The Szeged Index of Polyomino Chains of 4k-cycles

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Abstract—**In the theory of chemical graphs, several topological indices were defined or expanded or generalized by Wiener, Gutman and mathematicians and other chemists [1]. In this paper, we will calculate the indices of Szeged and vertex-PI for 4k-cycle polyomino chains.**

Index Terms—**Topological index, Szeged index, vertex-PI index, Wiener index, Polyomino chains.**

I. Introduction

In this section, with some definitions theorem from [2] and [3] for calculating the Szeged index of graph.
Suppose $G = (V, E)$ is a simple and connected graph Whose V is the set of vertices and E is the set of edges of G, we consider $e = uv$ to be edge of E. The distance between two vertices u, v is defined as the shortest path between them and is represented by $d(u, v)$. We define $n_u(e)$ to be the number of vertices of V such that these vertices are closer to the vertex u than the vertex v. We define $n_v(e)$ similarly.

The Szeged index is defined according to the definitions of $n_u(e)$ and $n_v(e)$ as follows:

$$Sz(G) = \sum_{e\in E(G)} n_u(e)n_v(e). \quad (1)$$

The calculations performed in [8], are about the relationship between the Szeged index and some other indices, including the edge-vertex and vertex-edge Szeged indices, and revised Szeged index on. The PI index is defined in the graph as follows Consider the edge e of the graph is the sum of all edges that are not spaced from the two ends e, Consider $m_u(e)$ the number of edges of the graph that are closer to the vertex u than the vertex v and Consider $m_v(e)$ the number of edges of the graph that are closer to the vertex v than the vertex u

The PI index the molecular graph G is labeled from

$$PI(G) = \sum_{e=uv} (m_u(e) + m_v(e)) \quad (2)$$

Similarly, the vertex-PI index is the sum of all vertices of the graph that are not spaced u, v,

$$PI_v(G) = \sum_{e=uv} (n_u(e) + n_v(e)), \quad (3)$$

khalifeh also expressed the vertex pi for molecular graphs [9]. Ashrafi in [10] for the infinite family examined the graphs of the fullerences graphs of the vertex PI index.

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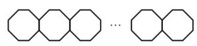

Fig. 1. The linear chain of 8-cycles

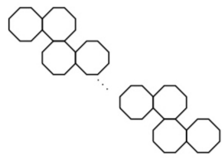

Fig. 2. The Zig-zag chain of 8-cycles

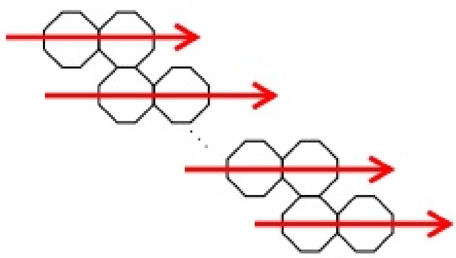

Fig. 3. I-type cut of polyomino chain

II. Calculation of the Szeged index of polyomino 4k-cycles chain

In this section, we first define the k-polyomino system. Each k-polyomino section consists of a number of 4k-cycle chains that are in a common rim and are connected to each other. Examples of linear and zig-zag chains can be seen in Figures 2 and 3. A linear chain consists of one part and the number of chains can be 1 to n and the zig-zag chain of each part consists of two chains, and the number of 4k-cycles in each segment is represented by $l(S)$. A k-polyomino chain has features that include the following: from the sections $S_1, S_2, ..., S_m$, where the length of each section is defined as $l(S_i) = l_i$ for $i = 1, 2, ..., m$, here

$$\sum_{i=1}^{m} l_i = n + m - 1, |V(G)| = n(4k - 2) + 2 \quad (4)$$

which we show the length of all segments as $l = (l_1, . . . , l_m)$[6]. To calculate the Szeged index for these chains, we used four types of cuts. In these cuts, the edges that are located on these straight cut lines have similar calculations that can be seen in Figures 3, 4, 5 and 6.

We start by the following theorem:

Theorem 1 Let $B_{n,k|l}$ be n 4k-cycle k-polyomino chain consisting of $m \geq 1$ segments $S_1, S_2, ..., S_m$ with lengths l_1, . . . , l_m.
(1) If m is odd then

