A Conjugate Gradient Direction-Based Method to Evaluate Reliability Analysis Problems

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Abstract—**Reliability is always considered as one of the most important factors in every system and it is commonly accepted to investigate reliability by employing reliability analysis problems. In this case, the need of existence of powerful reliability analysis methods has resulted in introducing different methods to solve first-order and higher-order reliability analysis problems. These methods have also been employed to ensure failure probability of a system is below an acceptable level. However, there are still disadvantages such as instability and inefficiency in some of these methods. The Hasofer - Lind and Rackwitz - Fiessler (HL-RF) method is still the most popular method to solve the first-order reliability analysis problems, but it will be shown in this paper that this method is not stable enough to solve highly non-linear problems. Also, it is found that the HL-RF method may show some inefficiency behaviour even when it converges to the optimum point. In this paper, a new reliability analysis method is introduced that solves first-order reliability analysis problems stably and efficiently. This method is based on the conjugate gradient direction that is used as a line search algorithm. The new reliability analysis method is called Conjugate Gradient Direction-Based (CGDB) method whose performance is compared with the performance of the existing HL-RF method by solving several numerical experiments at the end of this paper.**

Index Terms—**Reliability Analysis, Conjugate Gradient Method, First-Order Reliability Method, Non-deterministic Design Optimization**

I. INTRODUCTION

RELIABILITY theory has widely been applied into various research and industrial projects, including, but not limited to, radar systems, electricity networks, teeth X-ray images, car crash-worthiness, etc., to improve system safety and design reliability [4], [12], [13], [22]. These theories have also been used in non-deterministic optimization models to find the optimum solution while taking reliability factors into account. Applying reliability-related issues in engineering systems has however resulted in emerging more complicated mathematical problems. In this case, reliability analysis methods are of the most popular and powerful methods to tackle these problems [17].

One of the commonly used non-deterministic optimization models is the reliability-based design optimization (RBDO). Among three categories of RBDO problems are the decoupled approach, the reliability index approach (RIA) and the performance measure approach (PMA) [5], [9]. These approaches are designed based on two types of reliability analysis problems as first-order direct problems and first-order inverse problems that can be solved by a number of iterative methods.

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Furthermore, iterative approaches are applied in a new algorithm to solve economical problems related to a power system. A global optimization algorithm is used in this method that seeks the optimum solution in an iterative process [18]. In this method, once initial parameters are set, the current optimum value is updated to check whether it meets the relevant stopping criteria. If not, a three-phase sub-process, as mutation, crossover and selection, is applied before repeating the process of updating the optimum point.

Apart from these approaches, a reliability analysis method is proposed based on the Voronoi cells. These cells are used to partition a random-variate space by dividing the space according to the nearest neighbour rule. Given $(x_1, x_2, ..., x_d)$ denote independent random variables, the basic point set is shown by $P_U = \{u_j = (u_{1,j}, u_{2,j}, ..., u_{d,j}), j = 1, 2, ..., N\}$ where N indicates total number of sampling points and $u_{i,j}$ is j-th coordinate in i-th dimension (Xu 2018). Moreover, structural reliability and reliability analysis problems are investigated by using a multi-parameter correlation problem. For this purpose, a joint probability density function is first built among all random variables and then the correlation between these variables is described. It is reported that the Copula function applied into the RBDO problems results in significant improvements [9], [25].

Another interesting research is carried out to forecast wind speed in short-term where two main frequency sequences are first found in order to decomposing the data into various frequencies. These sub-sequenced frequencies are then categorized for training, verification, and test purposes [6]. Reliability considerations are also discussed in the literature where two independent random variables are used to define a system failure probability. This investigation, which has been implemented by employing the Weibull distribution, leads to maximizing the likelihood estimation of parameters of the Weibull distribution [23].

All the above-mentioned applications show that reliability-related issues are considered in many researches. However, there are still shortcomings to solve direct reliability analysis problems where the Hasofer - Lind and Rackwitz - Fiessler (HL-RF) method is still the most stable and efficient method. A new reliability analysis method is introduced in this paper to solve these problems. Stability and efficiency of this method is compared with the HL-RF method.

II. FIRST ORDER RELIABILITY METHOD

Non-deterministic characteristics play undeniable roles in modern designed systems resulting in existence of uncertainty in many problems. Reliability-related issues (in general) and reliability analysis problems (in particular) have attracted significant attention by researchers and engineers to take these uncertainties into account.

