly special kind of the fractional programming problem. Its solution usually has multiple levels and many local optimal solutions of non-global solutions. In this paper, an improved branch and bound global optimization is proposed for a class of sum of linear ratios problems. Firstly, the original problem is transformed into its equivalent form by introduced variables, and the equivalence of the two optimal solutions is proved. Secondly, the linear relaxation programming of the equivalent problem is constructed by using linear relaxation technique. The upper and lower bounds of the optimal value of the original problem are continuously improved by solving a sequence of linear relaxation programming problems, then we can get an approximate optimal solution of the original problem. Finally, with the combination of branch rules and bounding techniques, an improved branch-and-bound global optimization algorithm is designed to solve the optimal solution of the original problem. The numerical experimental results show that the improved algorithm is feasible and effective.**

***Index Terms*—Sum of linear rations sum of linear rations, linear relaxation technique, branch and bound global optimization**

I. Introduction

FRACTIONAL programming is an important type of non-linear programming problem which can be applied in many fields and has a widely range of practical applications. Linear fractional programming problem(LFP) is a special type of fractional programming problem, and it is also a NP-hard problem[1-2]. Researchers reckon that it has certain difficulties in the theoretical research and computational solution, because the original problem has multiple non-global local optimal solutions. Moreover, the original problem also has many applications. such as in the field of transportation and economy[3-4], multi-level random transportation problem[5], cluster analysis problem[6] and queuing theory[7]. so the problem (LFP) has attracted a lot of researchers for decades.

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We can consider that the sum of linear ratios fractional programming problem[8] such as the following form:

$$(LFP):\begin{cases} \min & f(x)=\sum_{i=1}^{p}\frac{P_i(x)}{D_i(x)} \\ s.t. & Ax\le b \\ & x\ge 0 \end{cases} \tag{1}$$

Where the numerator $P_i(x)$ and the denominator $D_i(x)$ are all affine functions defined on R^n , and we have

$$P_i(x)=p_i^T x+\alpha_i=\sum_{j=1}^{n}p_{ij}x_j+\alpha_i$$

$$D_i(x)=d_i^T x+\beta_i=\sum_{j=1}^{n}d_{ij}x_j+\beta_i \cdot$$

$$A\in R^{m\times n}, b\in R^m ,$$

let $S_0=\{x\in R^n \mid Ax\le b, x\ge 0\}$ be a non-empty bounded set.

For:

$\forall x\in S_0,\ p_i,d_i\in R^n,\ \alpha_i,\beta_i,p_{ij},d_{ij}\in R,\ i=1,2,\dots,p$, $p\ge 2$.

According to the features of fractional function $\frac{P_i(x)}{D_j(x)}$, $i=1,2,\dots,p$,the denominator $D_j(x)\neq 0$;that is $D_i(x)>0$ or $D_i(x)<0$ .If $D_i(x)<0$, by letting $\frac{P_i(x)}{D_j(x)}=\frac{-P_i(x)}{-D_i(x)}$, we have $-D_i(x)>0$.

If $P_i(x)$ is an arbitrary function, we can see that there will exists a positive number σ_i large enough to arrive at：

$$P_i(x)+\sigma_i D_i(x)>0$$

Thus, we can assume $P_i(x)>0$ [9].

For special sum of linear ratios problem, the earliest algorithm was proposed when the objective function only has the sum of two terms. For this problem, when $p=1$, Charnes and Cooper[10] proposed an effective simplex method; When $p=2$, Konno constructed a similar parametric simplex algorithm, which was used to solve large-scale optimization problems[11].When $p=3$, Konno developed parametric simplex algorithm and constructed an effective heuristic algorithm, When $p \ge 3$, solution methods are branch and bound algorithm, concave minimization algorithm, harmony search algorithm, outer approximation algorithm, and about cutting plane algorithm [12].

