Further Study for Inventory Models with Compounding

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Abstract—**Recently, one of the solutions to the inventory problem is the construction of an approximated inventory model, which can be used to derive closed-form optimal solutions. However, this method fails to provide a reasonable explanation for the derivation of the key steps to obtain an approximated formula for the exponential function. The purpose of our study is threefold. First, we refer to an exponential function upper bound to patch the motivation gap. Second, we provide two other proofs for the upper bound of the exponential function or a lower bound of the negative exponential function. Third, we cite a recently published paper that directly applies the Taylor's series expansion of the exponential function to obtain the desired approximated inventory model. Consequently, our discussions will help researchers realize these kind of inventory models.**

Index Terms—**Inventory model, Compounding, Economic ordering quantity, Taylor's series expansion**

I. Introduction

Çalışkan [1] developed a new inventory model with (i) setup cost, (ii) estimated holding cost, and (iii) the monetary value of the inventory with compounding interest. He first computed the interest for one replenishment, which will then generate interests for the remaining duration of the year. He considered partial cycles and then constructed a new inventory model. Çalışkan [2] wrote the second paper that investigates this new inventory model with compounding interest. A trend to develop approximated solutions for inventory models has recently emerged. For example, Çalışkan [3] studied the approximated inventory model proposed by Widyadana et al. [4] through a simple derivation to avoid applying the cost-difference comparison method developed by Wee et al. [5]. Çalışkan [6] also examined the approximated inventory model proposed by Chung and Ting [7] and then obtained a new approximated optimal solution. Moreover, Çalışkan [8] considered Ghare and Schrader [9] and Widyadana et al. [4] by using a different approach to find the optimal replenishment cycle length with a nested radical expression and a closed-form solution for cubic polynomial in an arccosine formation.

Moreover, Çalışkan [8] developed an approximated solution. Based on the above discussions, these kinds of inventory models are hot research topics in recent years. Hence, in this paper, we will provide two improvements for Çalışkan [2] to help practitioners realize this important paper proposed by Çalışkan [2].

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II. Notation and assumptions

To be compatible with Çalışkan [2], we adopted the same notation and assumption as that of Çalışkan [2].

Notation

D is the constant demand per year.

S is the setup cost per replenishment.

T is the duration for one replenishment.

$I(t)$ is the inventory level, with $I(0) = Q$ and $I(T) = 0$.

Q is the ordering quantity per replenishment with Q= DT.

c is the purchasing cost per item.

i is the holding cost that is expressed as a fraction of c per year and then the holding cost is ic.

r is the annual interest rate.

Assumptions

The average holding cost is estimated as the average of the maximum inventory level, $I(0) = Q$, and the ending inventory level, $I(T) = 0$, to imply the total holding cost is $ic\frac{Q}{2}T$ and the average holding cost is $ic\frac{Q}{2}$.

The lead time is neglected.

The average setup cost is $\frac{S}{T} = \frac{DS}{Q}$.

The average purchasing cost with compounding is shown as $c(e^r - 1)\left[\frac{Q}{1-e^{-rQ/D}} - \frac{D}{r}\right]$ that was discussed in Çalışkan [1].

III. Review of Çalışkan [2]

We cite from Çalışkan [2], "Considering continuous compounding of interest at an annual rate of r, Çalışkan [1] formulated the annual average total cost per year as follows.

$$TC(Q) = \frac{DS}{Q} + ic\frac{Q}{2} + c(e^r - 1)\left[\frac{Q}{1-e^{-rQ/D}} - \frac{D}{r}\right]. \quad (C1)$$

Taking the first derivative of (C1) results in the following.

$$\frac{d}{dQ}TC(Q) = \frac{-DS}{Q^2} + \frac{ic}{2} + c(e^r - 1) \times \left[\frac{1-e^{-rQ/D}-(rQ/D)e^{-rQ/D}}{\left(1-e^{-rQ/D}\right)^2}\right]. \quad (C2)"$$

Remark.

(C1) indicates that it is Equation (1) of Çalışkan [2]. In Çalışkan [2], he claims that solving for $\frac{d}{dQ}TC(Q)=0$ based on (C2), however, a closed-form solution cannot be derived for this.

Hence, he tried to find a new inventory system to approximate his inventory model of (C1), and then the closed-form optimal solution can be derived for his new inventory system.

We cite from Çalışkan [2], "The Taylor series expansion for the exponential function is as follows.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \quad \text{(C4)"}$$

We further cite from Çalışkan [2], "The following approximation is quite accurate for small values of x.

$$e^x \approx 1 + x + \frac{x^2}{2-x} = \frac{2-x}{2-x} + \frac{2x-x^2}{2-x} + \frac{x^2}{2-x} = \frac{2+x}{2-x}. \quad \text{(C5)"}$$

Substituting (C5) in (C1), we obtain the following.

$$TC(Q) = \frac{DS}{Q} + ic\frac{Q}{2} + c(e^r-1)\left[\frac{Q}{1-\frac{2-(rQ/D)}{2+(rQ/D)}} - \frac{D}{r}\right],$$

$$TC(Q) = \frac{DS}{Q} + ic\frac{Q}{2} + c(e^r-1)\left[\frac{2D+rQ}{2r} - \frac{D}{r}\right], \quad \text{(C6)}$$

$$TC(Q) = \frac{DS}{Q} + c[i+(e^r-1)]\frac{Q}{2}.$$

The optimal solution to (C6) is obvious and it is the same as the one derived in Çalışkan [1].

$$Q^* = \sqrt{\frac{2DS}{c[i+(e^r-1)]}}. \quad \text{(C7)"}$$

IV. Our comments for Çalışkan [2]

Çalışkan [2] did not provide us with any preparation for his computation of (C5). Based on (C4), researchers can recall the well-known approximated formula,

$$e^x \approx 1 + x + \frac{x^2}{2}. \quad (4.1)$$

However, in (C5), Çalışkan [2] directly mentioned that

$$e^x \approx 1 + x + \frac{x^2}{2-x}, \quad (4.2)$$

without any motivation to help readers. Çalışkan [2] should provide a motivation for readers why he changed from $\frac{x^2}{2}$ to $\frac{x^2}{2-x}$. In the following, we will try to provide some reasonable motivations for the solution procedure proposed by Çalışkan [2]. We cite Proposition 1 of Chung [10],

"Proposition 1.