A system performance function is often used to formulate an optimization problem as a reliability analysis problem to deal with these considerations. The performance function can either be employed in the objective function or in the constraints, depending on the type of the reliability analysis problem. A direct reliability analysis problem aims at minimizing the Euclidian norm of decision variables where the performance function is used in constraints, whereas an inverse problem is designed to minimize a performance function on a circular constraint [11], [14].

Although reliability analysis problems are often formulated in original random space (i.e. X-space), it is widely accepted to transfer the problem into a new space, which is the standard normalized random space (U-space), in order to decrease non-linearity of the problem [14]. For this purpose, all variables and relevant performance functions need to be transferred from X-space to U-space that is often done by the first-order reliability method (FORM). The FORM is based on the first-order Taylor expansion of the given performance function [8], [21], [24]:

$$T_{FORM} : X \longrightarrow U$$

The transformation T is defined based on the distribution function of design variables. For example, if a design variable is Normally distributed, then T would be defined as below:

$$u = T(x) = \frac{x - \mu}{\sigma} \tag{1}$$

where x is the original design variable, u is the standard normalized design variable, and μ and σ are the statistical parameters of the design variable x [25].

The FORM is also applied to solve reliability-based design optimization (RBDO) problems that are generally categorized as single-loop, double-loop, and decoupled problems defined as below [15]:

$$\begin{aligned} Min\ & cost(x_1, x_2, ..., x_n) \\ s.t.\quad & P_f < \Phi(-\beta_t) \end{aligned} \tag{2}$$

where a performance function is used to model the probabilistic constraint as P_f that is the system failure probability. Inner loop of a double-loop RBDO problem includes a reliability analysis problem to ensure the probabilistic constaint is considered when solving the RBDO problem.

Furthermore, radial-based sampling and Krigning are combined to introduce a specific structural reliability method. In this method, a search domain is first defined in which a failure point should be found. It is then aimed at getting closer to the origin of the random space by updating the limit-state (failure) point. The last limit state point is ultimately considered as the most probable point since it is supposed that no more failure point can be found. Moreover, other methods such as second-order reliability method (SORM) and Monte Carlo simulation (MCS) have been applied for standard normalizing a reliability analysis problem. It has been reported that these methods are very time-consuming and rarely can reduce non-linearity of a reliability analysis problem [5].

Reliability related concepts are also considered when designing renewable energy systems in the last couple of years. Optimization modelling of these systems are categorized as low-frequency and high-frequency models. Five different layers may be used to apply a basic adaptive neuro-fuzzy inference system (ANFIS) architecture on decomposited data that is related to a low-frequency model. However, a multiple-input single-output formulation can be used in a high-frequency model. In this case, real value estimation is supposed to be done by the obtained mathematical formulation [6].

In the next section, different reliability analysis problems and available methods to solve them will be illustrated briefly.

III. Reliability Analysis

One of the most important factors that is always considered in system design is safety, which is improved by investigating uncertainty. For this purpose, reliability analysis problems are designed to take system uncertainties into account, and reliability analysis methods are desigend to solve these problems [2]. Two types of reliability analysis problems are illustrated in the first subsection below. Then, available reliability analysis methods to solve these problems are discussed in the second subsection.

A. *Reliability Analysis Problems*

Reliability analysis problems are often formulated as a constrained minimization problem in which a system performance function is used. These problems are generally categorized as first-order direct and first-order inverse problems.

The first-order direct reliability analysis problem is formulated as below [1], [8]:

$$\begin{aligned} Min\quad & \|(u_1, u_2, ..., u_n)\| \\ s.t.\quad & G_U(u_1, u_2, ..., u_n) = 0 \end{aligned} \tag{3}$$

where $\|.\|$ is the Euclidean norm, $u = (u_1, u_2, ..., u_n)$ is the standard normalized design variable, and $G_U(u)$ is the standard normalized performance function. As discussed earlier, all variables and the performance function are transformed by FORM into the standard normalized random space (U-space) to reduce the problem's non-linearity.

This problem is displayed in [1] where the optimum solution is called the most probable failure point (MPFP). A first-order direct reliability analysis problem aims at finding the nearest point on the failure surface (i.e. $G_U(u) = 0$) to the origin of the U-space.