For the sum of linear ratios problem with some multinomial sum of objective functions, Hongwei Lin obtained the value space of each denominator by solving the linear programming problem, and formed these spaces into a super rectangle according to the Cartesian product, and proposed a branch and bound algorithm based on this super-rectangular [13].Yonghong Zhang and others, proposed a branch and bound algorithm to solve the linear ratio sum problem by using the techniques of equivalent transformation and linear transformation[14]; Yuelin Gao and Siqiao Jin, proposed a sum of linear ratios problem can be transformed into bilinear programming problems, Through the characteristics of convex envelope linearity and a binary concave envelope product function, the relaxed linear programming of bilinear programming problem can be given to determine the lower bound of the optimal value of the original problem[15]. Yanxia Chen, An improved branch and bound algorithm was proposed to solve a series of linear ratio problems, The main feature of the algorithm was using the monotonicity and concavity of exponential and logarithmic functions, proposed a new second-level relaxation programming to determine the lower bound of the optimal value of the original problem[16]; Jiao and Liu considered the sum of linear ratios problem as symbol. Converted the original problem to bilinear programming, and then used the definition of convex and concave envelopes to construct linear relaxation programming problem[17]; Danhua Li proposed a new branch and bound algorithm for sum of linear ratios problem, Firstly, the original problem was equivalently transformed, then the linear relaxation programming of the equivalent problem was constructed. Finally, an approximate optimal solution of the original problem was obtained by solving a series of linear programming problems[18].In this paper, An improved branch and bound algorithm is proposed, which is used to solve a class of linear ratio sum problems. Numerical examples show that the improved new algorithm has higher accuracy, better convergence and fewer iterations. The improved algorithm is proposed on the basis of a large number of algorithms in the traditional fractional programming literature.

The structure of this paper is organized as follows: In Chapter 2, the original problem is transformed, including the equivalent transformation of the original problem and the relaxed linear programming of the equivalent problem; In Chapter 3, the branching rule is given, and the description of the specific steps of the branch and bound algorithm and the proof of convergence are carried out. In Chapter 4, the numerical experiments are carried out and making a comparison with the numerical results and each other.

II. Linear Relaxation Programming

Before solving the linear ratio sum problem (LFP), we convert the original problem, which is mainly divided into two parts. The first part is to transform the original problem(LFP) into the linear programming problem(LP) by using transformation techniques, and the second part is to use the relaxation technique to convert the linear programming problem into the linear relaxation programming problem (RLP).

A. Equivalent Transformation

Before solving the original problem, we can solve that this following $2n$ linear programming problems[16]:

$$\begin{cases}\min & x_j \\ s.t. & x \in S_0\end{cases} \text{ and } \begin{cases}\max & x_j \\ s.t. & x \in S_0\end{cases} \tag{2}$$

In order to solving the optimal solution of the above problem, we can get the initial upper and lower bounds l_j^0 and u_j^0 of each variable x_j $(j=1,\dots,n)$, the initial box :

$$X^0 = \left\{ x \in R^n \mid l_j^0 \le x_j \le u_j^0, j = 1,2,\dots,n \right\}.$$

For each $i = 1,2,\dots,p$, let:

$$l_i = \min_{x\in S_0} p_i^T x + \alpha_i, u_i = \max_{x\in S_0} p_i^T x + \alpha_i,\quad L_i = \frac{1}{\max_{x\in S_0} d_i^T x + \beta_i},$$

$$U_i = \frac{1}{\min_{x\in S_0} d_i^T x + \beta_i}$$

obviously for each i, $l_i, u_i, L_i, U_i > 0$ .Introducing the variables:

$$y_i = p_i^T x + \alpha_i = P_i(x),$$

$$z_i = \frac{1}{d_i^T x + \beta_i} = \frac{1}{D_i(x)} (i = 1,2,\dots,p),$$

and converting the original problem into its equivalence problem as follows[17]:

$$(LP): \begin{cases} \min & \Psi_0(y,z) = \sum_{i=1}^{p} y_i z_i \\ \text{s.t.} & P_i(x) - y_i \ge 0 \\ & z_i D_i(x) \le 1 \\ & x \in S_0 \cap X^0, (y,z) \in \Omega \end{cases} \tag{3}$$

Among them,

$$\Omega = \left\{ (y,z) \in R^{2p} \mid l_i \le y_i \le u_i, L_i \le z_i \le U_i \right\}$$