(a) Let $g(x) = e^x - 1 - x - \frac{x^2}{2}$ for $x>0$. Then $g(x)>0$ for $x>0$.

(b) $e^x < \frac{2+x}{2-x}$ for $2 > x > 0$."

Based on Proposition 1, (b) of Chung [10], researchers can consider $\frac{2+x}{2-x}$ as an approximation for e^x. Consequently, researchers can evaluate that

$$e^x - 1 - x \approx \frac{2+x}{2-x} - 1 - x = \frac{x^2}{2-x}. \quad (4.3)$$

Based on the above discussion, we provide a reasonable motivation why Çalışkan [2] assumed that $e^x \approx 1 + x + \frac{x^2}{2-x}$. For completeness, we recall that there are two other sources for the approximation of $e^x \approx \frac{2+x}{2-x}$.

We cite from Wan and Chu [11], "LEMMA 1. $xe^{-x} + 2e^{-x} + x - 2 > 0$, for $x > 0$.
Proof. See appendix."
and we also cite from Wan and Chu [11],
"PROOF OF THE LEMMA 1. Recall the Remark 1 of Rachamadugu [12]. He considered the Taylor series of $rT(1+e^{rT}) + 2(1-e^{rT}) = \sum_{n=2}^{\infty}\frac{n-2}{n!}(e^{rT})^n$ then $rT(1+e^{rT}) + 2(1-e^{rT}) \geq 0$.

On the other hand, from Theorem 2(f) of Chung and Lin [13], they derived a very sophisticated procedure. First, they constructed a decreasing function, $\frac{1}{x} - \frac{1}{e^x-1}$ to verify that $0 < \frac{1}{x} - \frac{1}{e^x-1} < \frac{1}{2}$. By $\frac{1}{x} - \frac{1}{e^x-1} < \frac{1}{2}$, then $e^{-x} > \frac{2-x}{2+x}$.

Here, we offer simple proof. Let $p(x) = (2+x)e^{-x} - 2 + x$. For $x>0$, since $p(0)=0$ and $\frac{d}{dx}p(x) = e^{-x}(e^x - 1 - x) > 0$, we derive $p(x) > 0$."

Therefore, based on various principles, three different proofs were provided by Chung and Lin [13], Wan and Chu [11], and Chung [10] to find an upper bound of e^x or a lower bound for e^{-x} that can be treated as the possible sources for $e^x \approx \frac{2+x}{2-x}$.

We recall that Rachamadugu [12] studied an inventory model with net present value. He tried to prove the objective function is convex such that the zero of the first derivative is the optimal minimum solution. The objective function of Rachamadugu [12] is expressed as,

$$ANN(T) = \frac{Sr}{1-e^{-rT}} + \frac{hD}{r}\frac{e^{-rT}-1+rT}{1-e^{-rT}} + \frac{DprT}{1-e^{-rT}}, \quad (4.4)$$

and then he derived that

$$\frac{d^2}{dT^2}ANN(T) = \frac{Sr^3e^{-rT}(1+e^{-rT})}{(1-e^{-rT})^3} + \frac{D(h+pr)}{(1-e^{-rT})^3}[(2+rT)e^{-rT} - 2 + rT]. \quad (4.5)$$

It is well known that $1 - e^{-rT} > 0$, for $T > 0$. Hence, Rachamadugu [12] had verified that

$$(2+rT)e^{-rT} - 2 + rT > 0, \quad (4.6)$$

for $T>0$, to support his assertion of convexity. We can claim that the appearance of Equation (4.6) in Rachamadugu [12] is the most natural way to provide a lower bound for the negative exponential function.

Based on our citations, we provide three possible sources that researchers used a lower bound of e^{-x}, as

$$e^{-x} > \frac{2-x}{2+x}, \quad (4.7)$$

or an upper bound of e^x, as

$$\frac{2+x}{2-x} > e^x, \quad (4.8)$$

to present a reasonable motivation for the substitution

$$e^x \approx \frac{2+x}{2-x}. \quad (4.9)$$

Next, we will provide an alternative approach without referring to Proposition 1, (b) of Chung [10], or Lemma 1 of Wan and Chu [11].

We concentrate on the term "$\frac{Q}{1-e^{-rQ/D}}-\frac{D}{r}$" of (C1) and then we evaluate that

$$\frac{Q}{1-e^{-rQ/D}}-\frac{D}{r}=\left(\frac{D}{r}\right)\frac{e^{-rQ/D}-1+rQ/D}{1-e^{-rQ/D}}. \tag{4.10}$$

We apply the well-known result of Taylor's series expansion of the exponential function, that is, (C4), to directly imply that

$$e^{-x}-1+x\approx\frac{x^2}{2}, \tag{4.11}$$

and

$$1-e^{-x}\approx x. \tag{4.12}$$

Based on Equation (4.11), we know that

$$e^{-rQ/D}-1+rQ/D\approx\frac{(rQ/D)^2}{2}, \tag{4.13}$$

and owing to Equation (4.12), we derive that

$$1-e^{-rQ/D}\approx rQ/D. \tag{4.14}$$

We plug Equations (4.13) and (4.14) into Equation (4.10), then

$$\frac{Q}{1-e^{-rQ/D}}-\frac{D}{r}=\left(\frac{D}{r}\right)\frac{(rQ/D)^2/2}{rQ/D}=\frac{Q}{2}. \tag{4.15}$$

If we plug our results of Equation (4.15) into (C1), then we can imply the same results as derived by Çalışkan [2] in (C6) without referring to the interesting substitution, $e^x\approx\frac{2+x}{2-x}$, of (C5).