Moreover, the first-order inverse reliability analysis problem is formulated in the U-space as follows [11]:

$$\begin{aligned} Min\quad & G_U(u_1, u_2, ..., u_n) \\ s.t.\quad & \|(u_1, u_2, ..., u_n)\| = \beta \end{aligned} \tag{4}$$

where β is the (target) reliability index.

A figure in [1] shows an inverse reliability analysis problem that is designed to find a point on a circular constraint (with radius of reliability index β) minimizing the standard normalized performance function. The optimum solution of this problem is called the minimum performance target point (MPTP) that is here displayed by u^{MPTP}.

B. Reliability Analysis Methods

A first-order direct reliability analysis problem is solved to find the minimum distance from the origin of the U-space to the failure surface as a reliability index. The first-order Taylor series expansion was initially used to introduce a reliability analysis method in which the steepest descent direction is employed [1]. In this case, a reliability analysis method was proposed based on two researches conducted by Hasofer - Lind and Rackwitz - Fiessler. This method is therefore called the Hasofer - Lind and Rackwitz - Fiessler (HL-RF) method [19], [20]. The design point is iteratively updated in this method as below:

$$u^{(k+1)} = (u^{(k)}.n^{(k+1)})n^{(k+1)} + \frac{G_U(u^{(k)})}{\|\nabla G_U(u^{(k)})\|}n^{(k+1)} \quad (5)$$

where $n^{(k+1)}$ is the steepest descent direction, $u^{(k)}$ is the current design point, and $G_U(u^{(k)})$ is the standard normalized performance function value at the current design point. In this method, the origin of U-space is considered as the initial design point.

The steepest descent direction is also updated regularly as follows:

$$n^{(k+1)} = \frac{\nabla G_U(u^{(k)})}{\|\nabla G_U(u^{(k)})\|} \quad (6)$$

It can be seen in the literature that the HL - RF method is still the most popular and efficient method to solve the first-order direct reliability analysis problems [3], [14].

On the other hand, the existing methods to solve an inverse reliability analysis problem can be categorized into three groups as below:

1) Methods based on the steepest descent direction: There are four different methods in this category that the hybrid mean value (HMV) method, as a combination of all other methods in this group, is the most efficient one to solve a first-order inverse reliability analysis problem [26].
2) A method based on the conjugate gradient direction: This method, which is called the conjugate gradient analysis (CGA) method, is based on the conjugate gradient direction. It has been reported that this method is more efficient and stable than the HMV method [11].
3) Polar coordinate-based method: This method is called unconstrained polar reliability analysis (UPRA) method that transforms the first-order inverse reliability analysis problem into an unconstrained problem in the polar space and then solves it [10].

Many researchers have studied inverse reliability analysis problems and proposed new methods to enhance their efficiency and stability. However, there is a limited number of methods (mainly based on the HL-RF method) to solve a first-order direct reliability analysis problem. A new method is proposed in this paper to improve performance of these methods.

IV. Conjugate Gradient Direction-Based Method

A new reliability analysis method is introduced in this section to solve the first-order reliability analysis problems, which are illustrated in Subsection (3.1). As discussed earlier, the HL-RF method is the most efficient and stable existing method to solve these problems. However, it is observed and will be shown in this paper that the HL-RF method is not stable enough to solve highly non-linear problems. Also, it will be shown that another disadvantage of this method is inefficiency in some cases.

In the new method introduced in this paper, the conjugate gradient direction is employed to update each design point. So, this method is called the Conjugate Gradient Direction-Based (CGDB) method. Performance of the CGDB method is then compared with the performance of the HL-RF method by solving several numerical experiments.

In the CGDB method, a reliability analysis problem is first transformed from the original random space (i.e. X-space) to the standard normalized random space (i.e. U-space). This transformation is done by employing the first-order reliability method (FORM).

Once the problem is standard normalized, the origin of the U-space (i.e. $u^{(0)} = (0, 0, ..., 0)$) will be considered as the initial design point in order to start the iterative process. Then, all design points are updated by the conjugate gradient direction as a line search algorithm.