Theorem 1. If (x^*, y^*, z^*) is a global optimal solution of the problem (LP), then x^* is a global optimal solution to the problem (LFP), and for every $i = 1,2,\dots,p$, $y_i^* = P_i(x^*)$, , $z_i^* = \frac{1}{D_j(x^*)}$,Conversely, if x^* is a global optimal solution of the problem(LFP), then (x^*, y^*, z^*) is a global optimal solution to the problem(LP) [18], where:

$$y_i^* = P_i(x^*), z_i^* = \frac{1}{D_i(x^*)}, i = 1,2,\dots,p$$

Proof: With the counter-evidence, if (x^*, y^*, z^*) is a global optimal solution of the problem(LP), assuming that x^* is not the global optimal solution of the problem(LFP), then there is a feasible solution for(LFP) $\overline{x}$ with :

$$\sum_{i=1}^{s}\frac{p_i^T\overline{x}+\alpha_i}{d_i^T\overline{x}+\beta_i}\le\sum_{i=1}^{s}\frac{p_i^Tx^*+\alpha_i}{d_i^Tx^*+\beta_i},$$

let $\overline{y}_i=p_i^T\overline{x}+\alpha_i\overline{z}_i=\frac{1}{d_i^T\overline{x}+\beta_i},i=1,2,\ldots,p,$,Obviously $(\overline{x},\overline{y},\overline{z})$ is a feasible solution of the problem(LP), then the following inequalities are true:

$$\sum_{i=1}^{p}\frac{p_i^Tx^*+\alpha_i}{d_i^Tx^*+\beta_i}=\sum_{i=1}^{p}y_i^*z_i^*\le\sum_{i=1}^{p}\overline{y}_i\overline{z}_i=\sum_{i=1}^{p}\frac{p_i^T\overline{x}+\alpha_i}{d_i^T\overline{x}+\beta_i}\tag{4}$$

That is $\sum_{i=1}^{p}\frac{p_i^Tx^*+\alpha_i}{d_i^Tx^*+\beta_i}\le\sum_{i=1}^{p}\frac{p_i^T\overline{x}+\alpha_i}{d_i^T\overline{x}+\beta_i}$, which contradicts the inequality satisfied in the hypothesis, so x^* is a global optimal solution of the problem(LFP);

On the contrary, if x^* is a global optimal solution of the problem(LFP), according to:

$$y_i^*=p_i^Tx^*+\alpha_i,$$

$$z_i^*=\frac{1}{d_i^Tx^*+\beta_i}\,(i=1,\ldots,p)$$

(x^*,y^*,z^*) is a feasible solution of the problem (LP). Assuming that (x^*,y^*,z^*) is not a global optimal solution of the problem (LP), then there is a feasible solution $(\overline{x},\overline{y},\overline{z})$ of the problem(LP), the following inequality is established:

$$\sum_{i=1}^{p}\overline{y}_i\overline{z}_i\le\sum_{i=1}^{p}y_i^*z_i^*\tag{5}$$

Obviously $\overline{x}$ is a feasible solution of the problem(LFP),the above is:

$$\sum_{i=1}^{p}\frac{p_i^T\overline{x}+\alpha_i}{d_i^T\overline{x}+\beta_i}\le\sum_{i=1}^{p}\frac{p_i^Tx^*+\alpha_i}{d_i^Tx^*+\beta_i},$$

Which in contradiction with x^* is a global optimal solution of the problem (LFP), so (x^*,y^*,z^*) is a global optimal solution of the problem (LP). Then the proof is completed.

We can see that the problem (LFP) and (LP) have the same global optimum value by Theorem 1. After these transformations, the form of the sum of the ratio in the objective function is changed into the form of the sum of the linear, which can reduce the complexity of the objective function.

B. Linear Relaxation Technique

In this section, we describe a transformation technique to convert an equivalent problem into a linear relaxation programming problem by using the objective function and constraint function of the equivalent problem. The best solution for the linear relaxation problem is the possible solution of the original problem, which updates the bias of the optimal value of the original problem. The optimal value of a linear relaxation problem is the optimal value's lower bound of the original problem.