$$Sz(B_{n,k|l}) = \Big(l_1(2k-1)+1\Big)\Big((n-\tfrac{1}{2}l_1)(4k-2)+1\Big)\Big(l_1+1\Big)+$$
$$\sum_{i=2}^{[\frac{m}{2}]+1}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Big(l_{2i-1}+1\Big)\Bigg)+$$
$$2\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i}-3}\Bigg(\sum_{j=1}^{2i-1}4kl_j-2l_j+4tk-2t+4(2k-2ki+i-1)+2k\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-1}l_j2(2k-1)+2t(2k-1)+4(i-1)(-2k+1)+2k\Bigg)\Bigg)\Bigg)+$$
$$\sum_{i=0}^{l_1-2}2\Big(2k+(4ik-2i)\Big)\Big(4nk-2n)+2-2(k+(2ki-i))\Big)+$$
$$\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(2l_{2i-2}-1)(2k-1)+\tfrac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+4k(l_{2i-1}-1)-2(l_{2i-1}-1)+\tfrac{1}{2}\Big(4k(l_{2i}-1)-2(l_{2i}-1)\Big)+2k\Bigg)\Bigg)\Bigg)\Big(l_{2i}+1\Big)+$$
$$2\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i-1}-3}\Bigg(\sum_{j=1}^{2i-2}4l_jk-2l_j+4kt-2t+2(-4ki+2i+6k-3)+2k\Bigg)\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{2i-2}4l_jk-2l_j+4kt-2t+2(-4ki+2i+6k-3)+2k\Bigg)\Bigg)\Bigg)+$$
$$2\Bigg(\sum_{t=0}^{l_m-2}\Bigg(\sum_{j=1}^{m-1}4l_jk-2l_j+4kt-2t+2(4k-2km+m-2)+2k\Bigg)\Bigg(-2n+4nk+2-\Bigg(\sum_{j=1}^{m-1}2l_j(2k-1)+4kt-2t+2(4k-2km+m-2)+2k\Bigg)\Bigg)\Bigg)+$$
$$\sum_{i=1}^{l_1}2(k-1)\Big(2k+2(2ki-i-2k+1)\Big)\Big(-2n+4nk)+2-(2k+2(2ki-i-2k+1))\Big)+$$
$$\sum_{i=1}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i}-2}\Bigg(\sum_{j=1}^{2i-1}\Big(4kl_j-2l_j+4kt-2t+2k+4(-2ki+i+2k-1)\Big)\Bigg)\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{2i-1}\Big(4kl_j-2l_j+4kt-2t+2k+2(-2ki+i+2k-1)\Big)\Bigg)\Bigg)\Big(2k-2\Big)+$$
$$\sum_{i=2}^{[\frac{m}{2}]+1}\sum_{t=0}^{l_{2i-1}-1}\Bigg(\sum_{j=1}^{2i-2}\Big(4kl_j-2l_j+4kt-2t-8ki+6k+8k-4\Big)\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-2}\Big(4kl_j-2l_j-8ki+6k+8k-4\Big)\Bigg)\Bigg)\Big(2k-2\Big)+$$
$$2(m+1)\Bigg(k\Big(-2n+4nk+2-2k\Big)\Bigg)2(k-1)+\sum_{i=1}^{l_1-2}(2k-2)\Big(2k+4ki-2i\Big)\Big(-2n+4nk+2-2\Big(k+2ki-i\Big)\Big)+$$
$$\sum_{i=2}^{m}\Bigg(\sum_{t=0}^{l_i-3}\Bigg(\sum_{j=1}^{i-1}4kl_j-2l_j+4kt-2t+2(2k-2ki+i-1)+2k\Bigg)\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{i-1}4kl_j-2l_j+4kt-2t+2(2k-2ki+i-1)+2k\Bigg)\Bigg)(2k-2)\Bigg).$$

(2) If m is even then,

$$Sz(B_{n,k|l}) = \Big(l_1(2k-1)+1\Big)\Big((n-\tfrac{1}{2}l_1)(4k-2)+1\Big)\Big(l_1+1\Big)+$$
$$\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Big(l_{2i-1}+1\Big)\Bigg)+$$

$2\sum_{i=1}^{[\frac{m}{2}]-1}\Bigg(\sum_{t=0}^{l_{2i}-3}\Bigg(\sum_{j=1}^{2i-1} 4kl_j-2l_j+4kt-2t+4(-2ki+i+2k-1)+2k\Bigg)\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{2i-1} 4kl_j-2l_j+4kt-2t+4(-2ki+i+2k-1)+2k\Bigg)\Bigg)\Bigg)+$

$2\sum_{t=0}^{l_m-2}\Bigg(\sum_{j=1}^{m-1}\Bigg(4kl_j-2l_j+4kt-2t+6k-2km+m-2\Bigg)\Bigg)\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{m-1}\Bigg(l_j(4k-2)+4kt-2t+6k-2km+m-2\Bigg)\Bigg)\Bigg)+$

$\sum_{i=0}^{l_1-2} 4\Bigg(k+2ki-i\Bigg)\Bigg(2n(2k-1)+2-2(k+2ki-i)\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Bigg((l_{2j-1}+l_{2j})(4k-2)-8k+4\Bigg)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Bigg((l_{2i}-1)(4k-2)\Bigg)+2k\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Bigg((l_{2j-1}+l_{2j})(4k-2)-8k+4\Bigg)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Bigg((l_{2i}-1)(4k-2)\Bigg)+2k\Bigg)\Bigg)\Bigg(l_{2i}+1\Bigg)+$

$2\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i-1}-3}\Bigg(\sum_{j=1}^{2i-2} l_j(4k-2)+4kt-2t+2(-4ki+2i+7k-3)\Bigg)\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{2i-2} l_j(4k-2)+t(4k-2)+2(-4ki+2i+7k-3)\Bigg)\Bigg)\Bigg)+$

$\sum_{i=1}^{l_1} 2(k-1)\Bigg(2k+2(i-1)(2k-1)\Bigg)\Bigg(2n(2k-1)+2-(2k+2(i-1)(2k-1))\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i}-2}\Bigg(\sum_{j=1}^{2i-1}\Bigg((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Bigg)\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-1}\Bigg((l_j+t)(4k-2)+2k+4(i-1)(-2k+1)\Bigg)\Bigg)\Bigg)\Bigg(2k-2\Bigg)+$

$\sum_{i=2}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i-1}-2}\Bigg(\sum_{j=1}^{2i-2}\Bigg((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Bigg)\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-2}\Bigg((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Bigg)\Bigg)\Bigg)\Bigg(2k-2\Bigg)+$

$2(m+1)\Bigg(k\Bigg(4kn-2n+2-2k\Bigg)\Bigg)(2k-2)+\sum_{i=1}^{l_1-2}(2k-2)\Bigg(2k+i(4k-2)\Bigg)\Bigg(4kn-2n+2-2\Bigg(k+2ki-i\Bigg)\Bigg)+$

$\sum_{i=2}^{m}\Bigg(\sum_{t=0}^{l_i-3}\Bigg(\sum_{j=1}^{i-1} l_j(4k-2)+4kt-2t-4ki+2i+10k-4\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1} 2l_j(2k-1)+4kt-2t+2(-2ki+i+5k-2)\Bigg)\Bigg)(2k-2)\Bigg).$

Proof:

$B_{n,k|l}$ cuts are divided into I-type (see figure 3), II-type (see figure 4), III-type (see figure 5) and IV-type (see figure 6). An edge is called I-type (or, II-type, or III-type, or IV-type) if it intersects with I-type (or, II-type or, III-type, or IV-type) cut.