The initial design point is updated in the first step by using the gradient vector of the standard normalized performance function at the origin of the U-space. In this case, a vector is defined as below:

$$\alpha^{(1)} = \nabla G_U(u^{(0)})$$

Then, this vector is used to find a new direction in order to update the current design point, which is the origin as the initial design point.

$$n^{(1)} = \frac{\alpha^{(1)}}{\|\alpha^{(1)}\|}$$

As can be seen, in the first iteration, the conjugate gradient direction is indeed equivalent to the steepest descent direction. In other words, the first step of this method is similar to the first step of the HL-RF method because of the similarity between the conjugate gradient direction and the steepest descent direction. However, these will not be the same from the next iteration and so the main difference between these methods will emerge.

Therefore, a new design point is found by updating the initial design point as below:

$$u^{(1)} = \frac{G_U(u^{(0)})}{\|\nabla G_U(u^{(0)})\|}n^{(1)}$$

In the next step, a scalar factor $d^{(1)}$ should be calculated using the new design point $u^{(1)}$. This scalar factor was zero at the first step that resulted in the conjugate gradient direction to be as same as the steepest descent direction. The scalar factor is obtained as below:

$$d^{(1)} = \frac{\|\nabla G(u^{(1)})\|^2}{\|\nabla G(u^{(0)})\|^2} \quad (7)$$

Once the above factor $d^{(1)}$ and the current design point $u^{(1)}$ are found, it is required to obtain a new conjugate gradient direction, which is not as same as the steepest descent direction any more.

$$\alpha^{(2)} = \nabla G_U(u^{(1)}) + d^{(1)}\alpha^{(1)} \quad (8)$$

and then the direction gets updated as

$$n^{(2)} = \frac{\alpha^{(2)}}{\|\alpha^{(2)}\|} \tag{9}$$

It is now the time to calculate the next design point. As u and n are both n-dimensional vectors, their dot product needs to be found that will then be used as a coefficient in the update process of the design point. So, a new design point is computed as below:

$$u^{(2)} = (u^{(1)}.n^{(2)})n^{(2)} + \frac{G_U(u^{(1)})}{\|\nabla G_U(u^{(1)})\|}n^{(2)} \tag{10}$$

These steps should be repeated and a new scalar factor d should be calculated until convergence. In this method and based on the reliability analysis problem, the convergence criterion is defined as follows:

$$G_U(u^{(k+1)}) = 0 \tag{11}$$

Then, the most current design point for which the stopping criterion is met will be considered as the optimum solution.

The conjugate gradient direction-based (CGDB) method can be summarized as below:

1) Set $k = 0$ as the iteration counter, $u^{(k)} = (0, 0, ..., 0)$, $\alpha^{(k)} = (0, 0, ..., 0)$, and $d^{(k)} = 0$.
2) Calculate $\alpha^{(k+1)} = \nabla G_U(u^{(k)}) + d^{(k)}\alpha^{(k)}$
3) Calculate $n^{(k+1)} = \frac{\alpha^{(k+1)}}{\|\alpha^{(k+1)}\|}$
4) Calculate $u^{(k+1)} = (u^{(k)}.n^{(k+1)} + \frac{G_U(u^{(k)})}{\|\nabla G_U(u^{(k)})\|})n^{(k+1)}$
5) Check stopping criterion as $G_U(u^{(k+1)}) = 0$
6) If the stopping criterion holds, then stop; the current design point is the optimum solution.
   Otherwise, set $k = k+1$, calculate a new scalar factor as
   $$d^{(k)} = \frac{\|\nabla G(u^{(k)})\|^2}{\|\nabla G(u^{(k-1)})\|^2}$$
   Then, go back to Step 2.

Figure (1) displays these steps in a flowchart. Performance of this method will be compared with the HL-RF method in the next section to check stability and efficiency of the newly introduced CGDB method against the existing one.

