Bi-matrix Games with 2-tuple Linguistic Information

Dong Qiu, Li Xiang

Abstract—**A new type of bi-matrix game called the bi-matrix game with 2-tuple linguistic information is proposed and the linguistic linear programming model is used to obtain the mixed set. The problem of linguistic information is transformed into a mathematical problem that can be solved quickly, which reduces the uncertainty and complexity caused by linguistic information. This paper takes the media industry as an example to illustrate the superiority and effectiveness of the bi-matrix game with 2-tuple linguistic information.**

***Index Terms*—2-tuple linguistic model; linguistic quadratic linear programming problem; linguistic bi-matrix game.**

I. Introduction

GAME theory [21] developed in the 20th century. Its object of study is the tough competitive game derived from competition and game. That is, there is no cooperation and standard behavior between the two players, also called the two-person zero-sum game. The interests of both sides of the game are strictly opposite, and the loss of one party is the gain of the other party. But this game is relatively limited and does not apply to most economies and politics. This leads to a two person non-zero-sum game, also known as a bi-matrix game. Bi-matrix game is a classic two players non-zero-sum game, which has been successfully applied in the fields of competition, voting, and artificial intelligence. In the traditional bi-matrix game, the payoffs value of players is accurately known. However, in real life, we can only know vague values or oral descriptions. The birth of fuzzy sets solves many uncertain problems in life. In real life, it leads to many ambiguous concepts, such as fuzzy stress[22] in physics, the role of fuzzy soft set in flood prevention[30], application of fuzzy investment[9] and decision making[35]. It is based on fuzzy mathematics, and fuzzy set theory to study imprecision. In the objective world, there are many ambiguities. In order to solve this problem, Zadeh [36] put forward the fuzzy set theory for the first time and achieved great success. On this basis, Verma [29] proposed a new method to find the complete solution to fuzzy payment matrix games. Qiu et al. [31] established a multi-objective bi-matrix game model based on fuzzy objectives. They proved that the equilibrium solution of this game model problem could be transformed into the ideal solution of a multi-objective nonlinear programming problem. Especially in recent years, there have been many types for fuzzy numbers and their applications are more and more widely. Bhaumik et al. [6] looked at the problem of human trafficking, one of the latest problems in today's society, by using the prisoner's dilemma game of hesitation interval-valued intuitionistic fuzzy linguistic term set. Brikama et al. [4] proposed a way to solve multi-criteria matrix games with intuitionistic fuzzy objectives. Deli [8] studied the multi-attribute decision-making problem with generalized trapezoidal hesitant fuzzy numbers and developed two aggregation techniques. Fahmi et al. [10] considered the Bonferroni average operator of generalized trapezoidal hesitant fuzzy numbers and its application in decision problems.

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Quadratic programming is a special kind of mathematical programming problem in nonlinear programming. It has a program widely used in portfolio selection to solve constrained least squares problems and nonlinear optimization problems of sequential quadratic programming. At the same time, it provides a new idea for solving bi-matrix games. Kocken et al. [14] proposed a compensatory approach to solve multi-objective linear transportation problems, which is called the fuzzy cost coefficients method. Xu et al. [32] constructed two models for auxiliary bilinear programming models, and obtained the optimal strategy of the bi-matrix game through explicit calculation. In recent years, people have made various attempts in fuzzy bi-matrix game theory. Nishhizaki and Sakawa [20] first adopted linear functions as the date-marking membership function. Soon after, Li et al. [5] put forward a new method based on single-valued triangular neutron number(α, β, γ)- the ranking method of cut is applied to bi-matrix game theory. Jana and Roy [13] chose the elements of dual hesitation fuzzy as the payoff of matrix game and treated them as dual hesitation fuzzy matrix game. Nayak and Pal [18, 19] studied the bi-matrix game and multi-objective bi-matrix game represented by intuitionistic fuzzy set (IFS). Li [16] implemented the bilinear programming model to solve the bi-matrix game with intuitionistic fuzzy set benefits. Based on the dual programming theory, An and Li [1] developed two linear programming models to solve any constrained bi-matrix games

In addition, Arfi [2] first proposed a new game method based on linguistic fuzzy logic, including fuzzy concepts, linguistic advantages, and linguistic Nash equilibrium. Based on Arfi, this paper presents a 2-tuple linguistic framework for non-cooperative two persons non-sum matrix game problems. Single linguistic terms [2], 2-tuple linguistic terms [27] and interval 2-tuple linguistic terms [26] are widely used linguistic models. The first to specifically propose and analyze 2-tuple linguistic variables is Herrera and Martinez [11]. Wang [28] thinks the advantage of hesitant Fermatean 2-tuple linguistic information is that it can deal with higher levels of uncertainty. In recent years, the accuracy and interpretability of the 2-tuple linguistic model have been widely used in decision making.

In order to express complex fuzzy information more conveniently in a qualitative environment, Fan [10] combined GLDS method with a 2-tuple linguistic neutral number to develop a 2TLNN-GLDS method for multi-attribute group decision-making. Li and Zheng [16] proposed a linguistic decision-making method based on voting model to deal with the multi linguistic evaluation provided by decision-makers. Based on the voting of assessment, a multi linguistic decision matrix is designed. Xu and Liao [32] used probabilistic linguistic information as input for two-person zero-sum matrix game and solveed the problem of fuzzy information by using the triangular membership function. Shahzad and Nawaz [24] gave Einstein a weighted average operator of intuitionistic 2-tuple linguistic. Malhotra and Gupta [17] based on the concept of minimum distance measure and proposed a new 2-tuple method for unbalanced linguistic term sets.