Therefore it is enough to calculate $n_u(e)$ and $n_v(e)$ in each section.

(1) If m is odd, consider the following four cases.

Case 1. If edge e is I-type in the j-th 4k-cycle of the r-th segment (that is, e is the edge that passed the dotted line in figure 3). If you can see that r is odd and there are l_r+1 such edges in the r-th segment. If r is even then there are 2 such edges in j-th 4k-cycle of r-th segment, where $2\le j\le l_r-1$.

Subcase 1.1 If $r=1$. Next, $n_u(e)=\frac{1}{2}\Bigg(l_1(4k-2)+2\Bigg)$ and $n_v(e)=\Bigg(n(4k-2)+2-\frac{1}{2}\Bigg(l_1(4k-2)+2\Bigg)\Bigg)$.

Subcase 1.2 If $r=2s-1$, $2\le s\le[\frac{m}{2}]$. Then, we obtain $n_u(e)=\sum_{j=1}^{[\frac{m}{2}]-1}\Bigg((l_{2j-1}+l_{2j}-2)(4k-2)\Bigg)+(l_{2[\frac{m}{2}]-1}-1)(2k-1)+2k$, and $n_v(e)=\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{[\frac{m}{2}]-1}\Bigg((l_{2j-1}+l_{2j}-2)(4k-2)\Bigg)+(l_{2[\frac{m}{2}]-1}-1)(2k-1)+2k\Bigg)$.

Subcase 1.3 If $r=2s$, $1\le s\le[\frac{m}{2}]$ and $0\le t\le l_r-3$ in j-th 4k-cycle, $2\le j\le l_r-1$. Then, we have $n_u(e)=\Bigg(\sum_{j=1}^{2s-1} l_j(4k-2)+t(4k-2)+2(s-1)(-4k+2)+2k\Bigg)$, and $n_v(e)=\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2s-1} l_j(4k-2)+$

$t(4k-2)+2(s-1)(-4k+2)+2k\Big)\Big)$.

Subcase 1.4 If $r=m$. Then, we get $n_u(e)=\sum_{j=1}^{[\frac{m}{2}]}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_m-1)(2k-1)+2k$, and $n_v(e)=\Big(n(4k-2)+2-\Big(\sum_{j=1}^{[\frac{m}{2}]}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_m-1)(2k-1)+2k\Big)$.

Case 2. If the j-th 4k-cycle edge e of the r-th segment is II-type (that is, e is the edge that passed the dotted line in figure 4).
If r is even, you can see that there are l_r+1 such edges in the r-th segment. If r is odd then there are 2 such edges in j-th 4k-cycle of r-th segment, 2 ≤j≤ l_r-1.

Subcase 2.1 If $r=1$ and $0\le i\le l_1-2$ then $n_u(e)=\Big(2k+i(4k-2)\Big)$, and $n_v(e)=\Big(n(4k-2)+2-\Big(2k+i(4k-2)\Big)\Big)$.

Subcase 2.2 If $r=2s-1$, where $2\le s\le[\frac{m}{2}]$ and $0\le t\le l_r-3$. Then we obtain $n_u(e)=\Big(\sum_{j=1}^{r-1}l_j(4k-2)+4kt-2t-8ks+4s+14k-6\Big)$, and $n_v(e)=\Big(n(4k-2)+2-\Big(\sum_{j=1}^{r-1}l_j(4k-2)+4kt-2t-8ks+4s+14k-6\Big)\Big)$.

Subcase 2.3 If $r=2s$, where $1\le s\le[\frac{m}{2}]$. Then we get $n_u(e)=\sum_{j=1}^{s-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2s-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2s}-1)(4k-2)\Big)+2k$, and $n_v(e)=\Big(n(4k-2)+2-\Big(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2s-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2s}-1)(4k-2)\Big)+2k\Big)\Big)$

Subcase 2.4 If $r=m$ and $0\le t\le l_m-2$. Then we have $n_u(e)=\Big(\sum_{j=1}^{m-1}2l_j(2k-1)+4kt-2t-4km+2m+10k-2\Big)$, and $n_v(e)=\Big(4kn-2n+2-\Big(\sum_{j=1}^{m-1}2l_j(2k-1)+4kt-2t-4km+2m+10k-2\Big)\Big)$.

Case 3. If the j-th 4k-cycle edge e of the r-th segment is of III-type (that is, e is the edge that passed through the dotted line in figure 5).
For each r, we see that there are $2k-2$ such edges in the j-th 4k-cycle of the r-th segment, and 1 ≤j≤ l_r.

Subcase 3.1 If $r=1$, $1\le i\le l_1$ then $n_u(e)=\Big(2k+(i-1)(4k-2)\Big)$, and $n_v(e)=\Big(4kn-2n+2-\Big(4ki-2i-2k+2\Big)\Big)$.

Subcase 3.2 If $r=2s-1$, where $2\le s\le[\frac{m}{2}]$ and $0\le t\le l_{2s-1}-2$ then $n_u(e)=\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)$, and $n_v(e)=\Big(n(4k-2)+2-\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Big)$.

Subcase 3.3 If $r=2s$, where $1\le s\le[\frac{m}{2}]$ and $0\le t\le l_{2s}-2$ then we get $n_u(e)=\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)$, and $n_v(e)=\Big(n(4k-2)+2-\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Big)$.

Subcase 3.4 If $r=m$, where and $0\le t\le l_m-2$ then we have $n_u(e)=\sum_{j=1}^{m-1}\Big(4kl_j-2l_j+4kt-2t+2k+2(-2km+4k+m-2)\Big)$, and $n_v(e)=\Big(4kn-2n+2-\sum_{j=1}^{m-1}\Big(4kl_j-2l_j+2k+4kt-2t+2(-2km+4k+m-2)\Big)\Big)$.