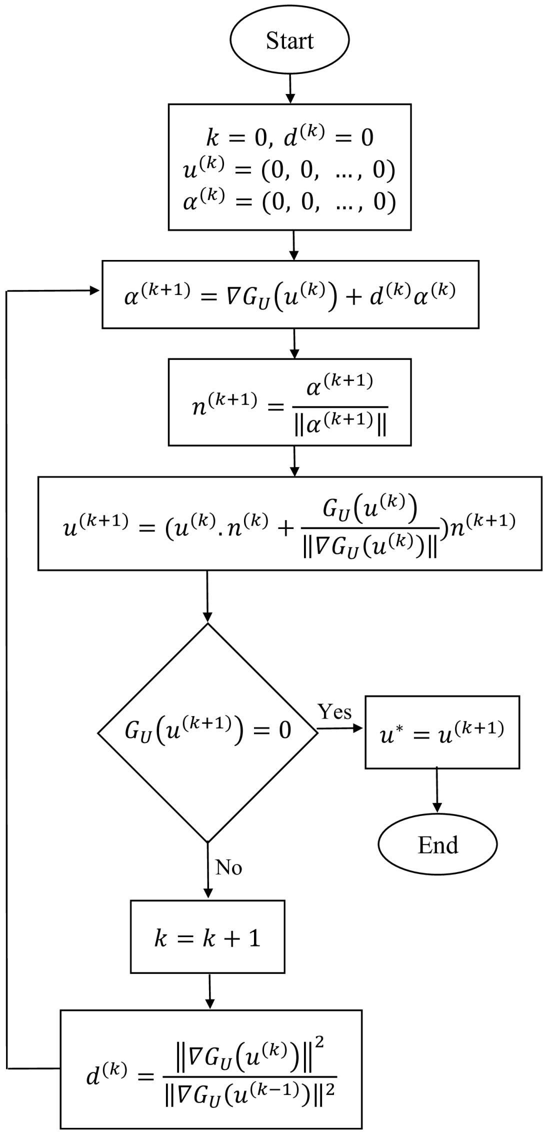

Fig. 1. Conjugate Gradient Direction-Based Method

V. Numerical Experiments

In this section, several first-order reliability analysis problems are solved by using the HL-RF method and also the new conjugate gradient direction-based (CGDB) method proposed in this paper. It is intended to compare their performance in order to find the most stable and efficient method to solve the first-order reliability analysis problems.

In each case, a performance function is needed to formulate the relevant reliability analysis problem. This problem will then be transformed into the standard normal random space (U-space) by employing the first-order reliability method (FORM).

Numerical results of each problem is provided in a detailed table. In the first subsection, a figure is also used to represent how design points are being updated in each problem. Dispersion of the design points as well as directions through which these points are updated are displayed in these figures. In this case, if a method diverges, it can be seen in the dot plots that how an oscilating behavior prevents a method from convergence by not letting the design points to get closer to each other.

A. Fixed Initial Design Points

This subsection includes problems with fixed initial design points and various non-linearity. Different levels of non-linearity of performance functions are considered in these problems to investigate potential effect of non-linearity on the methods' performance.

Numerical Experiment 1

In the first numerical experiment, the following system

TABLE I
RESULT SUMMARY OF PROBLEM 1

	HL-RF	CGDB
Iterations	163	25
(x_1^*, x_2^*)	(6.9426, 2.1723)	(6.9547, 2.1676)
β	8.4036	8.4036

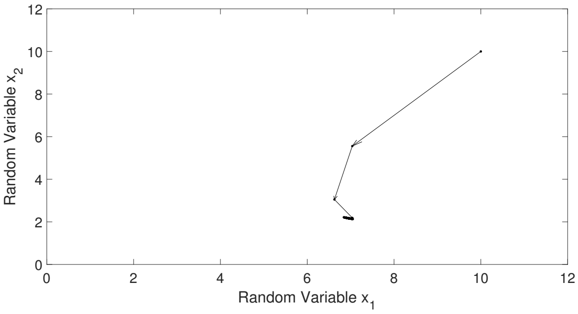

Fig. 2. Design points of the HL-RF method in the X-space - Numerical Experiment 1

performance function is considered:

$$G(x_1, x_2) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1$$

where both design variables (x_1, x_2) are Normally distributed with statistical parameters as $\mu = (10, 10)^T$ and $\sigma = (1, 1)^T$.

The performance function is then standard normalized by the first-order reliability method (FORM) based on the below relationships:

$$x_1 = u_1 + 10 \quad \& \quad x_2 = u_2 + 10$$

Therefore, the relevant first-order direct reliability analysis problem can be formulated (regarding the Model (3)) as follows:

$$\begin{aligned} &Min\|(u_1, u_2)\| \qquad (12) \\ &s.t. 5u_1^2 + 5u_2^2 + 6u_1u_2 + 96u_1 + 144u_2 + 924 = 0 \end{aligned}$$

This problem is now solved twice; first by the HL-RF method, and then by the CGDB method proposed in this paper. Both methods converge in this problem to find the optimum point but with different convergence rates and so various efficiency. The HL-RF method needs 163 iterations to reach to the optimum point while the CGDB method needs only 25 iterations to converge and finish the process.