Let:

$$\Omega=\left\{(y,z)\in R^{2p}\mid\ l_s\le y_i\le u_i,L_i\le z_j\le U_j\right\}=\Omega\times\Omega_2\times\cdots\times\Omega_p$$

Where $\Omega_i=\left\{(y_i,z_i)\in R^2\mid\ l_i\le y_i\le u_i,L_i\le z_i\le U_i\right\}$, for each (y_i,z_i) in Ω, l_i,u_i,L_i,U_i must satisfy the following relationships[19]:

$$0\le l_i\le u_i,0<L_i\le U_i\tag{6}$$

First, we have $y_i-l_i\ge0,z_i-L_i\ge0$ in Ω_i ,so

$$(y_i-l_i)(z_i-L_i)\ge0$$

Through expanding it, we can get:

$$y_iz_i\ge L_iy_i+l_iz_i-l_iL_i\,,\ i=1,2,\ldots,p\tag{7}$$

Similarly, we can obtain that:

$$y_i-u_i\le0,$$

$$z_i-U_i\le0\ \text{ in }\ \Omega_i\ ,$$

so $(y_i-u_i)(z_i-U_i)\ge0$, Through expanding it, then we can get:

$$y_iz_i\ge U_iy_i+u_iz_i-u_iU_i,i=1,2,\ldots,p\tag{8}$$

Let

$$e_i^1(y_i,z_i)=L_iy_i+l_iz_i-l_iL_i$$

$$e_i^2(y_i,z_i)=U_iy_i+u_iz_i-u_iU_i$$

$$e_i^l(y_i,z_i)=\max\left\{e_i^1(y_i,z_i),e_i^2(y_i,z_i)\right\}\tag{9}$$

Since $y_iz_i\ge e_i^1(y_i,z_i),y_iz_i\ge e_i^2(y_i,z_i)$ we can get:

$$y_iz_i\ge e_i^l(y_i,z_i)\tag{10}$$

Secondly, we have:

$$(y_i-l_i)(z_i-U_i)\le0,(y_i-u_i)(z_i-L_i)\le0\quad\text{in}\quad\Omega_i\ ,$$

Through expanding them, then we have:

$$y_iz_i\le U_iy_i+l_iz_i-l_iU_i$$
$$y_iz_i\le L_iy_i+u_iz_i-L_iu_i\tag{11}$$

Let

$$e_i^3(y_i,z_i)=U_iy_i+l_iz_i-l_iU_i$$

$$e_i^4(y_i,z_i)=L_iy_i+u_iz_i-L_iu_i$$

$$e_i^u(y_i,z_i)=\min\left\{e_i^3(y_i,z_i),e_i^4(y_i,z_i)\right\}\tag{12}$$

Since $y_iz_i\le e_i^3(y_i,z_i),y_iz_i\le e_i^4(y_i,z_i)$, we can get:

$$y_iz_i\le e_i^u(y_i,z_i)\tag{13}$$

Finally, to sum up:

$$e_i^l(y_i,z_i)\le y_iz_i\le e_i^u(y_i,z_i)\tag{14}$$

By means of a series of transformation of the above, we can get a linear relaxation programming (RLP), and the problem (LP) can get the following results [17]:

$$(RLP):\begin{cases} \min & \Psi_0^l(y,z)=\sum_{i=1}^{p} e_i^l(y_i,z_i) \\ s.t. & e_i^l(y_i,z_i)\ge L_i y_i + l_i z_i - l_i L_i \\ & e_i^l(y_i,z_i)\ge U_i y_i + u_i z_i - u_i U_i \\ & P_i(x)-l_i\ge 0 \\ & L_i D_i(x)\le 1 \\ & x\in S_0\cap X^0 \\ & l_i\le y_i\le u_i \\ & \frac{1}{M_i}\le z_i\le \frac{1}{m_i} \end{cases} \qquad (15)$$

Among them[18],

$$M_i=\sum_{j\in T^+} d_{ij}u_j+\sum_{j\in T^-} d_{ij}l_j+\beta_i,$$

$$m_i=\sum_{j\in T^+} d_{ij}l_j+\sum_{j\in T^-} d_{ij}u_j+\beta_i,$$

$$T^+=\left\{j|\ d_{jy}>0\right\},T^-=\left\{j|\ d_{jj}<0\right\}.$$

Theorem2.
Let $\eta_j=u_j^0-l_j^0, i=1,2,\ldots,n$, then for $\forall x\in S_0\cap X^0$, when $\eta_j\to 0$, we have:

$$h_0=\Psi_0(y,z)-\Psi_0^i(y,z)\to 0, i=1,2,\ldots,p \qquad [14]$$

Proof: Giving the definition of y_i and z_i, when $\eta_j=u_j^0-l_j^0\to 0$, there must be:

$$y_i^n-y_i^l=u_i-l_i\to 0,$$

$$z_i^u-z_i^l=U_i-L_i\to 0, i=1,2,\ldots,p.$$

Obviously when $u_i\to l_i, U_i\to L_i$, there is

$$\lim e_i^1(y_i,z_i)=\lim e_i^2(y_i,z_i)$$

$$\lim e_i^3(y_i,z_i)=\lim e_i^4(y_i,z_i) \qquad (16)$$

so

$$\lim e_i^l(y_i,z_i)=\lim y_i z_i=\lim e_i^n(y_i,z_i)$$

then the following formula is established:

$$h_0=\Psi_0(y,z)-\Psi_0^l(y,z)=\sum_{i=1}^{p} y_i z_i-\sum_{i=1}^{p} e_l^i(y_i,z_i)\to 0 \qquad (17)$$

Then the proof is completed.

According to the above discussion, it is obvious that the optimal value of the problem(RLP) can provide the lower bound for the optimal value of the problem(LP). For $\forall X^k\subseteq X^0$, the feasible domain and the optimal value of the problem(LP) are denoted by $D(X^k)$ and $G(LP)$. The feasible domain and the optimal value of the problem(RLP) are denoted by $\bar{D}(X^k)$ and $G(RLP)$, then we have[16]:

$$D(X^k)\subseteq \bar{D}(X^k), G(RLP)\le G(LP) \qquad (18)$$

III. PROPOSED ALGORITHM AND ITS CONVERGENCE

This section is divided into three parts, In the first part, the branching rule is introduced. The second part combines the branch rule with the bounding determination technique of the precondition, and an effective branch and bound algorithm is designed. The third part is the proof of the convergence of this algorithm.

A. Branching

To solve the problem(RLP), let $x^*=\left(x_1^*,x_2^*,\ldots,x_n^*\right)$ be the optimal solution of the variable x in the problem (RLP), let $X^k=\left\{x\in R^n|\ l_j^k\le x_j\le u_j^k, j=1,2,\ldots,n\right\}\subseteq X^0$ be an arbitrary split box. The branching rule is as follows:

Let $\rho_j^k=\min\left\{u_j^k-x_j^*, x_j^*-l_j^k\right\}, j=1,2,\ldots,n,$

$t^k=\arg\max\left\{\rho_j^k|\ j=1,2,\ldots,n\right\},$

$r_{t^k}=\frac{l_j^k+u_j^k}{2}, \quad j=1,2,\cdots,n$. Let

$$X_1^k=\left\{x\in R^n\ |l_j^k\le x_j\le u_j^k, j\ne t^k, l_t^k\le x_t\le r_{t^k}\right\} \quad (19)$$

$$X_2^k=\left\{x\in R^n\ |\ l_j^k\le x_j\le u_j^k, j\ne t^k, r_{t^k}\le x_t\le u_t^k\right\} \quad (20)$$

Through the above branching rule, the box X^k is divided into two sub-boxes X_1^k and X_2^k.

In the k iteration of the algorithm [18], Q_k represents a set of boxes has global optimal solutions . For each $\bar{X}\in Q_k$, The feasible solution of the original problem (LFP) can be obtained by solving the optimal solution and optimal value of the linear relaxation programming problem (RLP).

By improving the optimal value's upper bound f_k^{ll} of the optimal value of the original problem; we can obtain the optimal value's lower bound f_k^l of the original problem by solving its optimal value. A box with smaller optimal value is selected and divided into two parts. The corresponding solution of the problem is solved on each new sub box, and repeat the process until some criterion for convergence is met.

B. Branch and bound algorithm

The branch and bound algorithm of the original problem is stated as follows:

Step 1 (Initialization). Choose $\varepsilon\ge 0$, we can find an optimal solution $\left(x^0,y^0,z^0\right)$ and the optimal value $\Psi_0^l\left(y^0,z^0\right)$ on the feasible domain X^0 by solving the problem (RLP), where $y_i^0=p_i^T x^0+\alpha_i, z_i^0=\frac{1}{d_i^T x^0+\beta_i}, j=1,2,\ldots,p$. Set,

$f_0^{'} = \Psi_0^{!}\left(y^0, z^0\right), f_0^u = f\left(x^0\right)$.