Case 4. If edge e is IV-type in j-th 4k-cycle of r-th segment (figure 6). It is obviously the beginning and the end of each segment we have $n_u(e)=2k$ and $n_v(e)=n(4k-2)+2-2k$ that the number of such these 4k-cycles in $B_{n,l|k}$ are $m+1$. Now, we calculate the rest below.
For each r we see that there are $2k-2$ such edges in the j-th 4k-cycle of the r-th segment, where 1 ≤j≤ l_r.

Subcase 4.1 If $r=1$, where $1\le i\le l_1-2$ then we obtain $n_u(e)=\Big(2k+i(4k-2)\Big)$, and $n_v(e)=\Big(n(4k-2)+2-\Big(2k+i(4k-2)\Big)\Big)$.

Subcase 4.2 If $r=s$, where $2\le s\le m-1$ and $0\le t\le l_s-3$ then we get $n_u(e)=\Big(\sum_{j=1}^{s-1}2l_j(2k-1)+4kt-2t+$

$2(-2ks+s+5k-2)\Big)$, and $n_v(e)=\Bigg(4kn-2n+2-\Big(\sum_{j=1}^{s-1}2l_j(2k-2)+4kt-2t+2(-2ks+s+5k-2)\Big)\Bigg)$.

Subcase 4.3 If $r=m$, where $0\le t\le l_m-3$ then we have $n_u(e)=\Big(\sum_{j=1}^{m-1}l_j(4k-2)+4kt-2t+2(-2km+m+5k-2)\Big)$, and $n_v(e)=\Bigg(n(2k-1)2+2-\Big(\sum_{j=1}^{m-1}2l_j(2k-1)-2t+4kt+2(-2km+m+5k-2)\Big)\Bigg)$.

Therefore, by combining the above cases with the definition of the Szeged index, we have:

$Sz(B_{n,k|l})=\Big(l_1(-1+2k)+1\Big)\Big((n-\frac{1}{2}l_1)(-2+4k)+1\Big)\Big(l_1+1\Big)+$

$\sum_{i=2}^{[\frac{m}{2}]+1}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Bigg(n(-2+4k)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Big(l_{2i-1}+1\Big)\Bigg)+$

$2\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i}-3}\Big(\sum_{j=1}^{2i-1}l_j(4k-2)+4kt-2t-8ki+4i+10k-4\Big)\Bigg(4kn-2n+2-\Big(\sum_{j=1}^{2i-1}2l_j(2k-1)+4kt-2t-8ki+4i+10k-4\Big)\Bigg)\Bigg)+$

$\sum_{i=0}^{l_1-2}2\Big(2k+4ki-2i\Big)\Big(4kn-2n+2-(2k+4ki-2i)\Big)+$

$\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(-2+4k)-8k+4\Big)+(l_{2i-1}-1)2(-1+2k)+\frac{1}{2}\Big((l_{2i}-1)2(2k-1)\Big)+2k\Bigg)\Bigg)\Big(l_{2i}+1\Big)+$

$2\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i-1}-3}\Big(\sum_{j=1}^{2i-2}4kl_j-2l_j+4kt-2t-8ki+4i+14k-6\Big)\Bigg(4kn-2n+2-\Big(\sum_{j=1}^{2i-2}4kl_j-2l_j+4kt-2t-8ki+4i+14k-6\Big)\Bigg)\Bigg)+$

$2\Bigg(\sum_{t=0}^{l_m-2}\Big(\sum_{j=1}^{m-1}4kl_j-2l_j+4kt-2t-4km+2m+10k-4\Big)\Bigg(4kn-2n+2-\Big(\sum_{j=1}^{m-1}4kl_j-2l_j+4kt-2t-4km+2m+10k-4\Big)\Bigg)\Bigg)+$

$\sum_{i=1}^{l_1}2(k-1)\Big(2k+2(i-1)(2k-1)\Big)\Big(2n(2k-1)+2-(2k+2(i-1)(2k-1))\Big)+$

$\sum_{i=1}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i}-2}\Bigg(\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+4(i-1)(-2k+1)\Big)\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+2(i-1)(-2k+1)\Big)\Bigg)\Bigg)\Big(2k-2\Big)+$

$\sum_{i=2}^{[\frac{m}{2}]+1}\sum_{t=0}^{l_{2i-1}-2}\Bigg(\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Bigg)\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-2}\Big(2(l_j+t)(2k-1)+2k+(2i-2)(-4k+2)\Big)\Bigg)\Bigg)\Big(2k-2\Big)+$

$2(m+1)\Bigg(k\Big(n(4k-2)+2-2k\Big)\Bigg)(2k-2)+\sum_{i=1}^{l_1-2}(2k-2)\Big(2k+4ki-2i\Big)\Bigg(4kn-2n+2-\Big(2k+4ki-2i\Big)\Bigg)+$

$\sum_{i=2}^{m}\Bigg(\sum_{t=0}^{l_i-3}\Big(\sum_{j=1}^{i-1}4kl_j-2l_j+4kt-2t-4ki+2i+6k-4\Big)\Bigg(2n(2k-1)+2-\Big(\sum_{j=1}^{i-1}4kl_j-2l_j+4kt-2t-4ki+2i+6k-4\Big)\Bigg)(2k-2)\Bigg)$.

(2) If m is even then we consider the flowing four cases. Now, we do the same as step (1):

Case 5. If edge e is I-type in the j-th 4k-cycle of the r-th segment (that is, e is the edge that passed the dotted line in figure 3). If you can see that r is odd and there are l_r+1 such edges in the r-th segment. If r is even, then there are two such edges at the j-th 4k-cycle of the r-th segment, where $2\le j\le l_r-1$.