The reliability index, which is supposed to be found in the probelm by minimizing $\|(u_1, u_2)\|$, is however the same in both methods that is equal to 8.4036. Despite this similarity in the minimization process, the CGDB method takes much shorter time than the HL-RF method for convergence due to the fewer iterations required. Table I briefs the reulsts obtained in this problem.

Figures (2) and (4) show how design points move in the X-space and are updated by each method. Although the direction arrows of the HL-RF method get updated sooner than the CGDB method toward the region, where the optimum point will ultimately be found, the Figure (**??**) shows that many updates are required by the HL-RF method to get closer to the optimum point and finally to find it,

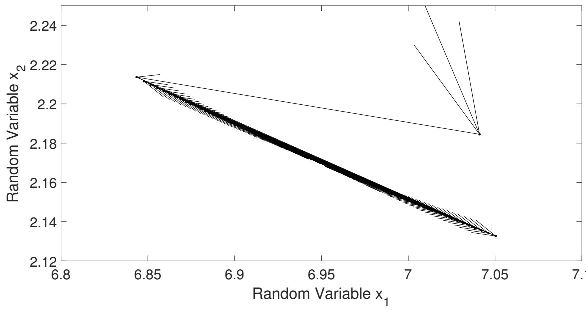

Fig. 3. Further details of the HL-RF method - Numerical Experiment 1

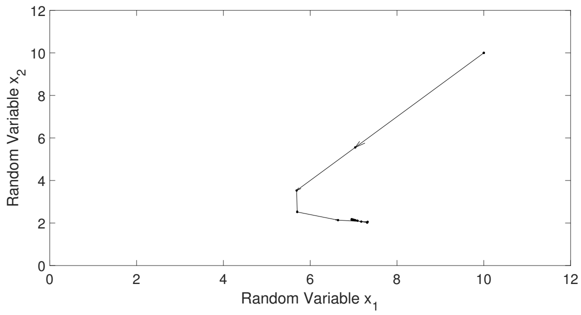

Fig. 4. Design points of the CGDB method in the X-space - Numerical Experiment 1

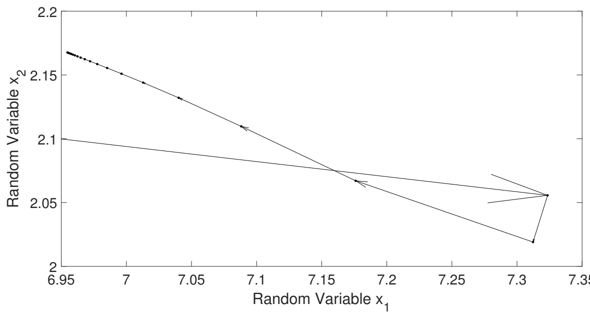

Fig. 5. Further details of the CGDB method - Numerical Experiment 1

while the CGDB method finds the optimum point and stops faster. Figures (3) and (5) are also provided to further clarify activities of each method in the last couple of iterations.

In both figures, as per the problem specifications, the initial design point can be seen as $(10, 10)$ that updates towards around $(7, 2)$ to find the optimum point. The HL-RF figure shows how intensive the design points are in this area, which is due to the number of iterations required, to ultimately find the optimum point that is $(6.9426, 2.1723)$, while it is shown in the CGDB figure the intensitiy in that area is much less as one-sixth of the iterations are required to find the optimum point, which is $(6.9547, 2.1676)$.

Hence, it can be concluded that the new CGDB method is more efficient than the HL-RF method to solve this problem, because less iterations as well as shorter time are required to find the same optimum solution.

