The Cholesky Decomposition of Matrices over the Symmetrized Max-Plus Algebra

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***Abstract*—This paper discusses the Cholesky decomposition in the symmetrized max-plus algebra. By using a link between the conventional algebra and the symmetrized max-plus algebra, we show the existence of the Cholesky decomposition of a matrix over the symmetrized max-plus algebra. A matrix has the Cholesky decomposition if it is symmetric and has principal leading submatrices whose determinant are positive. The results can be used to determine the solution of linear balance systems.**

***Index Terms*—symmetrized-max-plus-algebra, Cholesky-decomposition, balance-linear-systems, link**

I. Introduction

MAX-plus algebra is the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ where the basic operation is "max" (maximum) as addition and "plus" (usual addition) as multiplication, where $\mathbb{R}$ is the set of all real numbers. Every non zero element has no an inverse under addition operation. It is the main difference between the max-plus algebra and conventional algebra. Consequently, it is difficult to form matrices decompositions in $\mathbb{R}_{\max}$, e.g the Cholesky decomposition.

The symmetrization process can be done to solve the inverse problem in $\mathbb{R}_{\max}$. The result of the symmetrization process is called the symmetrized max-plus algebra and denoted by $\mathbb{S}$ [3]. Furthermore, $\mathbb{R}_{\max}$ can be viewed as the class of positive or zero in $\mathbb{S}$. By adopting the principle of extending the set of all natural numbers to the set of all integers to obtain inverse elements in $\mathbb{Z}$, we can obtain the elements called minus elements as inverse-like elements in $\mathbb{S}$.

There are two mappings between the symmetrized max-plus algebra and conventional algebra. That will be used as tools to solve some problems in $\mathbb{S}$, including the QR and singular value decomposition [2]. Schutter and Moor in [2] also showed that solving both decompositions by employing both mappings is easier than that without using both mappings as discussed in [1].

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In [2], both mappings are used as a link between $\mathbb{S}$ and conventional algebra. By using the link, in order to get the decompositions in $\mathbb{S}$, we first determine in conventional algebra a decomposition associated with a decomposition in $\mathbb{S}$, and then we take it back to $\mathbb{S}$ as the decomposition we desire. The QR and singular value decompositions obtained may be useful in discrete event systems.

In this paper, we determine the Cholesky decomposition of a matrix in $\mathbb{S}$ by using the link. Since $\mathbb{R}_{\max}$ can be viewed as the class of positive or zero elements in $\mathbb{S}$, the Cholesky decomposition in $\mathbb{R}_{\max}$ can be viewed as the Cholesky decomposition in $\mathbb{S}$. By using this method, as in conventional algebra, a computation to solve the linear balance systems $A \otimes x \nabla b$ for A that can be factorized as a Cholesky decomposition is simpler than that using the cofactor and generalized inverse methods as discussed in [3] and [4], respectively.

The paper is organized in 6 sections. Section 1 explains the topic under study. Section 2 discusses the Cholesky decomposition in conventional algebra. Section 3 and 4 discuss the symmetrized max-plus algebra and a link between the symmetrized max-plus algebra and conventional algebra, respectively. The main result will be discussed in Section 5 and 6, namely the existence of the Cholesky decomposition in $\mathbb{S}$ and its application to linear balance systems, respectively.

II. The Cholesky Decomposition in Conventional Algebra

In conventional algebra, matrix decomposition plays a very important role. The Cholesky decomposition and other decomposition are important matrix computations, because it is often difficult to do this computations explicitly. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive-definite if $x^T A x > 0$, for every $x \in \mathbb{R}^n$.

Definition 1. [6] *For a positive-definite symmetric matrix $A \in \mathbb{R}^{n\times n}$, there is a unique lower triangular matrix $G \in \mathbb{R}^{n\times n}$ whose diagonal entries are positive such that $A = GG^T$. The factorization $A = GG^T$ is called the Cholesky decomposition.*

Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$. The Cholesky decomposition of A is $A = \begin{bmatrix} \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$. The Cholesky decomposition plays an important role in solving systems of linear equation. Given a system of linear equations $Ax = b$ where $A = GG^T$ is the Cholesky decomposition of A. The solution of the system of linear equations, can be determined using the following steps: (i) determining the solution of the system of linear equations $Gy = b$, and (ii) determining the solution of the system of linear equations $G^T x = y$. By doing so, the computation process is easier and simpler.

III. THE SYMMETRIZED MAX-PLUS ALGEBRA

In this section, we discuss max-plus algebra and symmetrized max-plus algebra referring to [3] and [2]-[3], respectively. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. The operations on $\mathbb{R}_{\max}$ are defined as

$$a \oplus b = \max\{a, b\}$$
$$a \otimes b = a + b$$

where $\max\{a, -\infty\} = a$ and $a + (-\infty) = -\infty$ for all $a, b \in \mathbb{R}_{\max}$. For example, $2 \oplus 1 = \max\{2,1\} = 2$ and $2 \otimes 1 = 2 + 1 = 3$.

The mathematical system $(\mathbb{R}_{\max}, \oplus, \otimes)$ is called the max-plus algebra, where the additive identity element (zero element) is $\mathcal{E} = -\infty$ and the multiplicative identity element (unity element) is $e = 0$. Furthermore, $\mathbb{R}_{\max}$ is an idempotent commutative semi ring.