Subcase 5.1 If $r = 1$. Then, we have $n_u(e) = \frac{1}{2}\Big(l_1(4k-2)+2\Big)$, and $n_v(e) = \Big(n(4k-2)+2-\frac{1}{2}\Big(l_1(4k-2)+2\Big)\Big)$.

Subcase 5.2 If $r = 2s-1$, $2 \le s \le [\frac{m}{2}]$. Then, we obtain $n_u(e) = \sum_{j=1}^{[\frac{m}{2}]-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2[\frac{m}{2}]-1}-1)(2k-1)+2k$, and $n_v(e) = \Big(n(4k-2)+2-\Big(\sum_{j=1}^{[\frac{m}{2}]-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2[\frac{m}{2}]-1}-1)(2k-1)+2k\Big)$.

Subcase 5.3 If $r = m$, and $0 \le t \le l_m-2$. Then, we get $n_u(e) = \sum_{j=1}^{m-1}\Big((l_j+t)(4k-2)\Big)+(m-2)(-4k+2)+2k$, and $n_v(e) = \Big(2n(2k-1)+2-\Big(\sum_{j=1}^{m-1}\Big(4kl_j-2l_j+4kt-2t\Big)-4km+2m+10k-4\Big)\Big)$.

Case 6. If the j-th 4k-cycle edge e of the r-th segment is II-type (that is, e is the edge that passed the dotted line in figure 4).
If r is even, you can see that there are l_r+1 such edges in the r-th segment. If r is odd then there are 2 such edges in j-th 4k-cycle of r-th segment, $2 \le j \le l_r-1$.

Subcase 6.1 If $r = 1$, $0 \le i \le l_1-2$ then $n_u(e) = \Big(2k+i(4k-2)\Big)$, and $n_v(e) = \Big(n(4k-2)+2-\Big(2k+i(4k-2)\Big)\Big)$.

Subcase 6.2 If $r = 2s-1$, where $2 \le s \le [\frac{m}{2}]$ and $0 \le t \le l_r-3$. Then we obtain $n_u(e) = \Big(\sum_{j=1}^{r-1} 4kl_j-2l_j+4kt-2t-8ks+4s+14k-6\Big)$, and $n_v(e) = \Big(2n(2k-1)+2-\Big(\sum_{j=1}^{r-1} 2l_j(2k-1)+2t(2k-1)+2(2s-3)(-2k+1)+2k\Big)\Big)$.

Subcase 6.3 If $r = m$, where $1 \le s \le [\frac{m}{2}]$. Then we get $n_u(e) = \sum_{j=1}^{[\frac{m}{2}]-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{m-1}-1)(4k-2)+\frac{1}{2}\Big((l_m-1)(4k-2)\Big)+2k$, and $n_v(e) = \Big(n(4k-2)+2-\Big(\sum_{j=1}^{[\frac{m}{2}]-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{m-1}-1)(4k-2)+\frac{1}{2}\Big((l_m-1)(4k-2)\Big)+2k\Big)\Big)$.

Case 7. If the j-th 4k-cycle edge e of the r-th segment is of III-type (that is, e is the edge that passes through the dotted line in figure 5).
For each r, we see that there are $2k-2$ such edges in the j-th 4k-cycle of the r-th segment, and $1 \le j \le l_r$.

Subcase 7.1 If $r = 1$, $1 \le i \le l_1$ then $n_u(e) = \Big(2k+(i-1)(4k-2)\Big)$, and $n_v(e) = \Big(n(4k-2)+2-\Big(2k+2(i-1)(2k-1)\Big)\Big)$.

Subcase 7.2 If $r = 2s-1$, where $2 \le s \le [\frac{m}{2}]$ and $0 \le t \le l_{2s-1}-2$ then $n_u(e) = \sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)$, and $n_v(e) = \Big(n(4k-2)+2-\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Big)$.

Subcase 7.3 If $r = m$, where $1 \le s \le [\frac{m}{2}]$ and $0 \le t \le l_m-2$ then we get $n_u(e) = \sum_{j=1}^{m-1}\Big(4kl_j-2lj+4kt-2t-4km+2m+10k-4\Big)$, and $n_v(e) = \Big(2n(2k-1)+2-\sum_{j=1}^{m-1}\Big(4kl_j-2lj+4kt-2t-4km+2m+10k-4\Big)\Big)$.

Case 8. If the j-th 4k-cycle edge e of the r-th segment is IV-type (that is, e is the edge that is passes through the dotted line in the figure 6). It is obviously the beginning and the end of each segment we have $n_u(e) = 2k$ and $n_v(e) = n(4k-2)+2-2k$ that the number of such these 4k-cycles in $B_{n,l|k}$ are $m+1$. Now, we calculate the rest below. For each r we see that there are $2k-2$ such edges in the j-th 4k-cycle of r-th segment, where $1 \le j \le l_r$.

Subcase 8.1 If $r = 1$, where $1 \le i \le l_1-2$ then we obtain $n_u(e) = \Big(2k+i(4k-2)\Big)$, and $n_v(e) = \Big(n(4k-2)+2-\Big(2k+i(4k-2)\Big)\Big)$.

Subcase 8.2 If $r = s$, where $2 \le s \le m-1$ and $0 \le t \le l_s-3$ then we get $n_u(e) = \Big(\sum_{j=1}^{s-1} 4kl_j-2lj+4kt-2t+2(-2ks+s+5k-2)\Big)$, and $n_v(e) = \Big(2n(2k-1)+2-\Big(\sum_{j=1}^{s-1} 4kl_j-2lj+4kt-2t+2(-2ks+s+5k-2)\Big)\Big)$.