Step 2 (Termination). If $f_0^{ll} - f_0^{j} \le \varepsilon$, stop. x^0 is the global optimal solution of the problem(LFP). Otherwise, set $Q_0 = \left\{X^0\right\}$, the initial iteration number $k = 1$, and go to Step k.

Step. $k.k \ge 1$

Step $k1$ **(Branching).** According to the branching rule, the box X^k with the smaller optimal value is selected to be divided into two parts X_1^k and X_2^k, , and $Q_k := Q_k \setminus X^k$ is used to represent the set of new boxes that may have the global optimal solution. Set $R = \left\{X_1^k, X_2^k\right\}$.

Step $k2$ **(Bounding and Pruning).** For sub-boxes X_1^k and X_2^k, if $X_r^k \ne \phi$, let $y_i^{kJ} = p_i^T x_r^k + \alpha_i, z_j^{kr} = \dfrac{1}{d_j^T x_r^k + \beta_i}$, , where $r = 1, 2$.

Computing the optimal solution $\left(x_r^k, y^{kT}, z^{kT}\right)$ and the optimal value $f_k^l = \Psi_0^l\left(y^{kr}, z^{kr}\right)$ of the problem (RLP) on the sub-boxes X_1^k and X_2^k. If the optimal value is $f_{k-1}^l < f_k^l$, let $R = R \setminus \left\{X_r^k\right\}$, and there are two cases [13]:

Case 1: If $R = \phi$, go to Step k;

Case 2: If $R \ne \phi$,

let $Q_k = Q_k \cup R$, and update the upper bound $f_k^u = \min\left\{f_{k-1}^u, f\left(x_r^k\right)\right\}$. select x^k to make $f_k^u = f\left(x^k\right)$, and go to Step $k3$.

Step $k3$ **(Judgment Rule).**

Let:

$Q_{k+1} = Q_k \setminus \left\{\bar{X} : f_k^{ll}(\bar{X}) - f_k^i \le \varepsilon, \bar{X} \in Q_k\right\}$, if $Q_{k+1} = \phi$, then the algorithm terminates. x^k is a global optimal solution of the original problem, and f_k^u is the global optimal value of the original problem. otherwise, if $Q_{k+1} \ne \phi$, let $k = k + 1$. select X^k to make it satisfy $X^k = \arg\min_{\pi e_2} f^l(\bar{X})$, and return to Step k.

C. Convergence analysis

Theorem 3. If the proposed algorithm terminates in finite iterations, the global optimal solution of the problem (LFP) is obtained when the algorithm is terminated. If the algorithm generates an infinite branch and bound tree sequence, any limit point of the sequence is the global optimal solution of the problem (LFP)[21], and

$$\lim_{k\to\infty} f_k^u = \lim_{k\to\infty} f_k^l = v$$

Proof: If the algorithm terminates in a finite step, it may be assumed to terminate in iteration $k(k \ge 1)$. Let $\left(x^*, y^*, z^*\right)$ be the optimal solution for solving the problem(RLP) and let

$$T^0 = \left\{y \in R^p \mid l_i \le y_i \le u_i, i = 1, 2, ..., p\right\}$$
$$H^0 = \left\{z \in R^p \mid L_i \le z_i \le U_i, i = 1, 2, ..., p\right\}$$

For $T \subseteq T^0, H \subseteq H^0$ and optimal solutions x^*, we can get :

$$y_i^* = \sum_{j=1}^n p_{ij} x_j^* + \alpha_i, z_i^* = \frac{1}{\sum_{j=1}^n d_j x_j^* + \beta_j}, i = 1, 2, \ldots, p$$