Every non zero element in max-plus algebra has no an additive inverse. Thus, in max-plus algebra, a symmetrization process is needed to solve the problem. This process is carried out similarly to the process of expanding the set of all natural numbers into the set all of integers. The symmetrization process in max-plus algebra is carried out to obtain minus elements as inverse-like elements.

Let $P_{\mathcal{E}} \stackrel{\text{def}}{=} \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. The operations on $P_{\mathcal{E}}$ are defined as

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$$
$$(a, b) \otimes (c, d) = (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)$$

for all $(a, b), (c, d) \in P_{\mathcal{E}}$. The zero element is $(\mathcal{E}, \mathcal{E})$, the unity element is $(0, \mathcal{E})$ and the zero element is an absorbent under the multiplication. The mathematical system of $(P_{\mathcal{E}}, \oplus, \otimes)$ is an idempotent commutative semi ring and called the algebra of pairs.

Definition 2. [3] *Let* $u = (a, b) \in P_{\mathcal{E}}$.
1. *absolute value of* u *is* $|u|_{\oplus} = a \oplus b$
2. *minus of* u *is* $\ominus u = (b, a)$
3. *balance of* u *is* $u^{\bullet} = u \oplus (\ominus u) = (|u|_{\oplus}, |u|_{\oplus})$.

Theorem 3. [3] *For all* $u, v \in P_{\mathcal{E}}$ *then*
1. $u^{\bullet} = (\ominus u)^{\bullet} = (u^{\bullet})^{\bullet}$
2. $u \otimes v^{\bullet} = (u \otimes v)^{\bullet}$
3. $\ominus (\ominus u) = u$
4. $\ominus (u \oplus v) = (\ominus u) \oplus (\ominus v)$
5. $\ominus (u \otimes v) = (\ominus u) \otimes v$.

In conventional algebra, $x - x = 0$ for all $x \in \mathbb{R}$, but for all $u \in P_{\mathcal{E}}$, $u \ominus u = u^{\bullet} \neq (\mathcal{E}, \mathcal{E})$, except for $u = (\mathcal{E}, \mathcal{E})$. Thus, it is necessary to define a new relation to replace the relation $=$.

Definition 4. [3] *Let* $u = (a, b), v = (c, d) \in P_{\mathcal{E}}$. *The balance relation (denoted by* ∇*) in* $P_{\mathcal{E}}$ *is defined as* $u \nabla v$ *if and only if* $a \oplus d = b \oplus c$

The balance relation is reflexive and symmetric but not transitive, thus, it is not an equivalence relation. For example, $(4,3)\nabla(4,4)$ and $(4,4)\nabla(3,4)$, but $(4,3)\not\nabla(3,4)$. Therefore, it is not possible to define the quotient set of $P_{\mathcal{E}}$ under the ralation ∇. New relation must be defined to solve this problem.

Definition 5. [3] *Let* $u = (a, b), v = (c, d) \in P_{\mathcal{E}}$. *The relation* $\mathcal{B}$ *in* $P_{\mathcal{E}}$ *is defined as*

$$u\,\mathcal{B}\,v \text{ if only if } \begin{cases} (a, b)\nabla(c, d), \text{ if } a \neq b \text{ and } c \neq d \\ (a, b) = (c, d), \text{ if } a = b \text{ or } c = d \end{cases}$$

For all $u \in P_{\mathcal{E}}$, $u \ominus u\,\mathcal{B}\,(\mathcal{E}, \mathcal{E})$ except for $u = (\mathcal{E}, \mathcal{E})$. We have that $\mathcal{B}$ is an equivalence relation, thus, it is possible to define the quotient set of $P_{\mathcal{E}}$ under the relation $\mathcal{B}$. There are three kinds of equivalence classes generated by $\mathcal{B}$ namely:

$\overline{(w, -\infty)} = \{(w, x) \in P_{\mathcal{E}} | x < w\}$ called max-positive class,
$\overline{(-\infty, w)} = \{(x, w) \in P_{\mathcal{E}} | x < w\}$ called max-negative class,
$\overline{(w, w)} = \{(w, w) \in P_{\mathcal{E}}\}$ called balanced class.

Next, $\overline{(\mathcal{E}, \mathcal{E})}$ is called max-zero class and the quotient set of $P_{\mathcal{E}}$ under the relation $\mathcal{B}$ denoted by ${}^{P_{\mathcal{E}}}/_{\mathcal{B}} \stackrel{\text{def}}{=} \mathbb{S}$.

The mathematical system of $(\mathbb{S}, \oplus, \otimes)$ is an idempotent commutative semiring, where the zero element is $\bar{\mathcal{E}} = \overline{(\mathcal{E}, \mathcal{E})}$ and the unity element is $\bar{e} = \overline{(0, \mathcal{E})}$. Furthermore, $\mathbb{S}$ is called the symmetrized max-plus algebra. Hereafter, the class of $\overline{(w, -\infty)}$, $\overline{(-\infty, w)}$ and $\overline{(w, w)}$ are written as w, $\ominus w$ and $w^{\bullet}$ only, respectively. The set of all max-positive or zero, max-negative or zero and balanced classes are denoted by $\mathbb{S}^{\oplus}$, $\mathbb{S}^{\ominus}$ and $\mathbb{S}^{\bullet}$, respectively. Meanwhile, the set of all signed element is denoted by $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$. Note that $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet} = \mathbb{S}$, $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \{\overline{(\mathcal{E}, \mathcal{E})}\}$ and $\mathbb{S}^{\vee}_{*} = \mathbb{S}^{\vee} \backslash \mathbb{S}^{\bullet}$ which is the set of all elements that have multiplicative inverse.