Subcase 8.3 If $r = m$, where $0 \le t \le l_m - 3$, then we have $n_u(e) = \Bigg(\sum_{j=1}^{m-1} 4kl_j - 2lj + 4kt - 2t + 2(-2km + m + 5k - 2) \Bigg)$, and $n_v(e) = \Bigg(2n(2k-1) + 2 - \Bigg(\sum_{j=1}^{m-1} 4kl_j - 2lj + 4kt - 2t + 2(-2km + m + 5k - 2) \Bigg) \Bigg)$.
Therefore, by combining the above case with the definition of the Szeged index, we have:

$$Sz(B_{n,k|l}) = \Bigg(l_1(2k-1) + 1 \Bigg) \Bigg((n - \tfrac{1}{2}l_1)(4k-2) + 1 \Bigg) \Bigg(l_1 + 1 \Bigg) +$$
$$\sum_{i=2}^{[\frac{m}{2}]} \Bigg(\Bigg(\sum_{j=1}^{i-1} \Bigg((l_{2j-1} + l_{2j} - 2)(4k-2) \Bigg) + (l_{2i-1} - 1)(2k-1) + 2k \Bigg) \Bigg(n(4k-2) + 2 - \Bigg(\sum_{j=1}^{i-1} \Bigg((l_{2j-1} + l_{2j} - 2)(4k-2) \Bigg) + (l_{2i-1} - 1)(2k-1) + 2k \Bigg) \Bigg(l_{2i-1} + 1 \Bigg) \Bigg) +$$
$$2 \sum_{i=1}^{[\frac{m}{2}]-1} \Bigg(\sum_{t=0}^{l_{2i}-3} \Bigg(\sum_{j=1}^{2i-1} 2l_j(2k-1) + 4kt - 2t - 8ki + 4i + 10k - 4 \Bigg) \Bigg(2n(2k-1) + 2 - \Bigg(\sum_{j=1}^{2i-1} 2l_j(2k-1) + 4kt - 2t - 8ki + 4i + 10k - 4 \Bigg) \Bigg) \Bigg) +$$
$$2 \sum_{t=0}^{l_m-2} \Bigg(\sum_{j=1}^{m-1} \Bigg(4kl_j - 2lj + 4kt - 2t + 2(-2km + m + 5k - 2) \Bigg) \Bigg) \Bigg(2n(2k-1) + 2 - \Bigg(\sum_{j=1}^{m-1} \Bigg(4kl_j - 2lj + 4kt - 2t + 2(-2km + m + 5k - 2) \Bigg) \Bigg) \Bigg) +$$
$$\sum_{i=0}^{l_1-2} 4 \Bigg(k + i(2k-1) \Bigg) \Bigg(4kn - 2n + 2 - 2(k + 2ki - i) \Bigg) +$$
$$\sum_{i=1}^{[\frac{m}{2}]} \Bigg(\Bigg(\sum_{j=1}^{i-1} \Bigg((l_{2j-1} + l_{2j})(4k-2) - 8k + 4 \Bigg) + (l_{2i-1} - 1)(4k-2) + \tfrac{1}{2} \Bigg((l_{2i} - 1)(4k-2) \Bigg) + 2k \Bigg) \Bigg(n(4k-2) + 2 - \Bigg(\sum_{j=1}^{i-1} \Bigg((l_{2j-1} + l_{2j})(4k-2) - 8k + 4 \Bigg) + (l_{2i-1} - 1)(4k-2) + \tfrac{1}{2} \Bigg((l_{2i} - 1)(4k-2) \Bigg) + 2k \Bigg) \Bigg) \Bigg) \Bigg(l_{2i} + 1 \Bigg) +$$
$$2 \sum_{i=2}^{[\frac{m}{2}]} \Bigg(\sum_{t=0}^{l_{2i-1}-3} \Bigg(\sum_{j=1}^{2i-2} l_j(4k-2) + 4kt - 2t - 8ki + 4i + 14k - 6 \Bigg) \Bigg(2n(2k-1) + 2 - \Bigg(\sum_{j=1}^{2i-2} l_j(4k-2) + 4kt - 2t - 8ki + 4i + 14k - 6 \Bigg) \Bigg) \Bigg) +$$
$$\sum_{i=1}^{l_1} 2(k-1) \Bigg(2k + 2(i-1)(2k-1) \Bigg) \Bigg(2n(2k-1) + 2 - 2(k + (i-1)(2k-1)) \Bigg) +$$
$$\sum_{i=1}^{[\frac{m}{2}]} \sum_{t=0}^{l_{2i}-2} \Bigg(\sum_{j=1}^{2i-1} \Bigg((l_j + t)(4k-2) + 2k + (2i-2)(-4k+2) \Bigg) \Bigg) \Bigg(2n(2k-1) + 2 - \Bigg(\sum_{j=1}^{2i-1} \Bigg((l_j + t)(4k-2) + 2k + 4(i-1)(-2k+1) \Bigg) \Bigg) \Bigg) \Bigg(2k - 2 \Bigg) +$$
$$\sum_{i=2}^{[\frac{m}{2}]} \sum_{t=0}^{l_{2i-1}-2} \Bigg(\sum_{j=1}^{2i-2} \Bigg((l_j + t)(4k-2) + 2k + (2i-2)(-4k+2) \Bigg) \Bigg) \Bigg(n(4k-2) + 2 - \Bigg(\sum_{j=1}^{2i-2} \Bigg((l_j + t)(4k-2) + 2k + (2i-2)(-4k+2) \Bigg) \Bigg) \Bigg) \Bigg(2k - 2 \Bigg) +$$
$$2(m+1) \Bigg(2k \Bigg(4kn - 2n + 1 - k \Bigg) \Bigg) (2k-2) + \sum_{i=1}^{l_1-2} (2k-2) \Bigg(2k + i(4k-2) \Bigg) \Bigg(2n(2k-1) + 2 - 2 \Bigg(k + i(2k-1) \Bigg) \Bigg) +$$
$$\sum_{i=2}^{m} \Bigg(\sum_{t=0}^{l_i-3} \Bigg(\sum_{j=1}^{i-1} l_j(4k-2) + 4kt - 2t - 4ki + 2i + 10k - 4 \Bigg) \Bigg(n(4k-2) + 2 - \Bigg(\sum_{j=1}^{i-1} l_j(4k-2) + t(4k-2) + 2(i-2)(-2k+1) + 2k \Bigg) \Bigg) (2k-2) \Bigg) \quad \blacksquare$$