Based on our above derivation, we point out that by applying the well-known Taylor's series expansion of the exponential function, the desired approximated inventory system can be obtained without using the sophisticated derivation of (C5).

At last, there are several related papers: Hou et al. [14], Gaithuru et al. [15], Akafan et al. [16], Arwatchananukul et al. [17], Susanti et al. [18], Bindu et al. [19], and Wu et al. [20] that are worthy to mention.

V. Numerical examples

In Çalışkan [2], he did not provide numerical examples to illustrate his discussions. We refer to Çalışkan [21] to find data for parameters: $c=10$, $D\in\{500,10000\}$, $S=50$, $r\in\{0.25,0.20,0.15,0.10,0.05\}$, and we assume $i=0.05$. Hence, for $D=500$, we list the results in the next Table 1. From Table 1, we observe when $D=500$ that Q^* have a negative relationship with the annual interest rate, r, and $TC(Q^*)$ have a positive relationship with the annual interest rate, r.

Table 1. With $D=500$, and variation of r.

r	0.05	0.10	0.15	0.20	0.25
Q^*	222.20	179.51	153.63	135.73	122.35
$TC(Q^*)$	225.023	278.54	325.45	368.38	408.67

For $D=10000$, we list the results in the next Table 2.

Table 2. With $D=10000$, and variation of r.

r	0.05	0.10	0.15	0.20	0.25
Q^*	993.70	802.78	687.07	607.01	547.16
$TC(Q^*)$	1006.34	1245.68	1455.45	1647.43	1827.64

From Table 2, we notice when $D=10000$ that Q^* have a negative relationship with the annual interest rate, r, and $TC(Q^*)$ have a positive relationship with the annual interest rate, r.

VI. Another related problem

A colleague, Professor Robert Lin raised an open question: How did researchers use algebraic methods to solve the following minimum problem?

To find the minimum value and minimum point of

$$\frac{h+w}{n+1}x^{n+1}-wx+w\left(\frac{w}{h+w}\right)^{1/n}, \tag{6.1}$$

for $x\geq 0$, by algebraic methods.

Hence, we assume our objective function, denoted as f(x) such that we define our goal,

$$f(x)=\frac{h+w}{n+1}x^{n+1}-wx+w\left(\frac{w}{h+w}\right)^{1/n}. \tag{6.2}$$

We assume that a point, say a, attains the minimum, then we compute $f(x)-f(a)$ to imply that

$$f(x)-f(a)=\frac{h+w}{n+1}\left(x^{n+1}-a^{n+1}\right)-w(x-a)$$
$$=(x-a)g(x), \tag{6.3}$$

with an auxiliary function,

$$g(x)=\frac{h+w}{n+1}(x^n+x^{n-1}a+\cdots+a^{n-1}x+a^n). \tag{6.4}$$

From the fact that the objective function f(x) is a polynomial and $f(x)-f(a)$ is also a polynomial in the variable x, such that we replace x by $(x-a)+a$ rewriting the polynomial in the new variable $(x-a)$, and then we obtain

$$f(x)-f(a)=$$
$$a_0+a_1(x-a)+\sum_{k=2}^{n+1}a_k(x-a)^k. \tag{6.5}$$

In the following, we will try to prove two goals for the coefficient of the constant term and the coefficient of the linear term: $a_0=0$ and $a_1=0$.

If we plug $x=a$ into Equation (6.5), and then we derive that

$$0=a_0, \tag{6.6}$$

to show that our first goal, $a_0=0$, is verified.

Next, we assume that

$$\sum_{k=2}^{n+1}|a_k|=M. \tag{6.7}$$

We will divide our proof for the second goal into the following three cases: (i) $a_1>0$, (ii) $a_1<0$, and (iii) $a_1=0$.

For Case (i), with $a_1>0$, we will choose a special point, denoted as x_1 that satisfies

$$x_1=a-P, \tag{6.8}$$

where

$$P=\min\left\{1,\frac{a_1}{1+M}\right\}. \tag{6.9}$$

Based on Equation (6.9), we know that

$$|x_1-a|^k\leq|x_1-a|^2, \tag{6.10}$$

for $k=2,3,\ldots,n+1$.

We evaluate $f(x_1)-f(a)$ to derive that

$$f(x_1)-f(a)=a_1(x_1-a)+\sum_{k=2}^{n+1}a_k(x_1-a)^k$$
$$\leq a_1(x_1-a)+\sum_{k=2}^{n+1}|a_k|(|x_1-a|)^k$$
$$\leq a_1(x_1-a)+\sum_{k=2}^{n+1}|a_k|(|x_1-a|)^2$$
$$\leq a_1(x_1-a)+M(x_1-a)^2$$
$$=(x_1-a)[a_1+M(x_1-a)]$$

$$= -P[a_1 + M(-P)]$$
$$< (-P)\left[a_1 + M(-\frac{a_1}{1+M})\right]$$
$$= (-P)a_1\frac{1}{1+M} < 0. \qquad (6.11)$$

The finding of Equation (6.11) is violated our assumption that a is designed as the minimum point. Hence, Case (i) does not exist.