Theorem 6. [2] *Let* $x, y \in \mathbb{R}_{\max}$.
1. *if* $x > y$ *then* $x \oplus (\ominus y) = x$
2. *if* $x < y$ *then* $x \oplus (\ominus y) = \ominus y$
3. *if* $x = y$ *then* $x \oplus (\ominus y) = x^{\bullet}$

For example, $2 \oplus (\ominus 3) = \ominus 3$, $3 \oplus (\ominus 2) = 3$ and $3 \oplus (\ominus 3) = 3^{\bullet}$.

Theorem 7. [2]
1. *For all* $a, b, c \in \mathbb{S}$, $a \ominus c \nabla b$ *if and only if* $a \nabla b \oplus c$
2. *For all* $a, b \in \mathbb{S}^{\vee}$, *if* $a \nabla b$ *then* $a = b$

Theorem 8. (Weak Substitution) [3] *For all* $a, b, c \in \mathbb{S}$ *and* $x \in \mathbb{S}^{\vee}$, *if* $x \nabla a$ *and* $c \otimes x \nabla b$ *then* $c \otimes a \nabla b$.

The operations on a matrix over $\mathbb{S}$ can be performed as that on a matrix in conventional algebra. For example, if $A = \begin{bmatrix} 2 & \ominus 1 \\ \mathcal{E} & 4^{\bullet} \end{bmatrix}$ and $B = \begin{bmatrix} \ominus 2 & 3 \\ 1 & 2 \end{bmatrix}$, then $A \oplus B = \begin{bmatrix} 2^{\bullet} & 3 \\ 1 & 4^{\bullet} \end{bmatrix}$, $4 \otimes \begin{bmatrix} 2 & \ominus 1 \\ \mathcal{E} & 4^{\bullet} \end{bmatrix} = \begin{bmatrix} 6 & \ominus 5 \\ \mathcal{E} & 8^{\bullet} \end{bmatrix}$ and $A \otimes B = \begin{bmatrix} \ominus 4 & 5 \\ 5^{\bullet} & 6^{\bullet} \end{bmatrix}$. The zero matrix in $\mathbb{S}^{m \times n}$ is $[\mathcal{E}]$ where $\mathcal{E}_{ij} = \mathcal{E}$ for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$. The identity matrix is $[e] \in \mathbb{S}^{n \times n}$ where $e_{ij} = e$ for $i = j$ and $e_{ij} = \mathcal{E}$ for $i \neq j$. The balance of two matrices is given in the following definition.

Definition 9. [2] *For all* $A, B \in \mathbb{S}^{m \times n}$, $A \nabla B$ *if and only if* $a_{ij} \nabla b_{ij}$ *for* $i = 1,2, \dots, m$ *and* $j = 1,2, \dots, n$.

Let $A = \begin{bmatrix} 2 & \ominus 1 \\ \mathcal{E} & 4^{\bullet} \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3^{\bullet} \\ (-6)^{\bullet} & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & \ominus 1 \\ 1 & 4 \end{bmatrix}$. Since $a_{ij} \nabla b_{ij}$ for $i, j = 1,2$ then $A \nabla B$. Meanwhile, since $a_{12} \not\nabla C_{12}$ then $A \not\nabla C$.

IV. A LINK BETWEEN THE SYMMETRIZED MAX-PLUS ALGEBRA AND CONVENTIONAL ALGEBRA

In this section, we discuss a link between the symmetrized max-plus algebra and conventional algebra referring to [2]. This link is used to solve problems in the symmetrized max-plus algebra through solving in conventional algebra. The link consists of two mappings as explained in Definition 10 and 11.

Definition 10. [2] *A mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^+$ is defined as*

$$\mathcal{F}(a,\mu,s) = \begin{cases} |\mu| e^{as}, & if\ a \in \mathbb{S}^{\oplus} \\ -|\mu| e^{|a|_{\oplus} s}, & if\ a \in \mathbb{S}^{\ominus} \\ \mu e^{|a|_{\oplus} s}, & if\ a \in \mathbb{S}^{\bullet} \end{cases}$$

where $a \in \mathbb{S}, \mu \in \mathbb{R}_0$ and $s \in \mathbb{R}_0^+$.

Let f and g are functions. The function f is asymtotically equivalent to g in the neighborhood of ∞ denoted by $\sim g$ $x \to \infty$.

Definition 11. [2] *Let $f(s) \sim ve^{|a|_{\oplus} s}$ in the neighbourhood of ∞. The reverse function $\mathcal{R}$ is defined as*

$$\mathcal{R}(f) = \begin{cases} |a|_{\oplus}, & if\ v\ positive \\ \ominus\ |a|_{\oplus}, & if\ v\ negative \end{cases}$$

For example, $\mathcal{F}(2,1,s) = e^{2s}$, $\mathcal{F}(\ominus\ 2,1,s) = -e^{2s}$ and $\mathcal{F}(2^{\bullet},1,s) = e^{2s}$ for all $s \in \mathbb{R}_0^+$. Note that, $\mathcal{R}(\mathcal{F}(2,1,s)) = 2$, $\mathcal{R}(\mathcal{F}(\ominus\ 2,1,s)) = \ominus\ 2$ and $\mathcal{R}(\mathcal{F}(2^{\bullet},1,s)) = 2 \neq 2^{\bullet}$. If real valued-functions f and g are defined by $f(s) = 4e^{2s}$ and $g(s) = 4e^{2s} + e^{s}$, respectively, then $f(s) \sim g(s) \sim 4e^{2s}$, $s \to \infty$ and $\mathcal{R}(f) = \mathcal{R}(g) = 2$.