Firstly, the paper introduces the 2-tuple linguistic term set and explains its practicability. Secondly, the linguistic quadratic linear programming model is established and applied to the bi-matrix game with 2-tuple linguistic information. In general, the method proposed in this paper is desirable because it can be used to solve linguistic processing problems and linguistic game problems.

The rest of the paper is arranged as follows. In section II, the 2-tuple linguistic model and basic terminology are introduced. In section III, the basics and theorems of the matrix game between extraordinary and linguistic are proposed. In section IV, the quadratic linear programming formula for solving the kind of game is explained. In section V, an example is used to prove the applicability and effectiveness of the model boom. In section VI, conclusions are provided.

II. Preliminaries

In this section, we want to review some notions about 2-tuple linguistic variables and bi-matrix games, which will be used later.

A. 2-tuple Linguistic Model

The 2-tuple linguistic glossary is an important research content in the field of decision-making. Its related theories have good research value in the fields of economy, society, politics and other fields. Because objective things always have a certain degree of uncertainty, decision-makers usually tend to verbally express preference information. The 2-tuple linguistic term sets have gradually attracted academic attention because of their ability to process linguistic information.

The generalized linguistic model is composed of l_i, where l_i represents the cardinality $g+1$ from the predefined linguistic term set $LT = \{l_i : i = 0, 1, ..., g\}$, where the uncertainty granularity g in the index information is usually assumed as even and positive integer. The index of the closest linguistic term in LT is indicated by the value in $\{i = 0, 1, \dots, g\}$.

The totally ordered set LT has the following characteristics:

1. If $i > j$, then $l_i > l_j$;
2. If $l_i > l_j$, then max $(l_i, l_j) = l_i$;
3. If $l_i < l_j$, then min $(l_i, l_j) = l_i$.

Definition 1. *[12]* Let $\beta \in [0, g]$ be the aggregation result of a set of label indexes evaluated in LT. Hypothesis $i = [\beta]$, $\alpha = \beta - i$, where $\alpha \in [-0.5, 0.5)$ is called the symbolic translation.

The 2-tuple linguistic variable is translated by the linguistic term l_i and symbolic translation α, expressed as (l_i, α_i).

The 2-tuple linguistic term set (l_i, α_i) was first proposed by Herrera and Martinez [12], so that $l_i \in LT$ and $\alpha_i \in [-0.5, 0.5)$. And they further defined the lexicographic ordering in the 2-tuple linguistic variables.

Definition 2. *[9]* Let (l_i, α_i) and (l_j, α_j) be two 2-tuple linguistic variables, and each value represents a certain amount of linguistic information. Their order relationship is as follows:

(1) $(l_i, \alpha_i) < (l_j, \alpha_j)$ if $i < j$;

(2) $i = j$, i.e., l_i and l_j are also linguistics if $i = j$, i.e.,

(a) $(l_i, \alpha_i) = (l_j, \alpha_j)$ if $\alpha_i = \alpha_j$, that is, (l_i, α_i) and (l_j, α_j) express the same information;

(b) $(l_i, \alpha_i) < (l_j, \alpha_j)$ if $\alpha_i < \alpha_j$;

(c) $(l_i, \alpha_i) > (l_j, \alpha_j)$ if $\alpha_i > \alpha_j$.

Definition 3. *[12]* Let $\beta \in [0, g]$ represents the value of the aggregated symbolic result. Using the function $\Delta : [0, g] \rightarrow LT \times [-0.5, 0.5]$ to get an equivalent 2-tuple linguistic variable, defined by

$$\Delta(\beta) = (l_i, \alpha_i), \ where\ \alpha = \beta - i\ and\ i = [\beta]. \quad (1)$$

Moreover, the function Δ is a bijection, and its inverse is

$$\Delta^{-1} : LT \times [-0.5, 0.5) \rightarrow [0, g]\ as\ \Delta^{-1}(l_i, \alpha_i) = i + \alpha_i = \beta. \quad (2)$$

The sorting problem in Definition 2 can be simplified as follows:

$$if\ \Delta^{-1}(l_i, \alpha_i) \le \Delta^{-1}(l_j, \alpha_j),\ then\ (l_i, \alpha_i) \le (l_j, \alpha_j).$$

At present, there have been a large number of literature studies on the various operators of 2-tuple linguistic variables sets, so this section only introduces the weighted average operator which it will be used later .

Definition 4. *[25]* A set of 2-tuple linguistic variables is $\{(l_{r_i}, \alpha_{r_i}), r_i \in \{0, 1, \dots, g\}, i = 1, \dots, q\}$ and the weight vector is $\omega = (\omega_1, ..., \omega_q)^T$, satisfying $0 \le \omega_i \le 1$, $i = 1, \dots, q$, $\sum_{i=1}^{q} \omega_i = 1$. The weighted average operator can be defined as

$$\begin{aligned} & LWA\left[(l_{r_i}, \alpha_{r_i}) : i = 1, \cdots, q\right] \\ & = \omega_1 \left(l_{r_1}, \alpha_{r_1}\right) \oplus \omega_2 \left(l_{r_2}, \alpha_{r_2}\right) \oplus \cdots \oplus \omega_q \left(l_{r_q}, \alpha_{r_q}\right) \\ & = \Delta\left(\sum_{i=1}^{q} \omega_i \Delta^{-1}\left(l_{r_i}, \alpha_{r_i}\right)\right). \end{aligned} \quad (3)$$

Consequently,

$$\Delta^{-1}\left(\bigoplus_{i=1}^{q} (l_{r_i}, \alpha_{r_i})\omega_i\right) = \sum_{i=1}^{q} \omega_i \Delta^{-1}(l_{r_i}, \alpha_{r_i}). \quad (4)$$

In order to express the above concepts more accurately, we can consider the predefined linguistic term set.