Numerical Experiment 2

It is expected to find significant difference between performance of the above mentioned reliability analysis methods

TABLE II
RESULT SUMMARY OF PROBLEM 2

	HL-RF	CGDB
Iterations	NA	53
(x_1^*, x_2^*)	Diverged	(2.7274, 3.6143)
β	NA	7.3485

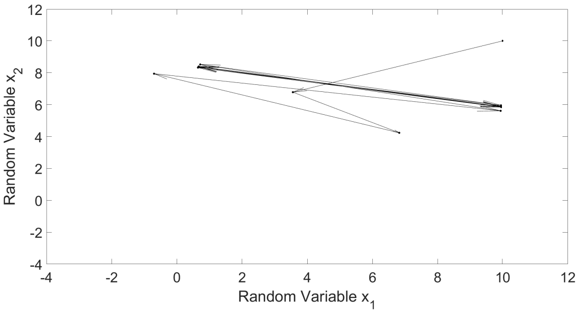

Fig. 6. Design points of the HL-RF method in the X-space - Numerical Experiment 2

when solving problems with highly non-linear performance function. High non-linearity may even result in instability of a method. So, a new performance function is used in the next experiment with the same set of design variables, i.e. both variables follow the Normal distribution, but with new standard deviations as $x_1 \sim N(10, 2)$ and $x_2 \sim N(10, 2)$.

$$G(x_1, x_2) = x_1^3 + x_2^3 - 67.5$$

Given the new standard deviations, the below transformations are used to standard normalize the problem:

$$x_1 = 2u_1 + 10 \quad \& \quad x_2 = 2u_2 + 10$$

Hence, the below optimization problem needs to be solved:

$$\begin{aligned} &Min\|(u_1, u_2)\| \\ &s.t. 8u_1^3 + 8u_2^3 + 120u_1^2 + 120u_2^2 + \\ &\qquad 600u_1 + 600u_2 + 1932.5 = 0 \end{aligned} \tag{13}$$

The CGDB method converges in this problem after 53 iterations and finds the optimum point $(2.7274, 3.6143)$ with the reliability index (i.e. the minimzed $\|(u_1, u_2)\|$) as 7.3485. But an instability behavior was noted when the HL-RF method applied to solve this problem.

The HL-RF method diverges in this problem as it fails to converge and find the optimum point even after 10,000 iterations. It starts oscillating between $(0.6636, 8.3733)$ and $(9.9481, 5.8725)$ in the 26-th iteration. and this oscillating behavior prevents the HL-RF method from convergence. These findings are summarized in the Table II.

Performance of the reliability analysis methods in this problem are displayed in the Figures (6) and (7). It is shown in the Figure (6) that direction arrows go back and forth between a couple of design points and so fail to find the optimum point. But the CGDB method produces arrows that are moving toward the optimum point one after another. It should be noted that the oscillating arrows of the HL-RF method do not even get close to the region of the optimum point that is found by the CGDB method.

Therefore, it can be concluded that the HL-RF method is divergent and unstable in this problem. On the other hand, if

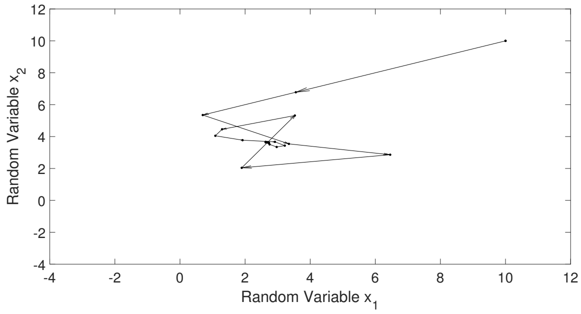

Fig. 7. Design points of the CGDB method in the X-space - Numerical Experiment 2

TABLE III
PERFORMANCE OF BOTH METHODS WITH RANDOMLY GENERATED INITIAL DESIGN POINTS

	Case 1	
	HL-RF Method	CGDB Method
Conv. Rate	616	1000
Iterations	6256	21

TABLE IV
PERFORMANCE OF BOTH METHODS WITH RANDOMLY GENERATED INITIAL DESIGN POINTS

	Case 2	
	HL-RF Method	CGDB Method
Conv. Rate	0	1000
Iterations	NA	25

the CGDB method is used to solve this problem, it would be found stable and efficient enough to reach the optimum point after 53 iterations with the reliability index as $\|(u_1, u_2)\| = 7.3485$.

In the next subsection, 1000 initial design points are randomly generated for each of the above problems. Then, the HL-RF and CGDB methods are applied to solve the reliability analysis problems to see how these methods perform when an initial design point is generated randomly in each case.

B. Randomly Generated Initial Design Points

The reliability analysis problems solved in the previous subsection are again considered with 1000 different initial design points, which are generated randomly. The main idea is to investigate performance of each reliability analysis method based on different initial design points to neutralize any impacts of the initial design points on the methods' performance.