Obviously x^* is a feasible solution of the original problem, and v is an optimal value of the problem(LFP), then

$$f\left(x^*\right) \ge v. \tag{21}$$

Since the algorithm terminates when $f_k^{nk} - f_k^l \le \varepsilon$, the update of the upper bound of the original problem is achieved by updating the function value of the original problem feasible solution, and through the algorithm step, we get the following formula:

$$f\left(x^*\right) - f_k^i \le \varepsilon, f_k^l \le v, \tag{22}$$

Hence, we have $v \le f\left(x^*\right) \le f_k^{'} + \varepsilon \le v + \varepsilon$, that is

$$v \le f\left(x^*\right) \le v + \varepsilon \tag{23}$$

Next is the proof of the second part, if the algorithm generates an infinite sequence $\left\{\left(x^k, y^k, z^k\right)\right\}$, then for each $k \ge 1$, $\left\{\left(x^k, y^k, z^k\right)\right\}$ is obtained by solving the relaxation problem (RLP). For some $T^k \subseteq T^0, H^k \subseteq H^0$ and optimal solutions

$$x^k \in S_0, y_i^k = \sum_{j=1}^n p_{jj} x_j^k + \alpha_1, \quad z_i^k = \frac{1}{\sum_{j=1}^n d_{jj} x_j^k + \beta_i},$$

$i = 1, 2, \ldots, p$. Obviously, sequence $\left\{x^k\right\}$ is a feasible solution of the original problem. Assuming $\bar{x}$ is the accumulation point of $\left\{x^k\right\}$, let us set $\lim_{k\to\infty} x^k = \bar{x}$, Obviously, $\bar{x}$ is also a feasible solution of the original problem, and

there is $f(\bar{x}) \ge v$. Since S_0 is a compact set, there must be $\bar{x} \in S_0$.

For each k of the sequence $\left\{x^k\right\}$ there are $T^{k+1} \subset T^k \subset T^0$, $H^{k+1} \subset H^k \subset H^0$, where T^k and H^k are defined as:

$$T^k = \left\{y \in R^p \mid l_i^k \le y_i \le u_i^k, i = 1, 2, ..., p\right\} \tag{24}$$

$$H^k = \left\{z \in R^p \mid L_i^k \le z_i \le U_i^k, i = 1, 2, ..., p\right\} \tag{25}$$

Then for some points $\bar{y} \in R^p, \bar{z} \in R^p$, we can get:

$$\lim_{k\to\infty} T^k = \bigcap_k T^k = \{\bar{y}\}, \lim_{k\to\infty} H^k = \bigcap_k H^k = \{z\}.$$

For each k, the $\left\{f_k^l\right\}$ obtained by the algorithm step is finite and $\lim_{k\to\infty} f_k^k \le v$. For $T^k \subseteq T^0, H^k \subseteq H^0$, Since x^k is the optimal solution of the problem $RLP(T^k, H^k)$ and f_k^l is equal to the optimal value of problem $RLP(T^k, H^k)$,

we can get:

$$\lim_{k\to\infty} l^k = \lim_{k\to\infty} u^k = \{\bar{y}\}, \lim_{k\to\infty} L^k = \lim_{k\to\infty} U^k = \{\bar{z}\}.$$

Hence, $\lim_{k\to\infty} f_k^l \le v \le f(\bar{x})$. By constantly branching, there is

$$\lim_{k\to\infty} f_k = v = f(x). \tag{26}$$

From the above, $\bar{x}$ is a global optimal solution of the problem (LFP), according to the algorithm steps, there is $f_k^u = f(x^k)$ in iteration k. Since $f(x)$ is a continuous function and there is $x^k \to \bar{x}(k\to\infty)$, we can get $\lim_{k\to\infty} f(x^k) = f(\bar{x})$.

Therefore, we have $\lim_{k\to\infty} f_k^u = \lim_{k\to\infty} f(x^k) = f(\bar{x}) = v$, that is $\lim_{k\to\infty} f_k^u = v$. Then the proof is completed.

IV. NUMERICAL EXPERIMENTS AND RESULTS

In this section, to verify the above algorithm is feasible, we selected 3 classical examples in recent literatures and compared the results with these three examples. The proposed algorithm is coded in MATLAB.