For Case (ii), with $a_1 < 0$, we will select a special point, denoted as x_2 that satisfies

$$x_2 = a + Q, \qquad (6.12)$$

where

$$Q = \min\left\{1, \frac{-a_1}{1+M}\right\}. \qquad (6.13)$$

Based on Equation (6.13), we still derive the following result as Equation (6.10) that

$$|x_1 - a|^k \le |x_1 - a|^2, \qquad (6.14)$$

for $k = 2,3,\dots,n+1$.

We compute $f(x_2) - f(a)$ to derive that

$$f(x_2) - f(a) = a_1(x_2 - a) + \sum_{k=2}^{n+1} a_k(x_2 - a)^k$$
$$\le a_1(x_2 - a) + \sum_{k=2}^{n+1} |a_k|(|x_2 - a|)^k$$
$$\le a_1(x_2 - a) + \sum_{k=2}^{n+1} |a_k|(|x_2 - a|)^2$$
$$\le a_1(x_2 - a) + M(x_2 - a)^2$$
$$= (x_2 - a)[a_1 + M(x_2 - a)]$$
$$= Q[a_1 + MQ]$$
$$< Q\left[a_1 + M(\frac{-a_1}{1+M})\right]$$
$$= Qa_1\frac{1}{1+M} < 0. \qquad (6.15)$$

The result of Equation (6.15) is against our assumption that a is designed as the minimum point. Hence, Case (ii) does not exist.

From Cases (i) and (ii) both do not exist, we prove our second goal, $a_1 = 0$.

After we verify that $a_0 = 0$ and $a_1 = 0$, we can rewrite Equation (6.5) as

$$f(x) - f(a) = \sum_{k=2}^{n+1} a_k(x - a)^k$$
$$= (x - a)^2 p(x), \qquad (6.16)$$

where

$$p(x) = \sum_{k=2}^{n+1} a_k(x - a)^{k-2}. \qquad (6.17)$$

Now we compare Equations (6.3) and (6.16) to find that g(x) must contain a factor of (x-a) such that we know $g(a) = 0$. Therefore, using Equation (6.4), it follows that the minimum point is derived as

$$a = \left(\frac{w}{h+w}\right)^{1/n}, \qquad (6.18)$$

and then we recall Equation (6.4) to imply that

$$p(x) = \frac{g(x)}{x-a}. \qquad (6.19)$$

Using Equation (6.19), we take a long division to derive another more exact expression of p(x) as

$$p(x) = \frac{h+w}{n+1}\sum_{k=0}^{n-1}(k+1)\left(\frac{w}{h+w}\right)^{k/n} x^{n-1-k}. \qquad (6.20)$$

From Equation (6.20), we know that $p(x)$ is a positive function for the domain $x > 0$ to ensure our solution of Equation (6.18) is the minimum point. For completeness, we compute the minimum value,

$$f\left(\left(\frac{w}{h+w}\right)^{1/n}\right) = \frac{w}{n+1}\left(\frac{w}{h+w}\right)^{1/n}. \qquad (6.21)$$

VII. The Simple Method of Çalışkan [3]

In this section, we will follow the simple method proposed by Çalışkan [3] to solve the minimum problem of Equation (6.2).

We assume x^* is the minimum solution for f(x) of Equation (6.2), and then two new expressions, x_+ and x_-, with

$$x_+ = x^* + \Delta x, \qquad (7.1)$$

and

$$x_- = x^* - \Delta x, \qquad (7.2)$$

where Δx is a small positive number.

We follow the solution procedure of Çalışkan [3] to compute $f(x_+) - f(x^*)$ and $f(x_-) - f(x^*)$, owing to $f(x^*)$ is the minimum value such that we know that

$$f(x_+) - f(x^*) \ge 0, \qquad (7.3)$$

and

$$f(x_-) - f(x^*) \ge 0. \qquad (7.4)$$

Based on Equation (6.2), we write down the detailed expression of Equations (7.3) and (7.4) as

$$\frac{h+w}{n+1}(x_+^{n+1} - (x^*)^{n+1}) - w(x_+ - x^*) \ge 0, \qquad (7.5)$$

and

$$\frac{h+w}{n+1}(x_-^{n+1} - (x^*)^{n+1}) - w(x_- - x^*) \ge 0. \qquad (7.6)$$

We divide Equation (7.5) by $x_+ - x^*$ to imply that

$$\frac{h+w}{n+1}\sum_{k=0}^{n} x_+^{n-k}(x^*)^k \ge w, \qquad (7.7)$$

because of $x_+ - x^* > 0$.

Remark. We must point out that in Çalışkan [3], he claimed that "can further be simplified as follows:" without explicitly informing the readers that he divided his computation by $Q_+ - Q^*$ that is $x_+ - x^*$ in this paper.

We divide Equation (7.6) by $x_- - x^*$ to imply that

$$\frac{h+w}{n+1}\sum_{k=0}^{n} x_-^{n-k}(x^*)^k \le w, \qquad (7.8)$$

because of $x_- - x^* < 0$.

Remark. We must point out that in Çalışkan [3], he claimed that "A similar analysis on ... will result in the following:" without explicitly informing the readers that he divided his computation by $Q_- - Q^*$ that is $x_- - x^*$ in this paper.

We follow the solution procedure of Çalışkan [3] to take the limit $\Delta x \to 0$ such that

$$x_+ \to x^*, \qquad (7.9)$$

and

$$x_- \to x^*. \qquad (7.10)$$

Based on Equation (7.9), we simplify Equation (7.7) as

$$(h + w)(x^*)^n \ge w. \qquad (7.11)$$

Similarly, applying Equation (7.10), we simplify Equation (7.8) as follows

$$(h + w)(x^*)^n \le w. \qquad (7.12)$$

We combine the results of Equations (7.11) and (7.12) to find that

$$(h + w)(x^*)^n = w, \qquad (7.13)$$

and then further simplify the finding of Equation (7.13) to show that

$$x^* = \left(\frac{w}{h+w}\right)^{1/n}, \qquad (7.14)$$

which is the same result of Equation (6.18) as we derived by algebraic methods.