Example 1. Defining LT={l_0: weak importance, l_1: weak-strong, l_2: strong importance, l_3: strong-very strong importance, l_4: very strong.}

Let $Z = \{(l_0, 0.3), (l_1, -0.34), (l_2, 0.11), (l_1, 0.125), (l_3, -0.43)\}$ be a 2-tuple linguistic term set. If $\beta = 0.66$, then $i = [\beta] = 1$ and $\alpha = \beta - i = -0.34$. Thus, the corresponding 2-tuple linguistic variable is $(l_1, -0.34)$. In contrast, a 2-tuple linguistic variable $(l_3, -0.43)$ can be converted to its numeric value using the (2) operator, such as $\Delta^{-1}(l_3, -0.43) = 2.57$.

Using the lexicographic ordering, the elements of set Z are arranged in the following order:

$(l_0, 0.3) < (l_1, -0.34) < (l_0, 0.125) < (l_2, 0.11) < (l_3, 0.43)$.

If the weight vector $\varphi = (0.07, 0.19, 0.24, 0.27, 0.13)$, then its weighted average operator is

$$\begin{aligned}
&(p_0, 0.3) * 0.07 \oplus (p_1, -0.34) * 0.19 \oplus (p_2, 0.11) * 0.24 \oplus \\
&\qquad (p_1, 0.125) * 0.27 \oplus (p_3, -0.43) * 0.13 \\
&= \Delta[0.3 * 0.07 + 0.66 * 0.19 + 2.11 * 0.24 + 1.125 * 0.27 + 2.57 * 0.13] \\
&= \Delta[1.29065] \\
&= (l_1, 0.29065).
\end{aligned}$$

B. Bi-matrix Game

In most cases, we study two-person zero-sum matrix game, where one party gains the other party's losses. But in real life, the interests of two players are not exactly the opposite, which is called the bi-matrix game. Famous examples of bi-matrix games include Prisoner's Dilemma, Gender War and so on.

Definition 5. Let $BG = (A, B, S^m, S^n)$ be a bi-matrix game, where

(i) S^m is a finite set of m mixed strategies of player 1;

(ii) S^n is a finite set of n mixed strategies of player 2;

(iii) Using real $m \times n$ matrices A and B to represent the pay-offs of player 1 and player 2, respectively.

Definition 6. The ordered pair of vectors $(x, y) \in S^m \times S^n$ is a mixed strategy, where

$$S^m = \{(x_1, \cdots, x_m) : \sum_{i=1}^{m} x_i = 1,\ x_i > 0\},$$

$$S^n = \{(y_1, \cdots, y_n) : \sum_{i=1}^{m} y_i = 1,\ y_i > 0\}.$$

Definition 7. *[18]* (*Equilibrium Solution*) Let $x^* \in S^m$, $y^* \in S^n$, if for any $x \in S^m$ and for any $y \in S^n$, we have

$$\begin{aligned} x^T A y^* \le x^{*T} A y^*, \\ x^{*T} A y \le x^{*T} A y^*. \end{aligned} \tag{5}$$

Then (x^*, y^*) is a equilibrium solution of the bi-matrix game BG.

The optimal strategies for the player 1 and 2 are also called x^* and y^* , respectively. The Nash equilibrium outcome of BG expresses as the pair of numbers $E = \langle x^{*T} A y^*, x^{*T} B y^* \rangle$ and the solution the bi-matrix game expresses as the triplet (x^*, y^*, E).

Definition 7 can be redefined, that is, $(x^*, y^*) \in S^m \times S^n$ is a Nash equilibrium solution of the BG if and only if x^* and y^* simultaneously solve the problems (6) and (7), where

$$\begin{gathered} max\ \ (x^*)^T A y \\ subject\ to, \\ e^T x = 1, \\ x \ge 0. \end{gathered} \tag{6}$$

and

$$\begin{gathered} max\ \ (x^*)^T B y \\ subject\ to, \\ e^T y = 1, \\ y \ge 0. \end{gathered} \tag{7}$$

Lemma 1. *[18]* (x^*, y^*) is an optimal solution of (8) if and only if there exist scalars α^* and β^* make $(x^*, y^*, \alpha^*, \beta^*)$ meet

$$\begin{gathered} x^{*T} A y^* = \alpha^* \\ x^{*T} B y^* = \beta^* \\ A y^* \le \alpha^* e \\ B^T x^* \le \beta^* e \\ e^T x^* = 1 \\ e^T y^* = 1 \\ x^* \ge 0 \\ y^* \ge 0. \end{gathered} \tag{8}$$

Therefore, by solving the appropriate quadratic programming problem discussed above, we can obtain the Nash equilibrium solution of the bi-matrix game BG.

III. 2-Tuple Linguistic Bi-matrix Game

In this section, we will introduce a new type of bi-matrix game which called 2-tuple linguistic information bi-matrix game and will prove that the bi-matrix game has a Nash equilibrium solution.