Theorem 12. [2] *For all $a,b,c \in \mathbb{S}$*

1. *If $a \oplus b = c$ then there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that* $\mathcal{F}(a,\mu_a,s) + \mathcal{F}(b,\mu_b,s) \sim \mathcal{F}(c,\mu_c,s),\ s \to \infty$
2. *If there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that* $\mathcal{F}(a,\mu_a,s) + \mathcal{F}(b,\mu_b,s) \sim \mathcal{F}(c,\mu_c,s),\ s \to \infty$ *then $a \oplus b\ \nabla\ c$*
3. *If $a \otimes b = c$ then there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that* $\mathcal{F}(a,\mu_a,s) \times \mathcal{F}(b,\mu_b,s) = \mathcal{F}(c,\mu_c,s),\ s \in \mathbb{R}_0^+$
4. *If there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that* $\mathcal{F}(a,\mu_a,s) \times \mathcal{F}(b,\mu_b,s) = \mathcal{F}(c,\mu_c,s), s \in \mathbb{R}_0^+$ *then $a \otimes b\ \nabla\ c$*

Theorem 13. [2] *For all A, B, C are matrix over $\mathbb{S}$*

1. *If $A \oplus B = C$ then there are M_A, M_B, M_C such that* $\mathcal{F}(A,M_A,s) + \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s),\ s \to \infty$
2. *If there are M_A, M_B, M_C such that* $\mathcal{F}(A,M_A,s) + \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s),\ s \to \infty$ *then $A \oplus B\ \nabla\ C$*
3. *If $A \otimes B = C$ then there are M_A, M_B, M_C such that* $\mathcal{F}(A,M_A,s) \times \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s),\ s \to \infty$
4. *If there are M_A, M_B, M_C such that* $\mathcal{F}(A,M_A,s) \times \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s),\ s \to \infty$ *then $A \otimes B\ \nabla\ C$*

The entries of a matrix over the symmetrized max-plus algebra will be expressed as the sum or series of exponential forms. Thus, we give definition of the kind of functions and their properties.

Definition 14. [2] *Let S_e be the set of real functions that are analytic and can be written as a (possibly infinite, but absolutely convergent) sum of exponentials in a neighborhood of ∞ :*

$S_e = \{f: A \to \mathbb{R} | A \subseteq \mathbb{R}, \exists K \in \mathbb{R}_0^+$ *such that* $[K,\infty) \subseteq A$ *and f satisfies analytic in $[K,\infty)$ and either*
$\forall x \geq K, f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x},\ n \in \mathbb{N}, \alpha_i \in \mathbb{R}_0, a_i \in \mathbb{R}_{\mathcal{E}}$ *for all i and $a_0 > a_1 > \cdots > a_n$* or
$\forall x \geq K, f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x}, \alpha_i \in \mathbb{R}_0, a_i \in \mathbb{R}$, $a_i > a_{i+1}, \lim_{i\to\infty} a_i = \mathcal{E}$ *and where the series converges absolutely for every $x \geq K$\}.*

Theorem 15. [2] *Every function $f \in S_e$ is asymptotically equivalent to an exponential in neighborhood of ∞ :*

$$f \in S_e\ \ then\ (x) \sim \alpha_0 e^{a_0 x},\ x \to \infty$$

Theorem 16. [2] *If f and g belong to S_e then ρf, $f+g$, $f-g$, fg, f^l and $|f|$ also belong to S_e for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$. Furthermore, if there is a real number P such that $f(x) \neq 0$ for all $x \geq P$ then the functions $\frac{1}{f}$ and $\frac{g}{f}$ restricted to $[P,\infty)$ also belong to S_e. If there is a real number Q such that $f(x) > 0$ for all $x \geq Q$ then the function $\sqrt{f}$ restricted to $[Q,\infty)$ also belongs to S_e.*

V. THE EXISTENCE OF THE CHOLESKY DECOMPOSITION OF MATRIX OVER THE SYMMETRIZED MAX-PLUS ALGEBRA

The main results of this paper are discussed in Section 5 dan 6. In Section 5, we show the existence of the Cholesky decomposition in the symmetrized max-plus algebra by using the link between the symmetrized max-plus algebra and conventional algebra by employing a function class in S_e as in [2] and the conventional Cholesky decomposition as in [5]-[6]. In Section 6, we show the application of the Cholesky decomposition to the linear balance systems.

Theorem 17. *If $A \in \mathbb{S}^{n\times n}$ with $\det(A(1:k,1:k))\ \not\nabla\ \mathcal{E}$ for $k = 1,2,\dots,n$ then there are unit lower triangular matrices $L, M \in (\mathbb{S}^{\vee})^{n\times n}$ and $D = \mathrm{diag}(d_1, d_2, \dots, d_n) \in (\mathbb{S}^{\vee})^{n\times n}$ such that $A\ \nabla\ L \otimes D \otimes M^T$*

Proof. If there is an entry that is not a signed element in $A \in \mathbb{S}^{n\times n}$, then we define $\hat{A} \in (\mathbb{S}^{\vee})^{n\times n}$ as

$$\hat{a}_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} \text{ signed} \\ |a_{ij}|_{\oplus}, & \text{if } a_{ij} \text{ non signed} \end{cases}$$

for all $i,j = 1,2,\dots,n$. If $\hat{A}\ \nabla\ L \otimes D \otimes M^T$ where $\hat{A}$ is a signed matrix, then $A\ \nabla\ L \otimes D \otimes M^T$. Thus, we assume that A is a signed matrix.