Each problem is solved 2000 times, first 1000 times solved by the HL-RF method with randomly generated initial design points and then 1000 times by the CGDB method with the same randomly generated initial design points. Performance of the methods are compared based on the obtained results of these experiments summarized in the Tables (III) and (IV).

For this purpose, the number of times (out of 1000) that the applied method shows stability (convergence) to solve the relevant reliability analysis problem is considered as the main measure to check stability/performance of each method. So, the first row of the Tables (III) and (IV) show the number of convergence cases for each problem out of 1000 times.

The CGDB method is stable in all 2000 cases as it converges to solve the problem using every single randomly generated initial design point in both problems. However, the HL-RF method fails to be stable as it finds the optimum point in 616 cases (out of 1000) when trying to solve the first problem, and it diverges in all the 1000 runs of the second problem where the performance function is a highly non-linear function.

The required iterations for solving the problems are also considered for the comparison between the two methods. In this case, the average number of iterations of the convergence cases are included in the Table. Regardless of the second case, where the HL-RF method never converges, a huge difference is found between performance of these methods in the first case. It is displayed in the Tables (III) and (IV) that 6256 iterations are required for the HL-RF method in average to find the optimum point (in those 616 convegent cases), while the average number of iterations in the CGDB method to solve the same 1000 problems is just 21 iterations.

It can be concluded that both the convergence rate (out of 1000) and the average of the required iterations for convergence indicate that the new CGDB method is more stable and efficient than the existing HL-RF method. Therefore, problems solved in this subsection once again show that the CGDB method is more stable and efficient than the HL-RF method.

VI. Conclusion

A new reliability analysis method is introduced in this paper that is based on the conjugate gradient direction as a line search algorithm. This method, which is called the Conjuage Gradient Direction-Based (CGDB) method, is employed to solve several reliability analysis problems to check its stability and efficiency. Performance of this method is compared with the Hasofer and Lind - Rackwitz and Fiessler (HL-RF) method as the most stable and efficient existing method.

To compare these methods, the problems are first solved by initial design points already used in the liteature. Each problem is solved by both HL-RF and CGDB methods. Then, rather using these fixed initial design points, 1000 randomly generated initial design points are employed to solve the problems by both mentioned methods. As 10,000 iterations are considered for possible convergence of each method to solve every problem, up to 10,000,000 iterations performed for each performance function per each case of randomly generated initial design points.

Based on the analysis done in this reseach, which are illustrated in the previous section, it can be concluded that the new CGDB method is more stable and efficient than the HL-RF method. When the fixed initial design points are employed in each problem, the CGDB method is both more efficient (by requiring fewer number of iterations to find the optimum point with the same reliability index) and more stable (as it converges while the HL-RF method diverges) than the HL-RF method.

Furthermore, the CGDB method shows better performance than the HL-RF method when 1000 randomly generated initial design points are used to solve each problem.

Hence, one can result that the newly proposed conjugate gradient direction-based (CGDB) method is more stable and efficient than the HL-RF method.

VII. Discussion and Future Works

Reliability related concepts are being used in a variety of research areas to design more reliable systems. In this case, reliability analysis problems and their methods play significant roles for system design improvments in the current industrial world.

There is no doubt that more advanced systems need more advanced reliability models to take as many factors into consideration as possible. However, this high level of advancement brings more and more complexity to solve the problems.

The topic discussed in this paper can be expanded into two different directions. In a theoretical expansion, this method can be further improved by employing other reliability related issues in problems. Also, this method can be applied into various industrial projects to improve system design with enhanced reliability in practice. Therefore, our future works will focus on these two directions.

For theoretically improvement of the new CGDB method introduced in this paper, it can be considered to apply this method to solving verh highly nonlinear problems with non-Gaussian distributed random variables as part of our future works. Although it may be needed to apply slight amendments to this method, it can also bring more improvements to the CGDB method enabling it to solving a wider range of reliability analysis problems.

On the other hand, another important future work will be based on a focus on renewable energy systems, like solar panels and wind turbines. A new reliability analysis model will be introduced for solar panels where a single diode solar cell will be studied by considering a photocurrent relationship. The main aim of this model will be to enhance reliability of single diode solar cells in order to ultimately provide more sustainable renewable energy systems in the future.

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