Definition 8. $\tilde{BG} = (S^m, S^n, LT, \tilde{A}, \tilde{B})$ is defined as a linguistic bi-matrix game, where S^m and S^n are the above strategy sets, $LT = \{l_1, l_2, ..., l_g\}$, the cardinality is $g+1$. There is the predefined linguistic term set of two players, the linguistic payoff matrixes of player 1 and player 2 are expressed as $\tilde{A}$ and $\tilde{B}$,

$$\tilde{A} = \begin{pmatrix} \tilde{a_{11}} & \tilde{a_{12}} & \cdots & \tilde{a_{1n}} \\ \tilde{a_{21}} & \tilde{a_2} & \cdots & \tilde{a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{a_{m1}} & \tilde{a_{m2}} & \cdots & \tilde{a_{mn}} \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} \tilde{b_{11}} & \tilde{b_{12}} & \cdots & \tilde{b_{1n}} \\ \tilde{b_{21}} & \tilde{b_2} & \cdots & \tilde{b_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{b_{m1}} & \tilde{b_{m2}} & \cdots & \tilde{b_{mn}} \end{pmatrix}.$$

where $\tilde{a}_{ij} = (\alpha_{ij}, l_{\alpha_{ij}})$, $\tilde{b}_{ij} = (\beta_{ij}, l_{\beta_{ij}})$.
According to Definition 3, their inverse are given by

$$\Delta^{-1}(\tilde{A}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$\Delta^{-1}(\tilde{B}) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{2} & \cdots & b_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}.$$

where $\Delta^{-1}(\tilde{a}_{ij}) = \alpha_{ij} + l_{\alpha_{ij}}$, $\Delta^{-1}(\tilde{b}_{ij}) = \beta_{ij} + l_{\beta_{ij}}$

A linguistic payoff refers to how instructive and benefits of playing a strategy for players, which can be expressed in LT at the linguistic scale.

Definition 9. The linguistic bi-matrix game $\tilde{BG}$ has an equilibrium solution $(x^*, y^*) \in S^m \times S^n$ if

$$\begin{aligned} x^T\Delta^{-1}(\tilde{A})y^* \leq x^{*T}\Delta^{-1}(\tilde{A})y^*, \forall x \in S^m, \\ x^{*T}\Delta^{-1}(\tilde{B})y \leq x^{*T}\Delta^{-1}(\tilde{B})y^*, \forall y \in S^n. \end{aligned} \tag{9}$$

Definition 10. *(Expected Payoff)* Let player 1 choose any mixed strategy $x \in S_1$, let player 2 choose any mixed strategy $y \in S_2$. Then the expected pay-offs for player 1 and player 2 are

$$E_A(x, y) = \overset{m}{\underset{i=1}{\oplus}} \left(x_i \left(\overset{n}{\underset{j=1}{\oplus}} \tilde{a}_{ij} y_j \right) \right), \tag{10}$$

$$E_B(x, y) = \overset{m}{\underset{i=1}{\oplus}} \left(x_i \left(\overset{n}{\underset{j=1}{\oplus}} \tilde{b}_{ij} y_j \right) \right). \tag{11}$$

Theorem 1. $(Nash\ Existence\ Theorem)$ At least one equilibrium solution exists in any linguistic bi-matrix game $\tilde{BG} = (S^m, S^n, LT, \tilde{A}, \tilde{B})$.

Proof. We can define

$$\begin{aligned} \tilde{c}_i(x, y) &= max\left(\tilde{A}_i y - x^T \tilde{A} y, 0\right), \\ \tilde{d}_j(x, y) &= max\left(x^T \tilde{B}_j - x^T \tilde{B} y, 0\right), \end{aligned}$$

for each $(x, y) \in S^m \times S^n$. According to Definition 3 and Definition 8, we have

$$\begin{aligned} &\tilde{c}_i = \Delta\left(\max\left(\Delta^{-1}\left(\tilde{A}_i - x^T\tilde{A}y\right), 0\right)\right) \\ &= \Delta\left(\max\left(\left(\overset{n}{\underset{j=1}{\oplus}} \tilde{a}_{ij}y_j - \overset{m}{\underset{i=1}{\oplus}} x_i \overset{n}{\underset{j=1}{\oplus}} \tilde{a}_{ij}y_j\right), 0\right)\right) \\ &= \Delta\left(\max\left(\left(\sum_{j=1}^{n} \Delta^{-1}(\tilde{a}_{ij})y_j - \sum_{i=1}^{m} x_i \sum_{j=1}^{n} \Delta^{-1}(\tilde{a}_{ij})y_j\right), 0\right)\right) \\ &= \Delta\left(\max\left(\left(\sum_{j=1}^{n} \left(a_{ij} + l_{\alpha_{ij}}\right)y_j - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \left(a_{ij} + l_{\alpha_{ij}}\right)y_j\right), 0\right)\right). \end{aligned}$$

Because function Δ is reversible, we have

$$\begin{aligned} &\Delta^{-1}(\tilde{c}_i(x, y)) \\ &= \max\left(\left(\sum_{j=1}^{n} \left(\alpha_{ij} + l_{\alpha_{ij}}\right) y_j - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \left(\alpha_{ij} + l_{\alpha_{ij}}\right) y_j\right), 0\right). \end{aligned}$$

Let $a_{ij} = \alpha_{ij} + l_{\alpha_{ij}}$, then $c_i(x, y) = \Delta^{-1}(\tilde{c}_i(x, y)) = max\left((A_i y - x^T Ay), 0\right)$.