VIII. EXAMINED BY AN ANALYTICAL APPROACH

Based on Equation (6.2), we demonstrate that applying analytical methods to derive the minimum point and minimum value as follows,

$$\frac{d}{dx}f(x) = (h + w)x^n - w, \tag{8.1}$$

and

$$\frac{d^2}{dx^2}f(x) = n(h + w)x^{n-1}. \tag{8.2}$$

From Equations (8.1) and (8.2) we know that the objective function is convex, and then we obtain that the minimum point denoted as x^*, by solving $\frac{d}{dx}f(x) = 0$ to derive that

$$x^* = \left(\frac{w}{h+w}\right)^{1/n}, \tag{8.3}$$

that is the identical result of Equation (6.18) which we obtained by algebraic methods. The second derivative of f(x) is positive to imply the solution of the first derivative is indeed the minimum solution.

IX. OUR COMMENTS FOR SECTION VII

In Section VII, we follow the "simple method" proposed by Çalışkan [3] to find the minimum point. To help readers realize the computations in Section VII, we rewrite our mimic procedure in an abstract expression as follows.

First, we assume that x^* is the minimum solution, and then compute

$$f(x_+) - f(x^*) \geq 0, \tag{9.1}$$

and

$$f(x_-) - f(x^*) \geq 0. \tag{9.2}$$

Second, we evaluate

$$\frac{f(x_+)-f(x^*)}{x_+-x^*} \geq 0, \tag{9.3}$$

and

$$\frac{f(x_-)-f(x^*)}{x_--x^*} \leq 0. \tag{9.4}$$

Third, we take the limit as $\Delta x \to 0$, then

$$\lim_{\Delta x\to 0}\frac{f(x_+)-f(x^*)}{x_+-x^*} \geq 0, \tag{9.5}$$

and

$$\lim_{\Delta x\to 0}\frac{f(x_-)-f(x^*)}{x_--x^*} \leq 0. \tag{9.6}$$

On the other hand, we recall that the definition of derivative for f(x) at x^* is expressed as

$$\frac{df}{dx}(\text{at } x^*) = \lim_{\Delta x\to 0}\frac{f(x)-f(x^*)}{x-x^*}. \tag{9.7}$$

Next, we solve the zeros for the first derivative to yield that

$$\frac{df}{dx}(\text{at } x^*) = 0, \tag{9.8}$$

that is

$$\lim_{\Delta x\to 0}\frac{f(x)-f(x^*)}{x-x^*} = 0. \tag{9.9}$$

Now, we compare Equations (9.5), (9.6), and (9.9) to find that Equation (9.5) is the right-hand limit of Equation (9.9) and Equation (9.6) is the left-hand limit of Equation (9.9).

Hence, we provide a reasonable explanation for the simple method proposed by Çalışkan [3] why this approach can obtain the optimal solution because the simple method proposed by Çalışkan [3] is a complicated version of the calculus.

Hence, we can advise researchers that do not follow the research technique proposed by Çalışkan [3] to write down the solution procedure of calculus with the zeros for the first derivative in a right-hand and left-hand limits process which is not a new approach, but a well-known method in its complex version.

X. APPLICATION OF OUR APPROACH

In this section, we will demonstrate that our approach can be applied to study other inventory models.

A. Review of Wee et al. [5]

We will first recall the solution process of Wee et al. [5], then review the criticism proposed by Çalışkan [23], and then our improvements for Wee et al. [5] and Çalışkan [23].

To be consistent with the expressions in this paper, we use the same notation as Çalışkan [2] to discuss the inventory model and solution procedure in Wee et al. [5] and criticism of Çalışkan [23].

For the first model in Wee et al. [5], there are no shortages such that the optimal replenishment policy is to balance the setup cost and the holding cost.

For a fixed planning horizon, T, Wee et al. [5] uniformly partition it into n parts, and then the ordering quantity, Q_n, that satisfies

$$nQ_n = DT. \tag{10.1}$$

The setup cost is S and the inventory level is $I(t) = Q_n - Dt$, for $0 \leq t \leq Q_n/D$.

The total holding cost for one replenishment is evaluated as

$$ic\int_0^{Q_n/D}(Q_n - Dt)\,dt = \frac{ic}{2D}Q_n^2. \tag{10.2}$$

The total cost for one replenishment is

$$S + \frac{ic}{2D}Q_n^2. \tag{10.3}$$

The duration period for one replenishment is Q_n/D, and then the average cost is derived by Wee et al. [5] as

$$\left(S + \frac{ic}{2D}Q_n^2\right)/Q_n/D = \frac{SD}{Q_n} + \frac{ic}{2}Q_n. \tag{10.4}$$

In Wee et al. [5], they denoted as

$$C(i,T) = \frac{SD}{Q_n} + \frac{ic}{2}Q_n, \tag{10.5}$$

for $i = n-1, n, n+1$.