Since $\det(A(1:k,1:k))\ \not\nabla\ \mathcal{E}$ for $k = 1,2,\dots,n-1$, then the LU-decomposition of A is guaranteed to be exist [7]. Thus, there are a unity lower triangular matrix $L \in (\mathbb{S}^{\vee})^{n\times n}$ and an upper triangular matrix $U \in (\mathbb{S}^{\vee})^{n\times n}$ such that $A\ \nabla\ L \otimes U$

Let $\tilde{A} = \mathcal{F}(A,M,\cdot)$ with $M \in \mathbb{R}_0^{n\times n}$, $m_{ij} \in \{-1,1\}$ for all $i,j = 1,2,\dots,n$. Entries of $\tilde{A}$ i.e $\tilde{a}_{ij}(s) = m_{ij} e^{|a_{ij}|_{\oplus} s}$ are belong to S_e. According to the steps in obtaining the LU decomposition, we have $\tilde{A}(s) = \tilde{L}(s)\tilde{U}(s)$, for $s \in \mathbb{R}_0^+$, which is the path of LU-decomposition of $\tilde{A}$.

Let $\tilde{D}(s) = \mathrm{diag}\left(\tilde{d}_1(s), \dots, \tilde{d}_n(s)\right)$ where $\tilde{d}_i(s) = \tilde{u}_{ii}(s)$ for $i = 1,\dots,n$. Since $\tilde{d}_i(s) = \tilde{u}_{ii}(s)$ then $\det\left(\tilde{D}(s)\right) \neq 0$. If $\tilde{U}(s) = \tilde{D}(s)\tilde{M}(s)^T$ then $\tilde{M}(s)^T = \tilde{D}(s)^{-1}\tilde{U}(s)$ and $\tilde{M}(s)^T$ is a unit upper triangular matrix. Thus, $\tilde{A}(s) = \tilde{L}(s)\tilde{U}(s) = \tilde{L}(s)\tilde{D}(s)\tilde{M}(s)^T$ for $s \in \mathbb{R}_0^+$ is the path of LDM^T decomposition of $\tilde{A}$.

If $L = \mathcal{R}(\tilde{L})$, $D = \mathcal{R}(\tilde{D})$ and $M^T = \mathcal{R}(\tilde{M}^T)$ then $A \nabla L \otimes D \otimes M^T$ where $L, M \in (\mathbb{S}^{\vee})^{n\times n}$ are unity lower triangular matricesand $D \in (\mathbb{S}^{\vee})^{n\times n}$ is a diagonal matrix. ■

Theorem 18. *If $A \in \mathbb{S}^{n\times n}$ is a symmetric matrix with $\det(A(1{:}k,1{:}k)) \,\nabla\!\!\!\!/\, \mathcal{E}$ for $k = 1,2,\dots,n$ and the LDM^T decomposition of A is $A \nabla L \otimes D \otimes M^T$ then $L = M$.*

Proof. Let $\tilde{A}(s) = \tilde{L}(s)\tilde{D}(s)\tilde{M}(s)^T$, for $s \in \mathbb{R}_0^+$, is the path of LDM^T decomposition of $\tilde{A}$. Note that $\tilde{M}(s)$ is a unit lower triangular matrix, implying $\det\left(\tilde{M}(s)\right) \neq 0$ and $\tilde{M}(s)\tilde{A}(s)\tilde{M}(s)^{-T} = \tilde{M}(s)^{-1}\tilde{L}(s)\tilde{D}(s)$. The matrices on the left and right hand side of the last equation are symmetric, lower triangular and diagonal, respectively.

Since $\tilde{D}(s)$ is a diagonal matrix then $\tilde{M}(s)^{-1}\tilde{L}(s)$ is also a diagonal matrix. Since the product of $\tilde{M}(s)^{-1}\tilde{L}(s)$ is a unit lower triangular matrix, $\tilde{M}(s)^{-1}\tilde{L}(s)$ must be a identity matrix. Thus, $\tilde{A}(s) = \tilde{M}(s)\tilde{D}(s)\tilde{M}(s)^T = \tilde{L}(s)\tilde{D}(s)\tilde{L}(s)^T$ is the path of LDL^T decomposition of $\tilde{A}$, for $s \in \mathbb{R}_0^+$. Let $L = \mathcal{R}(\tilde{L})$ and $D = \mathcal{R}(\tilde{D})$ then we have $A \nabla L \otimes D \otimes L^T$ where $L \in (\mathbb{S}^{\vee})^{n\times n}$ is a unity lower triangular matrix and $D \in (\mathbb{S}^{\vee})^{n\times n}$ is a diagonal matrix. ■