Similarly, let $b_{ij} = \beta_{ij} + l_{\beta_{ij}}$, then $d_i(x, y) = \Delta^{-1}(\tilde{d}_i(x, y)) = max\left((x^T B_j - x^T By), 0\right)$.

According to the above definitions, $c_i(x, y) \geq 0 (i = 1, 2, \ldots, m)$, therefore $x_i^{'} \geq 0$. Similarly, $d_j(x, y) \geq 0 (j = 1, 2, \ldots, n)$ and therefore $y_j^{'} \geq 0$.

Now defining a function $P : S^m \times S^n \to S^m \times S^n$, then $P(x, y) = (x^{'}, y^{'})$, where

$$x_i^{'} = \frac{c_i(x, y) + x_i}{\sum\limits_{i=1}^{m} c_i(x, y) + 1}, \quad y_j^{'} = \frac{d_i(x, y) + y_j}{\sum\limits_{j=1}^{n} d_j(x, y) + 1}.$$

Further can be obtained

$$e^T x^{'} = \sum_{i=1}^{m} x_i^{'} = 1, \ e^T y^{'} = \sum_{j=1}^{n} y_j^{'} = 1.$$

Therefore we know $S^m \times S^n$ is a compact convex set.

Since P: $S^m \times S^n \to S^m \times S^n$ is a continuous surjective set and $S^m \times S^n$ is a compact convex set. Then, according to Brouwer's fixed point theorem, we can know that P contains at least one fixed point (x^*, y^*), namely

$$P(x^*, y^*) = (x^{'}, y^{'}) = (x^*, y^*).$$

Now we show the point (x^*, y^*) exactly is the equilibrium solution of the linguistic bi-matrix game $\tilde{BG}$.

Equilibrium solution of the linguistic bi-matrix game $\tilde{BG}$ is not (x^*, y^*). Then either there exist some $\bar{x} \in S^m$ so that

$$\bar{x}\,\Delta^{-1}(\tilde{A})y^* - x^{*T}\Delta^{-1}(\tilde{A})y^* > 0,$$

or there exist some $\bar{y} \in S^n$ such that

$$x^{*T}\Delta^{-1}(\tilde{B})\,\bar{x} - x^{*T}\Delta^{-1}(\tilde{B})y^* > 0.$$

And we also have

$$\begin{aligned} &\bar{x}\,Ay^* - x^{*T}Ay^* > 0, \\ &x^{*T}B\,\bar{x} - x^{*T}By^* > 0. \end{aligned}$$

Assuming that the first one is true, it means that there are such i, we have

$$A_{i.}y^* - x^{*T}Ay^* > 0,$$

there is $i = i_0$ satisfying $c_i > 0$. We have $\sum\limits_{i=1}^{m} c_i > 0$, because $c_i \geq 0$ for all i and $c_{i_0} > 0$.

Now we have

$$x^{*T}Ay^* = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij} y_j^* = \sum_{i=1}^{m} x_i^* \left(\sum_{j=1}^{n} a_{ij} y_j^*\right),$$

representing the weighted arithmetic mean of m scalars $\sum\limits_{j=1}^{n} a_{ij}y_j^*$ $(i = 1, \ldots, m)$, and therefore

$$x^{*T}Ay^* \geq \min\left(\sum_j a_{1j}y_j^*, \cdots, \sum_j a_{mj}y_j^*\right) = \sum_j a_{kj}y_j^*$$
$$for\ some\ 1 \leq k \leq m.$$

It represents that $x^{*T}Ay^* - A_{k.}y^* \geq 0$ and $x_k^* \geq 0$.

We can get $x_k^* > 0$, otherwise, $\sum\limits_j a_{kj}y_j^*$ cannot appear in the minimization. Thus it follows from $c_k(x^*, y^*) = 0$ that $x_k^* > x_k^{'} = \frac{x_k^*}{1 + \sum c_{i1}(x^*, y^*)}$ and $x^{'} \neq x^*$.

With similar reasoning, we can know that $y^{'} \neq y^*$. Thus $(x^{'}, y^{'}) \neq (x^*, y^*)$ in the second case, it contradicts that (x^*, y^*) is a fixed point.

Thus, a equilibrium solution of the linguistic bi-matrix game $\tilde{BG}$ is (x^*, y^*).

Lemma 2. (x^*, y^*) is a Nash equilibrium solution if and only if it is a solution of (12) and (13).

$$x^{*T}\Delta^{-1}(\tilde{A})y^* = \max_x \left\{x^T\Delta^{-1}(\tilde{A})y^* : e^T x = 1, x \geq 0\right\}, \tag{12}$$

$$x^{*T}\Delta^{-1}(\tilde{B})y^* = \max_y \left\{x^{*T}\Delta^{-1}(\tilde{B})y : e^T y = 1, y \geq 0\right\}. \tag{13}$$

Proof. Sufficiency: (x^*, y^*) is a solution of (12) and (13), then for any $x \in S^m$ and any $y \in S^n$, we have

$$x^T\Delta^{-1}(\tilde{A})y^* \leq x^{*T}\Delta^{-1}(\tilde{A})y^*,$$
$$x^{*T}\Delta^{-1}(\tilde{B})y \leq x^{*T}\Delta^{-1}(\tilde{B})y^*.$$

Therefore, according to Definition 9, $(x^*, y^{*)}$ is a Nash equilibrium solution.