Wee et al. [5] applied the "cost-difference comparisons" proposed by Minner [24] to assume n is the optimal partition number for this given finite planning horizon, T, and then

$$C(n-1,T) \geq C(n,T), \tag{10.6}$$

and

$$C(n+1,T) \geq C(n,T), \tag{10.7}$$

Wee et al. [5] simplified Equations (9.6) and (9.7) to yield that

$$\frac{2DS}{ic} \leq Q_{n-1}Q_n, \tag{10.8}$$

and

$$Q_{n+1}Q_n \leq \frac{2DS}{ic}. \tag{10.9}$$

Wee et al. [5] mentioned that if the planning horizon T and the optimal partition number n both went to infinite, then $\lim_{n\to\infty}\frac{Q_{n-1}}{Q_n} = \lim_{n\to\infty}\frac{n}{n-1} = 1$ and $\lim_{n\to\infty}\frac{Q_{n+1}}{Q_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1$ to imply that $Q_{n-1} = Q_n = Q_{n+1}$. By observing Equations (10.8) and (10.9), then the optimal ordering quantity converges to the economic ordering quantity $\sqrt{\frac{2DS}{ic}}$.

B. IMPROVMENTS OF ÇALIŞKAN [23]

Next, we provide a brief review of Çalışkan [23]. We recall that Çalışkan [23] improved Wee et al. [5] on the following three issues:

(i) Çalışkan [23] presented a detailed derivation to show how did Wee et al. [5] obtain Equations (10.8) and (10.9) from Equations (10.6) and (10.7).

(ii) Çalışkan [23] tried to explain how to show that Q_n converges to $\sqrt{2DS/ic}$.

(iii) Using Jensen's inequality (Jensen [25]), Çalışkan [23] proved that the inventory model of Equation (10.5) is convex.

Owing to Equation (10.1), researchers knew that $Q_{n^*-1} = DT/(n^*-1)$, $Q_{n^*} = DT/n^*$, and $Q_{n^*+1} = DT/(n^*+1)$ such that

$$Q_{n^*-1} > Q_{n^*} > Q_{n^*+1}. \tag{10.10}$$

Çalışkan [23] computed that

$$C(n-1,T) - C(n,T) = \left(\frac{SD}{Q_{n^*-1}} + \frac{ic}{2}Q_{n^*-1}\right) - \left(\frac{SD}{Q_{n^*}} + \frac{ic}{2}Q_{n^*}\right)$$
$$= \frac{SD}{Q_{n^*-1}Q_{n^*}}(Q_{n^*} - Q_{n^*-1}) + \frac{ic}{2}(Q_{n^*-1} - Q_{n^*}) \geq 0. \tag{10.11}$$

Çalışkan [23] divided Equation (9.11) by $Q_{n^*-1} - Q_{n^*}$ to yield Equation (9.8).

Similarly, Çalışkan [23] computed that

$$C(n^*+1,T) - C(n^*,T)$$
$$= \left(\frac{SD}{Q_{n^*+1}} + \frac{ic}{2}Q_{n^*+1}\right) - \left(\frac{SD}{Q_{n^*}} + \frac{ic}{2}Q_{n^*}\right)$$
$$= \frac{SD}{Q_{n^*-1}Q_{n^*}}(Q_{n^*} - Q_{n^*+1}) + \frac{ic}{2}(Q_{n^*+1} - Q_{n^*}) \geq 0. \tag{10.12}$$

Çalışkan [23] divided Equation (10.12) by $Q_{n^*+1} - Q_{n^*}$ to yield Equation (9.9).

Çalışkan [23] rewrote Equations (9.8) and (9.9) as follows,

$$\frac{2DS}{ic} \leq \left(\frac{n^*}{n^*-1}\right) Q_{n^*}^2, \tag{10.13}$$

and

$$\left(\frac{n^*}{n^*+1}\right) Q_{n^*}^2 \leq \frac{2DS}{ic}. \tag{10.14}$$

Çalışkan [23] combined Equations (10.13) and (10.14) to show that

$$\left(\frac{n^*}{n^*+1}\right) Q_{n^*}^2 \leq \frac{2DS}{ic} \leq \left(\frac{n^*}{n^*-1}\right) Q_{n^*}^2. \tag{10.15}$$

Çalışkan [23] took the limit of $n^* \to \infty$ to evaluate that

$$\lim_{n^*\to\infty}\left(\frac{n^*}{n^*+1}\right) Q_{n^*}^2 \leq \frac{2DS}{ic} \leq \lim_{n^*\to\infty}\left(\frac{n^*}{n^*-1}\right) Q_{n^*}^2, \tag{10.16}$$

and then Çalışkan [23] wrote that

$$Q_{n^*}^2 = \frac{2DS}{ic}, \tag{10.17}$$

and then Çalışkan [23] showed that

$$Q_{n^*} = \sqrt{\frac{2DS}{ic}}. \tag{10.18}$$

Later, we will point out that the discussion for issue (ii) contained severe questionable results.

For issue (iii), we recall the Jensen's inequality that $f(t)$ is a convex function if and only if $f(\alpha a + (1-\alpha)b) \leq \alpha f(a) + (1-\alpha)f(b)$, for $a \neq b$, where a and b are any two points in the domain of $f(t)$ and for any α, with $0 < \alpha < 1$.