Corollary 1 Let $B_{n,k|l}$ be n 4k-cycle k-polyomino chain consisting of $m \ge 1$ segments $S_1, S_2, ..., S_m$ with lengths $l_1, \ldots, l_m$.
(1) If m is odd, then

$$PI_v(B_{n,k|l}) = \Bigg(\Bigg(l_1(2k-1) + 1 \Bigg) + \Bigg((n - \tfrac{1}{2}l_1)(4k-2) + 1 \Bigg) \Bigg) \Bigg(l_1 + 1 \Bigg) +$$

$\sum_{i=2}^{[\frac{m}{2}]+1}\Bigg(\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Bigg)\Big(l_{2i-1}+1\Big)\Bigg)+$

$2\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i}-3}\Bigg(\Bigg(\sum_{j=1}^{2i-1}4kl_j-2l_j+4kt-2t-8ki+4i+10k-4\Bigg)+\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{2i-1}4kl_j-2l_j+4kt-2t-8ki+4i+10k-4\Bigg)\Bigg)\Bigg)\Bigg)+$

$\sum_{i=0}^{l_1-2}2\Bigg(\Big(2k+i(4k-2)\Big)+\Big(2n(2k-1)+2-2(k+i(2k-1))\Big)\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)\Bigg)\Bigg)\Big(l_{2i}+1\Big)+$

$2\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i-1}-3}\Bigg(\Bigg(\sum_{j=1}^{2i-2}2l_j(2k-1)+4kt-2t-8ki+4i+14k-6\Bigg)+\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{2i-2}l_j(4k-2)+4kt-2t-8ki+4i+14k-6\Bigg)\Bigg)\Bigg)\Bigg)+$

$2\Bigg(\sum_{t=0}^{l_m-2}\Bigg(\Bigg(\sum_{j=1}^{m-1}l_j(4k-2)+t(4k-2)+2(-2km+m+5k-2)\Bigg)+\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{m-1}l_j(4k-2)+4kt-2t+2(-2km+m+5k-2)\Bigg)\Bigg)\Bigg)\Bigg)+$

$\sum_{i=1}^{l_1}4(k-1)\Bigg(\Big(2ki-i-k+1\Big)+\Big(4kn-2n+2-(2ki-i-k+1)\Big)\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i}-2}\Bigg(\sum_{j=1}^{2i-1}\Bigg(\Big((l_j+t)(4k-2)-8ki+4i+10k-4\Big)\Bigg)+\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)-8ki+4i+10k-4\Big)\Bigg)\Bigg)\Bigg)\Big(2(k-1)\Big)+$

$\sum_{i=2}^{[\frac{m}{2}]+1}\sum_{t=0}^{l_{2i-1}-2}\Bigg(\Bigg(\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-2}\Big(2(l_j+t)(2k-1)+2k+4(i-1)(-2k+1)\Big)\Bigg)\Bigg)\Bigg)\Big(2k-2\Big)$

$+2(m+1)\Bigg(\Bigg(k+\Big(2kn-n+1-k\Big)\Bigg)\Bigg)(2k-2)+\sum_{i=1}^{l_1-2}(2k-2)\Bigg(\Big(2k+i(4k-2)\Big)+\Bigg(4kn-2n+2-\Big(2k+4ki-2i\Big)\Bigg)\Bigg)+$

$\sum_{i=2}^{m}\Bigg(\sum_{t=0}^{l_i-3}\Bigg(\Bigg(\sum_{j=1}^{i-1}l_j(4k-2)+4kt-2t-4ki+2i+10k-4\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}l_j(4k-2)+4kt-2t-4ki+2i+10k-4\Bigg)\Bigg)\Bigg)(2k-2)\Bigg).$

(2) If m is even then

$PI_v(B_{n,k|l})=\Bigg(\Bigg(l_1(2k-1)+1\Bigg)+\Bigg((n-\frac{1}{2}l_1)(4k-2)+1\Bigg)\Bigg)\Big(l_1+1\Big)+$

$\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)+\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j}-2)(4k-2)\Big)+(l_{2i-1}-1)(2k-1)+2k\Bigg)\Bigg)\Big(l_{2i-1}+1\Big)\Bigg)+$

$2\sum_{i=1}^{[\frac{m}{2}]-1}\Bigg(\sum_{t=0}^{l_{2i}-3}\Bigg(\Bigg(\sum_{j=1}^{2i-1}2l_j(2k-1)+4kt-2t-8ki+4i+10k-4\Bigg)+\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{2i-1}2l_j(2k-1)+4kt-2t-8ki+4i+10k-4\Bigg)\Bigg)\Bigg)\Bigg)+$

$2\sum_{t=0}^{l_m-2}\Bigg(\Bigg(\sum_{j=1}^{m-1}\Big(2l_j(2k-1)+4kt-2t-4km+2m+10k-4\Big)\Bigg)+\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{m-1}\Big(l_j(4k-2)+4kt-2t-4km+2m+10k-4\Big)\Bigg)\Bigg)\Bigg)+$

$\sum_{i=0}^{l_1-2}2\Bigg(2\Big(k+2ki-i\Big)+\Big(2n(2k-1)+2-2(k+2ki-i)\Big)\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\Bigg(\Bigg(\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{i-1}\Big((l_{2j-1}+l_{2j})(4k-2)-8k+4\Big)+(l_{2i-1}-1)(4k-2)+\frac{1}{2}\Big((l_{2i}-1)(4k-2)\Big)+2k\Bigg)\Bigg)\Bigg)\Big(l_{2i}+1\Big)+$