Necessary: (x^*, y^*) is a solution of the Nash equilibrium solution, we have

$$x^T\Delta^{-1}(\tilde{A})y^* \leq x^{*T}\Delta^{-1}(\tilde{A})y^*, \forall x \in S^m,$$
$$x^{*T}\Delta^{-1}(\tilde{B})y \leq x^{*T}\Delta^{-1}(\tilde{B})y^*, \forall y \in S^n.$$

Namely

$$x^{*T}\Delta^{-1}(\tilde{A})y^* = \max_x \left\{x^T\Delta^{-1}(\tilde{A})y^* : e^T x = 1, x \geq 0\right\},$$
$$x^{*T}\Delta^{-1}(\tilde{B})y^* = \max_y \left\{x^{*T}\Delta^{-1}(\tilde{B})y : e^T y = 1, y \geq 0\right\}.$$

Therefore, (x^*, y^*) is a solution of (12) and (13).

Theorem 2. $(Equivalence\ theorem)$. Let $\tilde{BG} = \left(S^m, S^n, LT, \tilde{A}, \tilde{B}\right)$ be a bi-matrix game. (x^*, y^*) is an equilibrium solution of $\tilde{BG}$ if and only if it is a solution of the following quadratic programming problem(QPP),

$$\begin{gathered} max\ x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta \\ subject\ to, \\ \Delta^{-1}(\tilde{A})y \leq \alpha, \\ \Delta^{-1}(\tilde{B}^T)x \leq \beta, \\ e^T x = 1, \\ e^T y = 1, \\ x \geq 0, y \geq 0, \\ \alpha \in R, \beta \in R. \end{gathered}$$

In addition, if the solution of the above problem is $(x^*, y^*, \alpha^*, \beta^*)$ then

$$\alpha^* = x^{*T}\Delta^{-1}(\tilde{A})y^*, \beta^* = x^{*T}\Delta^{-1}(\tilde{B})y^*,$$
$$x^{*T}\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y^* - \alpha^* - \beta^* = 0.$$

Proof. Suppose the set of all possible solutions to the above mentioned problems are S. Then according to Theorem 1 and Lemma 1, we get $S \neq 0$. Now for all $(x, y, \alpha, \beta) \in S$, we have

$$\begin{aligned} & x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta \\ & = x^T\Delta^{-1}\left(\tilde{A}\right)y + x^T\Delta^{-1}\left(\tilde{B}\right)y - \alpha - \beta \\ & = x^T\Delta^{-1}\left(\tilde{A}\right)y - \alpha e^T x + x^T\Delta^{-1}\left(\tilde{B}\right)y - \beta e^T y \\ & = x^T\left(\Delta^{-1}\left(\tilde{A}\right)y - \alpha\right) + y^T\left(\Delta^{-1}\left(\tilde{B}\right)x - \beta\right) \leq 0, \end{aligned}$$

which implies that

$$\max_{x,y,\alpha,\beta}\left(x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta\right) \leq 0.$$

Sufficiency: Now suppose that a equilibrium solution of the linguistic bi-matrix game $\tilde{BG}$ is (x^*, y^*). Then the solution of (5) and (6) is (x^*, y^*), where $\alpha^* = x^{*T}\Delta^{-1}(\tilde{A})y^*$ and $\beta^* = x^{*T}\Delta^{-1}(\tilde{B})y^*$. Hence $(x^*, y^*, \alpha^*, \beta^*) \in S$ and $x^{*T}\Delta^{-1}(\tilde{A} + \tilde{B})y^* = \alpha^* + \beta^*$. However, since $\max\limits_{x,y,\alpha,\beta}\left(x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta\right) \leq 0$, so we have

$$\begin{aligned} & x^{*T}\Delta^{-1}(\tilde{A}+\tilde{B})y^* - \alpha^* - \beta^* \\ & = \max_{x,y,\alpha,\beta}\left(x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta\right) = 0, \end{aligned}$$

which proves the result.

Necessary: Suppose a solution of quadratic programming problem is $(x^*, y^*, \alpha^*, \beta^*)$. Since

$$\max_{x,y,\alpha,\beta}\left(x^T\Delta^{-1}\left(\tilde{A}+\tilde{B}\right)y - \alpha - \beta\right) \leq 0,$$

so for the equilibrium solution $(x^*, y^*, \alpha^*, \beta^*)$ we have

$$x^{*T}\Delta^{-1}(\tilde{A}+\tilde{B})y^* \leq \alpha^* + \beta^*.$$

But from Definition 1 and Lemma 2, there exists $(\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}) \in S$ such that

$$\hat{x}^T\Delta^{-1}(\tilde{A}+\tilde{B})\hat{y} = \hat{\alpha} + \hat{\beta}.$$

Therefore we get the maximum value of the problem of square programming. Due to $(x^*, y^*, \alpha^*, \beta^*)$ is optimal, we have

$$x^{*T}\Delta^{-1}(\tilde{A}+\tilde{B})y^* = \alpha^* + \beta^*.$$

Given quadratic programming problem as $x^{*T}\Delta^{-1}(\tilde{A})y^* - \alpha^* = 0$, $x^{*T}\Delta^{-1}(\tilde{B})y^* - \beta^* = 0$. Therefore, (x^*, y^*) is the solution of (5) and (6), which gives that an equilibrium solution of the linguistic bi-matrix game $\tilde{BG}$ is (x^*, y^*).