Example 1(see [17]):

$$\begin{aligned} \max\ & \frac{-0.9x_1+1.8x_2+1.8}{3x_1-4x_2+5} + \frac{-0.4x_1+0.3x_2-0.4}{-2x_1+x_2+3} \\ s.t.\ & x_1+x_2 \le 1.50 \\ & x_1 - x_2 \le 0 \\ & 0 \le x_1, x_2 \le 1 \end{aligned} \tag{27}$$

Example 2(see [18] and [6]):

$$\begin{aligned} \max\ & \frac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50} + \frac{3x_1+4x_3+50}{4x_1+4x_2+5x_3+50} \\ s.t.\ & +\frac{x_1+2x_2+4x_3+50}{x_1+5x_2+5x_3+50} + \frac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50} \\ s.t.\ & 2x_1+x_2+5x_3 \le 10 \\ & x_1+6x_2+2x_3 \le 10 \\ & 9x_1+7x_2+3x_3 \ge 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \tag{28}$$

Example 3(see [18] and [6]):

$$\begin{aligned} \max\ & \frac{3x_1+5x_2+3x_3+50}{3x_1+4x_2+5x_3+50} + \frac{3x_1+4x_2+50}{4x_1+3x_2+2x_3+50} \\ s.t.\ & +\frac{4x_1+2x_2+4x_3+50}{5x_1+4x_2+3x_3+50} \\ s.t.\ & 2x_1+x_2+5x_3 \le 10 \\ & x_1+6x_2+2x_3 \le 10 \\ & 9x_1+7x_2+3x_3 \ge 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \tag{29}$$

Taking the precision $\varepsilon = 10^{-6}$ or $\varepsilon = 10^{-4}$ respectively, the approximate solutions and approximations of the accuracy of the examples 1 to 3 are obtained, and the number of iterations and running time of the algorithm are counted. These results show in the following three tables:

Table I Result comparison table

Example	1	
Literature	*This article*	*Article* [17]
ε	10^{-6}	10^{-6}
Optimal solution	(0，1)	(0，1)
The optimal value	3.575	3.575
Number of iterations Time(s)	2	1
Time(s)	0.24915	_

Table II Result comparison table

Example	2	
Literature	*This article*	*Article* [16]
ε	10^{-4}	10^{-4}
Optimal solution	(1. 11, 0, 0)	(0, 0.062, 1.875)
The optimal value	4.0907	4.000
Number of iterations Time(s)	2	58
Time(s)	0.41822	2.968694

Table III Result comparison table

Example	3		
Literature	*This article*	*Article* [6]	*Article* [18]
ε	10^{-4}	10^{-4}	10^{-4}
Optimal solution	(0.462, 0.833, 0)	(0, 3.3333, 0)	(0, 1.6667, 0)
The optimal value	2.9838	3.0029	3.0009
Number of iterations Time(s)	3	80	64
Time(s)	0.49573	8.566259	7.4100

The symbol "—" in Table I indicates that the corresponding numerical result is not given in the reference. We can get some information from the specific experimental results: in the first example, the optimal solution and the optimal value of this paper are equal to the article [17], and the number of iterations is similar; in the second example, comparing with the article[6], this paper not only has better results, but also has fewer iterations and less running time than the article[6], We can easily tell that the results in this paper are not as good as those obtained in the article[18], but the number of iterations is similar to the running time; in the third example, comparing with the article[6] and the article[18], The optimal values obtained are similar, but there are fewer iterations and less running time than the article[6] and the article[18]. In summary, comparing with the reference, although the optimal values obtained are similar, the algorithm has fewer iterations and less running time.

V. Conclusions

In the paper, Though these three examples of the comparison results, we get that our algorithm can be used to solve problems of the global, and our algorithm is feasible and effective. In lemma assumptions and the experimental steps, we verified the feasibility of the algorithm by using the programmed experimental data. Therefore, we can use this improved branch and bound method to solve this linear ratio sum problem. An improved branch-and-bound algorithm is discussed for a kind of linear ratio sum problems. This algorithm mainly uses new transformation techniques and relaxation techniques to obtain the optimal solution of the original problem. From the results of the numerical experiments, it can be seen that a number of iterations and the running time of the optimal solution obtained in this paper are less than those in the reference from these three examples,. Consistently, the optimal value 4.0907 obtained in Example 2 is better than the literature [21], and the optimal value 2.9838 obtained in Example 3 is not as good as the optimal value obtained in the literature [21, 41], but it is not much different. The results Indicates that the algorithm is feasible and efficient.

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