For completeness, we recall the proof in Çalışkan [23] to verify the convexity through Jensen's inequality. With $0 < \alpha < 1$, and $Q_n^a \neq Q_n^b$, he computed

$$\alpha C(Q_n^a) + (1-\alpha)C(Q_n^b) - C(\alpha Q_n^a + (1-\alpha)Q_n^b)$$
$$= \alpha\left(\frac{SD}{Q_n^a} + \frac{ic}{2}Q_n^a\right) + (1-\alpha)\left(\frac{SD}{Q_n^b} + \frac{ic}{2}Q_n^b\right)$$
$$- \frac{SD}{\alpha Q_n^a + (1-\alpha)Q_n^b} - \frac{ic}{2}\left[\alpha Q_n^a + (1-\alpha)Q_n^b\right],$$
$$= \alpha\frac{SD}{Q_n^a} + (1-\alpha)\frac{SD}{Q_n^b} - \frac{SD}{\alpha Q_n^a + (1-\alpha)Q_n^b},$$
$$= \frac{SD\Omega}{Q_n^a Q_n^b(\alpha Q_n^a + (1-\alpha)Q_n^b)}, \tag{10.19}$$

where Ω is an abbreviation to stand for

$$\Omega = \alpha Q_n^b(\alpha Q_n^a + (1-\alpha)Q_n^b) + (1-\alpha)Q_n^a(\alpha Q_n^a + (1-\alpha)Q_n^b) - Q_n^a Q_n^b. \tag{10.20}$$

Çalışkan [23] further simplified the expression of Ω in Equation (10.20) as

$$\Omega = \alpha(1-\alpha)(Q_n^a - Q_n^b)^2. \tag{10.21}$$

Based on Equations (10.19) and (10.21), Çalışkan [23] claimed that

$$\alpha C(Q_n^a) + (1-\alpha)C(Q_n^b) - C(\alpha Q_n^a + (1-\alpha)Q_n^b) > 0. \tag{10.22}$$

Hence, by the Jensen's inequality, Çalışkan [23] presented a patchwork for Wee et al. [5] to verify the objective function of this inventory model is a convex function.

C. OUR COMMENTS FOR ÇALIŞKAN [23]

We agree that Çalışkan [23] provided a detailed derivation for ordinary readers on how researchers can obtain Equations (10.8) and (10.9) from Equations (10.6) and (10.7).

However, there are severe questionable findings in the derivations proposed by Çalışkan [23] that are related to the issue (ii).

We recall Equation (10.16) and then Çalışkan [23] shows that

$$\left[\lim_{n^*\to\infty}\left(\frac{n^*}{n^*+1}\right)\right] Q_{n^*}^2 \leq \frac{2DS}{ic} \leq \left[\lim_{n^*\to\infty}\left(\frac{n^*}{n^*-1}\right)\right] Q_{n^*}^2, \tag{10.23}$$

and then owing to $\lim_{n^*\to\infty}\left(\frac{n^*}{n^*+1}\right) = 1$, and $\lim_{n^*\to\infty}\left(\frac{n^*}{n^*-1}\right) = 1$, Çalışkan [23] derived that

$$Q_{n^*}^2 \leq \frac{2DS}{ic} \leq Q_{n^*}^2. \tag{10.24}$$

Based on Equation (10.24), Çalışkan [23] found that $Q_{n^*}^2 = \frac{2DS}{ic}$ as Equation (10.17).

We must point out that Çalışkan [23] overlooked to consider the following problem:

$$\lim_{n^*\to\infty} Q_{n^*}^2 = ?. \tag{10.25}$$

Çalışkan [23] should prove that the sequence $(Q_{n^*}^2)$ or the sequence (Q_{n^*}) is convergent, and then denoted as

$$\lim_{n\to\infty} Q_{n^*} = Q_\infty. \tag{10.26}$$

Consequently, researchers obtain that

$$Q_\infty^2 = \frac{2DS}{ic}, \tag{10.27}$$

and then the desired result of the optimal solution,

$$Q_\infty = \sqrt{\frac{2DS}{ic}}, \tag{10.28}$$

appears.

However, how did researchers prove the convergence of the sequence (Q_{n^*}) that will be a serious open question for future practitioners. Therefore, we can claim that the patchwork proposed by Çalışkan [23] related to the issue (ii) is incomplete.

We agree that Çalışkan [23] applied Jensen's inequality to show the objective function is convex. However, we can provide a simple improvement.

We directly compute

$$\frac{d}{dQ_n}C(Q_n) = \frac{d}{dQ_n}\left(\frac{SD}{Q_n} + \frac{ic}{2}Q_n\right), \tag{10.29}$$

to imply that

$$\frac{d}{dQ_n}C(Q_n) = \frac{-SD}{Q_n^2}, \tag{10.30}$$

and then

$$\frac{d^2}{dQ_n^2}C(Q_n)=\frac{2SD}{Q_n^3}>0. \quad (10.31)$$

From the second derivation of $C(Q_n)$ is positive, we know that the objective function is convex. Our above discussion is straight and many researchers may claim it is too simple that should be written in a research paper. Hence, we can conclude that the third improvement of the issue (iii) related to the convexity of the objective function is redundant.

D. Review of Minner [24]

Minner [24] is the source paper to develop the "cost-difference comparisons" approach. Minner [24] considered the replenishment interval, T/n, for a finite planning horizon, T and then uniformly partitioned into n equal parts. On the other hand, Wee et al. [5] followed Minner [24]. Wee et al. [5] considered the ordering quantity, DT/n, such that $Q_n=DT/n$.

Based on Equation (9.5), Wee et al. [5] partitioned the finite planning horizon, T, into n equal parts, with $nQ_n=DT$. Hence, there are two sequences in Wee et al. [5]. The first sequence is the partition number, (n), and the second one is the ordering quantity, (Q_n).