Suppose $A = \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix}$ is a symmetric matrix with $\det(A(1{:}1,1{:}1)) \,\nabla\!\!\!\!/\, \mathcal{E}$. If $M_A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\tilde{L} = \tilde{M}_1^{\,-1} = \begin{bmatrix} 1 & 0 \\ e^{3s} & 1 \end{bmatrix}$ and $\tilde{U} = \tilde{M}_1\tilde{A} = \begin{bmatrix} e^s & e^{2s} \\ 0 & -e^{7s} + e^{5s} \end{bmatrix}$ where $\tilde{A} = \tilde{L}\tilde{U}$. Since $\tilde{L}$ is a unity triangular matrix whose entries belong to S_e, then inverse of $\tilde{L}$ is guaranteed to be exist. It yields $\tilde{L}^{-1}\tilde{A}\left(\tilde{L}^{-1}\right)^T = \tilde{U}\left(\tilde{L}^{-1}\right)^T = \begin{bmatrix} e^s & 0 \\ 0 & -e^{7s} + e^{5s} \end{bmatrix}$ where $\tilde{U}\left(\tilde{L}^{-1}\right)^T$is a diagonal matrix and $\tilde{d}_{ii} = \tilde{u}_{ii}$, for $i = 1,2$ Therefore, $\tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T$ where $\tilde{D} = \begin{bmatrix} e^s & 0 \\ 0 & -e^{7s} + e^{5s} \end{bmatrix}$ $\sim \begin{bmatrix} e^s & e^{2s} \\ 0 & -e^{7s} \end{bmatrix}$, $s \to \infty$. Let $L = \mathcal{R}(\tilde{L})$ and $D = \mathcal{R}(\tilde{D})$. Then $L = \begin{bmatrix} 0 & \mathcal{E} \\ 3 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & \mathcal{E} \\ \mathcal{E} & \ominus 7 \end{bmatrix}$ where $L \otimes D \otimes L^T = \begin{bmatrix} 1 & 4 \\ 4 & 7^{\bullet} \end{bmatrix} \nabla \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix} = A$.

Theorem 19. (The Cholesky Decomposition in $\mathbb{S}$)

If $A \in \mathbb{S}^{n\times n}$ is a symmetric matrix with $\det(A(1{:}k,1{:}k)) \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $k = 1,2,\dots,n$, then there is a lower triangular matrix $B \in (\mathbb{S}^{\vee})^{n\times n}$ whose diagonal entries $b_{ii} \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $i = 1,2,\dots,n$, such that $A \nabla B \otimes B^T$

Proof. By Theorem 18, since A is a symmetric matrix and $\det(A(1{:}k,1{:}k)) \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $k = 1,2,\dots,n$, then there are a unity lower triangular matrix $L \in (\mathbb{S}^{\vee})^{n\times n}$ and a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \dots, d_n) \in (\mathbb{S}^{\vee})^{n\times n}$ such that $A \nabla L \otimes D \otimes L^T$. We assume that A is a signed matrix.

Let $\tilde{A} = \mathcal{F}(A, N, \cdot)$ with $N \in \mathbb{R}_0^{n\times n}$, $n_{ij} = 1$ for all $i,j = 1,2,\dots n$. The entries in $\tilde{A}$ are $\tilde{a}_{ij}(s) = \gamma_{ij}e^{c_{ij}s}$ for all $s \in \mathbb{R}_0^+$, where $\gamma_{ij} \in \{-1,1\}$ and $c_{ij} = |c_{ij}|_{\oplus} \in \mathbb{R}_{\max}$ for all $i,j = 1,2,\dots,n$. According to the steps in obtaining the LDL^T decomposition, we get $\tilde{A}(s) = \tilde{L}(s)\tilde{D}(s)\tilde{L}(s)^T$, for $s \in \mathbb{R}_0^+$, which is the path of the LDL^Tdecomposition of $\tilde{A}$. Note that $\tilde{A}(s)$ is a symmetric matrix with $\det(A(1{:}k,1{:}k)) > 0$ for $k = 1,2,\dots,n$ and $\tilde{L}(s)$ is a unity lower triangular matrix. Thus, the value of diagonal entries of $\tilde{D}(s)$ must be positive.

Let $\tilde{B}(s) = \tilde{L}(s)\tilde{D}'(s)$ with $\tilde{D}'(s)$ is a diagonal matrix whose diagonal entries are the square roots of the diagonal entries of $\tilde{D}(s)$. Thus, the diagonal entries of $\tilde{D}'(s)$ is also positive. Note that $\tilde{L}(s)$ is a unity lower triangular matrix and $\tilde{D}'(s)$ is a diagonal matrix whose diagonal entries are positive. Thus, $\tilde{B}(s)$ is a lower triangular matrix whose diagonal entries are positive. It yields

$$\tilde{A}(s) = \tilde{L}(s)\tilde{D}(s)\tilde{L}(s)^T = \left(\tilde{L}(s)\tilde{D}'(s)\right)\left(\tilde{L}(s)\tilde{D}'(s)\right)^T,$$

for $s \in \mathbb{R}_0^+$. If $\tilde{B}(s) = \left(\tilde{L}(s)\tilde{D}'(s)\right)$ then $\tilde{A}(s) = \tilde{B}(s)\tilde{B}(s)^T$ is the path of the Cholesky decomposition of $\tilde{A}(s)$. Let $B = \mathcal{R}(\tilde{B})$. We have $A \nabla B \otimes B^T$ where $B \in (\mathbb{S}^{\vee})^{n\times n}$ is a lower triangular matrix and $b_{ii} \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $i = 1,2,\dots,n$. ■