IV. Quadratic Linear Programming Formula of a 2-tuple Linguistic Bi-matrix Game

In this section, we will introduce how to solve the 2-tuple linguistic information bi-matrix game with linguistic quadratic linear programming. Suppose the linguistic terms set is $LT = \{l_0, l_1, ..., l_g\}$, where g is a positive even integer.

The proposed model and solution steps are as follows:

Step 1 : Using Theorem 2 to establish quadratic programming model,

$$\begin{gathered} (M1)\ max\ x^T(\tilde{a}_{ij} + \tilde{b}_{ij})y - \tilde{\alpha} - \tilde{\beta} \\ subject\ to \\ \tilde{a}_{11}y_1 \oplus \tilde{a}_{12}y_2 \oplus \cdots \oplus \tilde{a}_{1n}y_n \leq \tilde{\alpha} \\ \vdots \\ \tilde{a}_{m1}y_1 \oplus \tilde{a}_{m2}y_2 \oplus \cdots \oplus \tilde{a}_{mn}y_n \leq \tilde{\alpha} \\ \tilde{b}_{11}x_1 \oplus \tilde{b}_{21}x_2 \oplus \cdots \oplus \tilde{b}_{m1}x_m \leq \tilde{\beta} \\ \vdots \\ \tilde{b}_{1n}x_1 \oplus \tilde{b}_{2n}y_2 \oplus \cdots \oplus \tilde{b}_{mn}x_m \leq \tilde{\beta} \\ x_1 + x_2 + \cdots + x_m = 1 \\ y_1 + y_2 + \cdots + y_n = 1 \\ x \leq 0, y \leq 0 \\ \alpha \in R, \beta \in R. \end{gathered}$$

Step 2 : Applying Δ^{-1} to the model (M1), it can be expressed as

$$\begin{gathered} max\ x^T(\Delta^{-1}(\tilde{a}_{ij}+\tilde{b}_{ij})y-\Delta^{-1}(\tilde{\alpha})-(\tilde{\beta}) \\ subject\ to, \\ \Delta^{-1}\left(\tilde{a}_{11}\right)y_1+\Delta^{-1}\left(\tilde{a}_{12}\right)x_2+\cdots+\Delta^{-1}\left(\tilde{a}_{1n}\right)y_n\leq\Delta^{-1}\left(\alpha\right) \\ \vdots \\ \Delta^{-1}\left(\tilde{a}_{m1}\right)y_1+\Delta^{-1}\left(\tilde{a}_{m2}\right)y_2+\cdots+\Delta^{-1}\left(\tilde{a}_{mn}\right)y_n\leq\Delta^{-1}\left(\alpha\right) \\ \Delta^{-1}\left(\tilde{b}_{11}\right)x_1+\Delta^{-1}\left(\tilde{b}_{21}\right)x_2+\cdots+\Delta^{-1}\left(\tilde{b}_{mn}\right)x_m\leq\Delta^{-1}\left(\beta\right) \\ \vdots \\ \Delta^{-1}\left(\tilde{b}_{1n}\right)x_1+\Delta^{-1}\left(\tilde{b}_{2n}\right)x_2+\cdots+\Delta^{-1}\left(\tilde{b}_{mn}\right)x_m\leq\Delta^{-1}\left(\beta\right). \end{gathered}$$

Step 3 : Set $\Delta^{-1}(\tilde{\alpha})=\alpha,\ \Delta^{-1}(\beta)=\beta,\ \Delta^{-1}(\tilde{a}_{ij})=a_{ij},\ \Delta^{-1}(\tilde{b}_{ij})=b_{ij},\ i=1,2,...,m,\ j=1,2,...,n$. Then we have

$$\begin{gathered} (M2)\ max\ x^T\left(a_{ij}+b_{ij}\right)y-\alpha-\beta \\ subject\ to, \\ a_{11}y_1+a_{12}y_2+\cdots+a_{1n}y_n\leq\alpha \\ \vdots \\ a_{m1}y_1+a_{m2}y_2+\cdots+a_{mn}y_n\leq\alpha \\ b_{11}x_1+b_{21}x_2+\cdots+b_{n1}x_n\leq\beta \\ \vdots \\ b_{1n}x_1+b_{2n}x_2+\cdots+b_{mn}x_n\leq\beta. \end{gathered}$$

The payoffs (α^*,β^*) and the optimal hybrid strategy (x^*,y^*) of players are obtained through LINGO software, they can be expressed as

$$\begin{gathered} \alpha^*=x^*Ay^*,\ \beta^*=x^*By^*, \\ x^*=(x_1,...,x_m),\ y^*=(y_1,...,y_n). \end{gathered}$$

Step 4 : Based on the (1) of Definition 3, the payoffs $\tilde{\alpha}=\Delta(\alpha^*)=([\alpha^*],\alpha^*-[\alpha^*]),\ \tilde{\beta}=\Delta(\beta^*)=([\beta^*],\beta^*-[\beta^*])$ can be obtained.

In general, we propose a linguistic quadratic linear programming model according to Theorem 2. The fuzzy linguistic information is transformed into specific values by introducing function Δ^{-1}. Then solving the quadratic linear programming through lingo, it is transformed into 2-tuple linguistic information through function Δ. After the first transformation, the solution of linguistic information is transformed into the solution of ordinary quadratic linear programming, and the uncertainty and complexity of the linguistic are greatly reduced. In the second transformation, specific values are transformed into linguistic information, and the uncertainty of the final result reflects the purpose of this article.