Wee et al. [5] tried to minimize the average cost for the first replenishment cycle, $[0,T/n]$. We recall Minner [24] that try to minimize the total cost for the entire planning horizon, $[0,T]$. Hence, the objective function in Minner [24] will be expressed as

$$n\left(S+\frac{ic}{2D}Q_n^2\right). \quad (10.32)$$

Owing to $Q_n=DT/n$, Equation (10.32) is rewritten as

$$C(n,T)=nS+\frac{icD}{2n}T^2. \quad (10.33)$$

Consequently, for a given T, the optimal corresponding partition number is denoted as $n^*(T)$, Minner [24] applied the "cost-difference comparisons" to compute

$$C(n^*-1,T)\ge C(n^*,T), \quad (10.34)$$

and

$$C(n^*+1,T)\ge C(n^*,T), \quad (10.35)$$

and then Minner [24] showed that

$$\frac{icD}{2S}T^2\ge n^*(n^*-1), \quad (10.36)$$

and

$$n^*(n^*+1)\ge\frac{icD}{2S}T^2. \quad (10.37)$$

Minner [24] combined Equations (10.36) and (10.37) to show that

$$\sqrt{\frac{2S}{icD}}\sqrt{\frac{n^*-1}{n^*}}\le\frac{T}{n^*(T)}\le\sqrt{\frac{2S}{icD}}\sqrt{\frac{n^*+1}{n^*}}. \quad (10.38)$$

Minner [24] mentioned that let the horizon length T (and $n^*(T)$) tend to infinity, then

$$\lim_{n^*\to\infty}\sqrt{\frac{n^*-1}{n^*}}=\lim_{n^*\to\infty}\sqrt{\frac{n^*+1}{n^*}}=1. \quad (10.39)$$

Minner [24] obtained the optimal cycle length converges to the economic order interval $EOI=\sqrt{2S/icD}$ as the lower and the upper bound converge, and the optimal lot size is

$$EOQ=D(EOI)=\sqrt{2DS/ic}. \quad (10.40)$$

E. Our improvement for Minner [24]

For a given finite planning horizon, T, the optimal partition number is denoted by $n^*(T)$. We provide our revisions for Minner [24] in the following.

We amend Equations (10.36), (10.37), and (10.38) as follows,

$$\frac{icD}{2S}T^2\ge[n^*(T)](n^*(T)-1), \quad (10.41)$$

$$[n^*(T)](n^*(T)+1)\ge\frac{icD}{2S}T^2, \quad (10.42)$$

and

$$\sqrt{\frac{2S}{icD}}\sqrt{\frac{n^*(T)-1}{n^*(T)}}\le\frac{T}{n^*(T)}\le\sqrt{\frac{2S}{icD}}\sqrt{\frac{n^*(T)+1}{n^*(T)}}. \quad (10.43)$$

Based on Equation (10.43), we show that taking the limit of $n^*\to\infty$ in Equation (10.39) proposed by Minner [24] is improper.

We claim that Minner [24] should inform the readers that he construct a new function, denoted as $n^*:(0,\infty)\to\{1,2,\ldots\}$ with $n^*:T\to n^*(T)$ that satisfies

$$C(n^*(T)-1,T)\ge C(n^*(T),T), \quad (10.44)$$

and

$$C(n^*(T)+1,T)\ge C(n^*(T),T). \quad (10.45)$$

Then, Equations (10.41), (10.42), and (10.43) are derived.

Based on Equation (10.42), we know that

$$\lim_{T\to\infty}n^*(T)=\infty. \quad (10.46)$$

Consequently, we derive that

$$\lim_{T\to\infty}\sqrt{\frac{n^*(T)-1}{n^*(T)}}=\lim_{T\to\infty}\sqrt{\frac{n^*(T)+1}{n^*(T)}}=1. \quad (10.47)$$

We compute the limit of $T\to\infty$ for Equation (10.43) and apply Equation (10.46) to obtain that

$$\sqrt{\frac{2S}{icD}}\le\lim_{T\to\infty}\frac{T}{n^*(T)}\le\sqrt{\frac{2S}{icD}}. \quad (10.48)$$

According to Equation (10.48), then

$$\lim_{T\to\infty}\frac{T}{n^*(T)}=\sqrt{\frac{2S}{icD}}, \quad (10.49)$$

the desired economic order interval EOI appears.

Based on our above discussions, we show that Çalışkan [23] did not provide a valuable patchwork for Wee et al. [5], because the core problem why did the sequence (Q_n) converge is not solved. Moreover, through Jensen's inequality, Çalışkan [23] provided a lengthy enhancement for the convexity of the objective function. However, this enhancement can be easily derived by the second derivative.

F. Direction for future research

In this section, we point out that Wee et al. [5] contained questionable findings, and then Çalışkan [23] did not provide a sufficient improvement. Moreover, we present revisions for Minner [24].

In Minner [24], he offered an open question to apply the "cost-difference comparisons" approach to solve inventory models with linear and fixed backorders. We can predict that this open question will be a hot research problem for future researchers.

XI. Conclusion

In this study, we provide reasonable motivation for an interesting approximation employed by Çalışkan [2] using three possible sources. Moreover, we show that researchers can directly apply Taylor's series expansion of the exponential function to obtain their desired approximated inventory model without recalling the interesting approximation of Çalışkan [2]. Our paper is a significant clarification of Çalışkan's work [2].

Moreover, we provide a positive answer to an open question proposed by Professor Lin by three different approaches: (i) Algebraic methods, (ii) A simple method proposed by Çalışkan [3], and (iii) An analytical method. We observe that algebraic methods are fundamental but lengthy because addition, subtraction, multiplication, and division are the only allowable tools for pure algebraic methods. On the other hand, we demonstrate how to follow the simple method proposed by Çalışkan [3] to solve this minimum problem. Additionally, we illustrate how to apply calculus to solve the same minimum problem. We also point out that the simple method proposed by Çalışkan [3] is a complicated version of calculus that is not a useful approach to solving minimum problems. At last, we apply our analytic approach to study a related problem that have examined by Minner [24], Wee et al. [5], and Çalışkan [23]. We point out that Çalışkan [23] did not presented sufficient improvements for Wee et al. [5], and then we offer revisions for Minner [24] to recognize their research issues. Our discussions will help researchers realize these kind of inventory models.

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