It is difficult to determine the Cholesky decomposition of a matrix over $\mathbb{R}_{\max}$ in the $\mathbb{R}_{\max}$ sense, because there is no additive inverse for every element in $\mathbb{R}_{\max}$, except for the zero element. If the Cholesky decomposition in $\mathbb{R}_{\max}$ is carried out in $\mathbb{S}$, then the problem concerning the absence of the additive inverse in $\mathbb{R}_{\max}$ can be solved by using the minus operator. Since $\mathbb{R}_{\max}$ is a proper subset of $\mathbb{S}$ then $(\mathbb{R}_{\max})^{n\times n}$ is also a proper subset of $\mathbb{S}^{n\times n}$. Thus, the Cholesky decomposition of a matrix over $\mathbb{R}_{\max}$ is a special case of Theorem 19, and can be determined by a process in $\mathbb{S}$ through the link as defined in Definition 10 and 11.

Corollary 20 (The Cholesky Decomposition in $\mathbb{R}_{\max}$)

If $A \in (\mathbb{R}_{\max})^{n\times n} \subset \mathbb{S}^{n\times n}$ is a symmetric matrix with $\det(A(1{:}k,1{:}k)) \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $k = 1,2,\dots,n$ then there is a lower triangular matrix $B \in (\mathbb{S}^{\vee})^{n\times n}$ whose diagonal entries $b_{ii} \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$, $i = 1,2,\dots,n$ such that $A \nabla B \otimes B^T$

Proof. The proof of this theorem is analogous to that of Theorem 19. ■

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \in (\mathbb{R}_{\max})^{n\times n} \subset \mathbb{S}^{n\times n}$ is a symmetric matrix with $\det(A(1{:}k,1{:}k)) \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $k = 1,2$. If $M_A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\tilde{A}(s) = \mathcal{F}(A, M_A, s) = \begin{bmatrix} e^s & e^{2s} \\ e^{2s} & e^{5s} \end{bmatrix}$, for $s \in \mathbb{R}_0^+$. Since $\tilde{M}_1 = \tilde{I}_2 - \tilde{\tau}^{(1)}\tilde{e}_1^T$ we have $\tilde{L} = \tilde{M}_1^{\,-1} = \begin{bmatrix} 1 & 0 \\ e^s & 1 \end{bmatrix}$, $\tilde{U} = \tilde{M}_1\tilde{A} = \begin{bmatrix} e^s & e^{2s} \\ 0 & -e^{3s} + e^{5s} \end{bmatrix}$, where $\tilde{A} = \tilde{L}\tilde{U}$. Note that $\tilde{L}$ is a unity lower triangular matrix. Thus, inverse of $\tilde{L}$ is guaranteed to be exist and $\tilde{L}^{-1}\tilde{A}\left(\tilde{L}^{-1}\right)^T = \tilde{U}\left(\tilde{L}^{-1}\right)^T = \begin{bmatrix} e^s & 0 \\ 0 & -e^{3s} + e^{5s} \end{bmatrix} = \tilde{D}$. Since $\tilde{D} = \begin{bmatrix} e^s & 0 \\ 0 & -e^{3s} + e^{5s} \end{bmatrix}$ then $\tilde{D} \sim \begin{bmatrix} e^s & 0 \\ 0 & e^{5s} \end{bmatrix}$, $s \to \infty$. The path of LDL^T decomposition of $\tilde{A}$ is $\tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T$ with $\tilde{L} = \begin{bmatrix} 1 & 0 \\ e^s & 1 \end{bmatrix}$.

Since $\tilde{D}' = \begin{bmatrix} \sqrt{e^s} & 0 \\ 0 & \sqrt{e^{5s} - e^{3s}} \end{bmatrix}$, we have $\tilde{B} = \tilde{L}\tilde{D}' = \begin{bmatrix} \sqrt{e^s} & 0 \\ e^s\sqrt{e^s} & \sqrt{e^{5s} - e^{3s}} \end{bmatrix}$ which is a lower triangular matrix whose diagonal entries are positive. Thus, the path of the Cholesky decomposition of $\tilde{A}$ is $\tilde{A} = \tilde{B}\tilde{B}^T$ with $\tilde{B} = \begin{bmatrix} \sqrt{e^s} & 0 \\ e^s\sqrt{e^s} & \sqrt{e^{5s} - e^{3s}} \end{bmatrix} \sim \begin{bmatrix} e^{\frac{1}{2}s} & 0 \\ e^{\frac{3}{2}s} & e^{\frac{5}{2}s} \end{bmatrix}$, $s \to \infty$. Let $B = \mathcal{R}(\tilde{B})$.

We have $B = \begin{bmatrix} \frac{1}{2} & \mathcal{E} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$, where $B \otimes B^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A \nabla A$. The cholesky decomposition of $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \in (\mathbb{R}_{\max})^{n\times n}$ can be determined in terms of $\mathbb{S}$.

Let $P = \begin{bmatrix} 1 & 2 \\ 2 & 5^{\bullet} \end{bmatrix} \in \mathbb{S}^{n\times n}$ with $P \nabla \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A$. The Cholesky decomposition of P is $P \nabla B \otimes B^T$ with $B = \begin{bmatrix} \frac{1}{2} & \mathcal{E} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$.

VI. APPLICATION OF THE CHOLESKY DECOMPOSITION IN THE LINEAR BALANCE SYSTEMS

The linear balance systems in $\mathbb{S}$ plays a similar role to the system of linear equations in conventional algebra. The linear balance system in $\mathbb{S}$ can be also used to solve a system of linear equations in $\mathbb{R}_{\max}$ which can not be solved in $\mathbb{R}_{\max}$.