V. An Example of the Media Industry Problem

In this section, we consider decision-making issues in the media industry to show the effectiveness and applicability of a given method to real-life problems. The two TV companies, T1 and T2, aim to increase TRPs by increasing the number of viewers. Assuming that the management of the two companies is rational, they will choose the best strategies to maximize their TRPs without the need for cooperation. Let the managers of T1 and T2 TV stations decide what TV programs to broadcast during the peak hours of the day (from 6 p.m. to 10 p.m.). They chose two strategies, one is the TV series (ε_1) and the other is the reality show (ε_2). The above question can be seen as a bi-matrix game, that is, the TV station companies T1 and T2 are considered players 1 and 2 respectively. They may use the strategies of ε_1 and ε_2. Due to the uncertainty and ambiguity of information, the administrators of the two companies are often unable to accurately predict the number of viewers. To deal with uncertainty, a 2-tuple linguistic term set S is used to express ambiguous audience numbers.

S = { s_0: Extremely Few(EF), s_1: Very Few(VF), s_2: Few(F), s_3: Several(S), s_4: Medium(M), s_5: Many(M), s_6: So Many(SM), s_7: Large(L) }

$$\tilde{A}=TV\ Serials\ Reality\ Show\begin{bmatrix} TV Serials & Reality Show \\ (s_4,-0.2) & (s_5,0.3) \\ (s_6,0.2) & (s_2,0.4) \end{bmatrix},$$

$$\tilde{B}=TV\ Serials\ Reality\ Show\begin{bmatrix} TV Serials & Reality Show \\ (s_5,0.1) & (s_3,0.3) \\ (s_2,0.4) & (s_6,-0.1) \end{bmatrix}.$$

The entries in these matrices are the approximate number of millions of viewers who watch $T1$ or $T2$ at a specified time. Other viewers may watch other small TV stations. Here $(s_4,-0.2)$ is an approximate number of people in matrix $\tilde{A}$. This shows that when $T1$ and $T2$ companies use the strategy ε_1(TV Serials) simultaneously, they support the expected number of viewers "Medium" of T1 TV station. The other elements in the matrix $\tilde{A}$ and $\tilde{B}$ also have similar explanations.

Firstly, the function Δ^{-1} is applied to matrices $\tilde{A}$ and $\tilde{B}$ to obtain general matrices A and B

$$\Delta^{-1}(\tilde{A})=A=\begin{bmatrix} 3.8 & 5.3 \\ 6.2 & 2.4 \end{bmatrix},$$

$$\Delta^{-1}(\tilde{B})=B=\begin{bmatrix} 5.1 & 3.3 \\ 2.4 & 5.9 \end{bmatrix}.$$

Then using the quadratic linear programming model $M(1)$ to solve the following problems,

$$\begin{gathered} Max\quad 8.9x_1y_1+8.6x_1y_2+8.4x_2y_1+8.3x_2y_2-\alpha-\beta \\ subject\ to \\ 3.8x_1+5.3x_2\leq\alpha \\ 6.2x_1+2.4x_2\leq\alpha \\ 5.1y_1+2.4y_2\leq\beta \\ 3.3y_1+5.9y_2\leq\beta \\ x_1+x_2-1=0 \\ y_1+y_2-1=0 \\ x_i\geq0\,(i=1,2) \\ y_i\geq0\,(i=1,2). \end{gathered}$$

Solving (6) with the LINGO software we obtained optimal solution.

Thus, for player 1, the mixed strategy is $(x_1^*=0.547,x_2^*=0.453)$, and the optimal value is $\alpha^*=\Delta(4.479)=(s_4,0.479)$. For player 2, the optimal strategy is $(y_1^*=$

TABLE I
SOLUTION OF THE QUADRATIC LINEAR PROGRAMMING PROBLEM

x_1^*	x_2^*	α^*	y_1^*	y_2^*	β^*
0.547	0.453	4.479	0.660	0.340	4.183

$0.660, y_2^* = 0.340)$, and the optimal value is $\beta^* = \Delta(4.183) = (s_4, 0.183)$.

According to the predefined language set, more people watch T1 TV station than T2 TV station, indicating that the strategy of the first TV station is more effective. In the payment matrix, it can be observed that T1 TV station has great competitive advantages in TV series and T2 TV station has great advantages in reality show. Therefore, if both TV stations want more viewers, they can stagger the peak and focus on more attractive programs. More importantly, compared with the model proposed by seikh et al. [23], this model has 14 constraints, but this paper has only 6 constraints. Therefore, the amount of calculation greatly simplifies the binary language information model, but the result is the same.

VI. CONCLUSION

In this paper, we propose and studie the bi-matrix game with 2-tuple linguistic information, which makes the game theory consist of fuzzy linguistic information. A linguistic quadratic linear programming model for solving 2-tuple linguistic bi-matrix game is proposed to obtain the optimal payoffs and optimal strategy of two players. Then a new function Δ is defined to convert the calculated payoffs in the model into 2-tuple linguistic information. Finally, the effectiveness of the game is verified by the game problem in the media industry.

In the following research, we can explore the bi-matrix game of interval-valued language and the bi-matrix game of hesitant language variables. These two games can expand the scope of language use and make the fuzzy bi-matrix game get more accurate payoffs.

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