$2\sum_{i=2}^{[\frac{m}{2}]}\Bigg(\sum_{t=0}^{l_{2i-1}-3}\Bigg(\Bigg(\sum_{j=1}^{2i-2}2l_j(2k-1)+4kt-2t-8ki+4i+14k-6\Bigg)+\Bigg(2n(2k-1)+2-\Bigg(\sum_{j=1}^{2i-2}2l_j(2k-1)+4kt-2t-8ki+4i+14k-6\Bigg)\Bigg)\Bigg)\Bigg)+$

$\sum_{i=1}^{l_1}2(k-1)\Bigg(\Big(2k+2(i-1)(2k-1)\Big)+\Big(2n(2k-1)+2-2(k+(i-1)(2k-1))\Big)\Bigg)+$

$\sum_{i=1}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i}-2}\Bigg(\Bigg(\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-1}\Big((l_j+t)(4k-2)+2k+4(i-1)(-2k+1)\Big)\Bigg)\Bigg)\Bigg)\Big(2k-2\Big)+$

$\sum_{i=2}^{[\frac{m}{2}]}\sum_{t=0}^{l_{2i-1}-2}\Bigg(\Bigg(\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Bigg)+\Bigg(n(4k-2)+2-\Bigg(\sum_{j=1}^{2i-2}\Big((l_j+t)(4k-2)+2k+(2i-2)(-4k+2)\Big)\Bigg)\Bigg)\Bigg)\Big(2k-2\Big)+$

$2(m+1)\Bigg(k+\Big(2kn-n+1-k\Big)\Bigg)(2k-2)+\sum_{i=1}^{l_1-2}(2k-2)\Bigg(\Big(2k+4ki-2i\Big)+\Bigg(4kn-2n+2-2\Big(k+2ki-i\Big)\Bigg)\Bigg)+$

$\sum_{i=2}^{m}\Bigg(\sum_{t=0}^{l_i-3}\Bigg(\Bigg(\sum_{j=1}^{i-1}2l_j(2k-1)+4kt-2t-4ki+2i+10k-4\Bigg)+\Bigg(4kn-2n+2-\Bigg(\sum_{j=1}^{i-1}l_j(4k-2)+4kt-2t-4ki+2i+10k-4\Bigg)\Bigg)\Bigg)(2k-2)\Bigg).$

Proof: Similar to Theorem 1. ■

Example 1

Consider a polyomino with $k=2$ and length $l_1=2$, $l_2=3$ and $l_3=2$, see figure 7. Therefore, $m=3$ and $n=5$. Please indicate it as $B_{5,2}$.

Clearly, by four types of cuts we get $Sz(B_{5,2})=(7*8*25+2*16*16+3*7*25)+(2*4*28+4*16*16+2*4*28)+(2*4*28+2*10*22+2*16*16+2*10*22+2*4*28)+(2*4*28+2*4*28+2*16*16+2*4*28+2*4*28)=6282$

By Theorem 1, we will have:

Corollary 2 (Linear chain). For $m=1$ and $l_1=n$, graphs $B_{n,k|l}$ are linear chains $L_{n,k}$ with n 4k-cycles. Of Theorem 1 is the Szeged index of the linear chain $L_{n,k}$ given by the following equation

$$Sz(L_{n,k})=(n+1)\Big(n(2k-1)+1\Big)^2+(4k-2)\Bigg(\sum_{i=1}^{n}2\Big(2ki-(i+k-1)\Big)\Bigg(4kn-2n+2-2\Big(2ki-(i+k-1)\Big)\Bigg)\Bigg).$$

Corollary 3 (Linear chain). For $m=1$ and $l_1=n$, graph $B_{n,k|l}$ are linear chains $L_{n,k}$ with n 4k-cycles. Of Theorem 1 is the vertex-PI index of the linear chain $L_{n,k}$ given by: by

$$PI_v(L_{n,k})=2(n+1)\Big(n(2k-1)+1\Big)+(4k-2)\Bigg(\sum_{i=1}^{n}\Bigg(2\Big(2ki-(i+k-1)\Big)+\Bigg(4kn-2n+2-2\Big(2ki-(i+k-1)\Big)\Bigg)\Bigg)\Bigg).$$

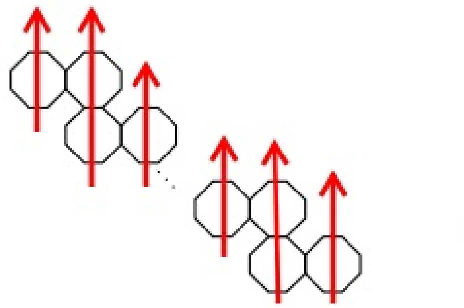

Fig. 4. II-type cut of polyomino chain

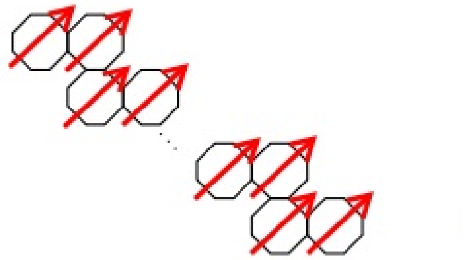

Fig. 5. III-type cut of polyomino chain

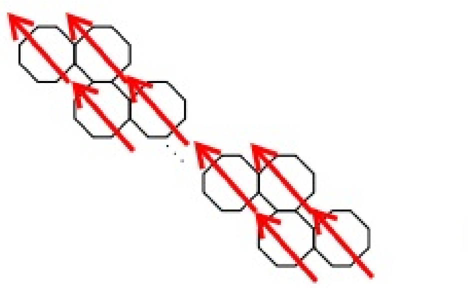

Fig. 6. IV-type cut of polyomino chain

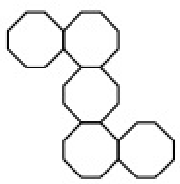

Fig. 7. The graph $B_{5,2}$

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