Suppose that $A \otimes x \nabla b$ is a linear balance systems in the symmetrized max-plus algebra, where the Cholesky decomposition of A can be determined. In this Section, we show that the use of using the Cholesky decomposition is simpler than that of cofactor and generalized inverse methods as in [3]-[4], respectively to compute the solution for the linear balance systems $A \otimes x \nabla b$.

Theorem 21. [3] *Let $A \in \mathbb{S}^{n\times n}$ with $\det(A) \in \mathbb{S}^{\vee} - \{\mathcal{E}\}$, $b \in \mathbb{S}^n$ and $\mathrm{cof}(A)^T \otimes b \in (\mathbb{S}^{\vee})^n$. Then there is a unique solution of $A \otimes x \nabla b$ and it satisfies*

$$x \nabla (\mathrm{cof}(A)^T \otimes b)^T \otimes \det(A)^{-1}$$

Theorem 22. [4] *Let $A \in \mathbb{S}^{n\times n}$. If X is any matrix satisfying $A \otimes X \otimes A \nabla A$, then $A \otimes x \nabla b$ has a solution if and only if $A \otimes X \otimes b \nabla b$, in which case the most general solution is*

$$x = X \otimes b \oplus (I_n \ominus X \otimes A) \otimes h$$

where h is arbitrary.

Algorithm 23. *Let the linear balance systems $A \otimes x \nabla b$ where $A \in \mathbb{S}^{n\times n}$ is a symmetric matrix and $\det\big(A(1{:}k, 1{:}k)\big) \in \mathbb{S}^{\oplus} - \{\mathcal{E}\}$ for $k = 1,2,\ldots,n$. If the Cholesky decomposition of A is $A \nabla B \otimes B^T$ then the solution of the linear balance systems $A \otimes x \nabla b$ can be solved by taking the following steps*:

1. *Solving the linear balance systems $B \otimes y \nabla b$, where B is a lower triangular matrix. For $y \in (\mathbb{S}^{\vee})^n$, supposing $y \nabla y_0$.*
2. *Solving the linear balance systems $B^T \otimes x \nabla y_0$, where B^T is an upper triangular matrix. Supposing $x \nabla x_0$.*
3. *If $x_0 \in (\mathbb{S}^{\vee})^n$, then for all x with $x \nabla x_0$ is a solution.*
   If $x_0 \notin (\mathbb{S}^{\vee})^n$, then $x = x_0$ is a solution.

Let $A \otimes x \nabla b$ is a linear balance systems where the Cholesky decomposition of A can be determined. By using Algorithm 23, the computation to determine solution of the linear balance systems becomes easier and simpler than that by using Theorem 21 and Theorem 22. Since B is a a lower triangular matrix, it only requires a back weak-substitution process to solve the solution of the linear balance systems.

Let $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \nabla \begin{bmatrix} 3 \\ \ominus 1 \end{bmatrix}$ is a linear balance system. Note that the Cholesky decomposition of $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \nabla B \otimes B^T$ with $B = \begin{bmatrix} \frac{1}{2} & \mathcal{E} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$. By using Algorithm 23, the solution of $\begin{bmatrix} \frac{1}{2} & \mathcal{E} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \nabla \begin{bmatrix} 3 \\ \ominus 1 \end{bmatrix}$ is $y_1 \nabla \frac{5}{2}$. Then, by doing a back weak-substitution, $y_2 \nabla \ominus \frac{3}{2}$. Furthermore, the solution of $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \mathcal{E} & \frac{5}{2} \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \nabla \begin{bmatrix} \frac{5}{2} \\ \ominus \frac{3}{2} \end{bmatrix}$ is $x_2 \nabla \ominus -1$. Then, by doing a back weak-substitution, we have $x_1 \nabla 2$. Note that $x_0 = \begin{bmatrix} 2 \\ \ominus -1 \end{bmatrix} \in (\mathbb{S}^{\vee})^2$. Thus, every x that satisties $x \nabla \begin{bmatrix} 2 \\ \ominus -1 \end{bmatrix}$ is solution of the linear balance systems $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \nabla \begin{bmatrix} 3 \\ \ominus 1 \end{bmatrix}$.

Let $A \otimes x = b$ is a linear equation systems in the max-plus algebra, where the Cholesky decomposition of A can be determined in $\mathbb{R}_{\max} \subset \mathbb{S}$. Often times, we can not determine the solution of the linear equation systems $A \otimes x = b$ in the max-plus algebra. Then, we can view the linear equation systems $A \otimes x = b$ in $\mathbb{R}_{\max}$ as the linear balance systems $A \otimes x \nabla b$ in $\mathbb{S}$.

VII. CONCLUSION

The existence of Cholesky decomposition of matrices over the symmetrized max-plus algebra can be determined by using a link between conventional algebra and the symmetrized max-plus algebra. This is a class of functions that are analytic and can be written as sums or series of exponentials in neighborhood of ∞. The Cholesky decomposition of matrices over the max-plus algebra can be determined as a special case of the Cholesky decomposition in the symmetrized max-plus algebra. The Cholesky decomposition can be used to simplify the computations in determining the solution of the linear balance systems. This is a consequence of the form of matrix in the Cholesky decomposition which is a triangular matrix.

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