Bipolar Fuzzy Comparative UP-Filters

T. Gaketem, P. Khamrot, P. Julatha and A. Iampan

Abstract—**The concept of UP-algebras was introduced by Iampan in 2017 as a generalization of KU-algebras. In UP-algebras, several types of bipolar fuzzy sets (BFSs) are introduced and studied. Therefore, in this article, we will introduce a new concept of BFSs called bipolar fuzzy comparative UP-filters (BFCUPFs) and investigate their essential properties. The relationship between comparative UP-filters (CUPFs) and BFCUPFs is shown. Finally, we'll express a BFCUPF in terms of its cuts, fuzzy sets, and neutrosophic sets.**

Index Terms—**UP-algebra, bipolar fuzzy set, comparative UP-filter, bipolar fuzzy comparative UP-filter, cut.**

I. Introduction and Preliminaries

THE concept of UP-algebras was introduced by Iampan [1] in 2017 as a generalization of KU-algebras [2]. Many researchers study the concept of fuzzy sets and its generalization on UP-algebras as interval-valued fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, neutrosophic sets, BFSs, hesitant fuzzy sets, cubic sets, etc.

Fuzzy sets are a kind of tools use solve uncertainties mathematical structure to supersede a collection of objects whose boundary is vague with initiated by Zadeh in 1965 [3]. In fuzzy set theory, there are numerous types of fuzzy set extensions, such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, and so on. The fuzzy set is expanded by a bipolar-valued fuzzy set with a membership degree range of $[-1, 0] \cup [0, 1]$, which was studied by Zhang in 1994, [4]. Jun and Song [5] utilized the notion of BFSs in BCH-algebras in 2008. Many researchers studied BFSs in algebraic structures such that in 2011, Lee and Jun [6] studied bipolar fuzzy a-ideals of BCI-algebras. Jun et al. [7] investigated bipolar fuzzy CI-algebras in 2012. In 2018, Kawila et al. [8] introduced the concept of bipolar fuzzy UP-algebras. The concept of doubt bipolar fuzzy H-ideals of BCK/BCI algebras was presented by Al-Masarwah and Ahmad [9]. They distinguished between the BCK/BCI-algebras' strong doubt positive t-level cut set and strong doubt negative s-level cut set. In 2021, Muhiuddin and Al-Kadi [10] studied bipolar fuzzy implicative ideals of BCK-algebras. Gaketem and Khamrot [11] introduced the concepts of bipolar fuzzy weakly interior ideals of semigroups. The relationship between bipolar fuzzy weakly interior ideals and bipolar fuzzy left (right) ideals, and the relationship between bipolar fuzzy weakly interior ideals and bipolar fuzzy interior ideals are also discussed. Gaketem et al. [12] introduced the concept of bipolar fuzzy implicative UP-filters (BFIUPFs) in UP-algebras.

The remainder of the article is structured as follows. After this, in Section 1, definitions, examples, and some properties of UP-algebras are provided and used in the next section. Section 2 will introduce the concept of BFCUPFs in UP-algebras and investigate some properties of BFCUPFs. Finally, we'll express a BFCUPF in terms of its cuts, fuzzy sets, and neutrosophic sets. Section 3 will include a summary and a plan.

Before we discussed the concept of UP-algebras, and basic properties for the study of next sections.

A *UP-algebra* [1] is defined as $\mathcal{X} = (\mathcal{X}, \diamond, 0)$ of type $(2, 0)$, where $\mathcal{X}$ is a nonempty set, $\diamond$ is a binary operation on $\mathcal{X}$, and 0 is a fixed element of $\mathcal{X}$ if it satisfies the followings: $\forall x, y, z \in \mathcal{X}$,

$$(y \diamond z) \diamond ((x \diamond y) \diamond (x \diamond z)) = 0, \tag{1}$$
$$0 \diamond x = x, \tag{2}$$
$$x \diamond 0 = 0, \tag{3}$$
$$x \diamond y = 0, y \diamond x = 0 \Rightarrow x = y. \tag{4}$$

According to [1], UP-algebras are a generalization of KU-algebras, as we well know (see [2]).

Unless otherwise indicated, we will assume that $\mathcal{X}$ is a UP-algebra $(\mathcal{X}, \diamond, 0)$.

The binary relation $\leq$ on $\mathcal{X}$ is defined as follows: $\forall x, y \in \mathcal{X}$,

$$x \leq y \Leftrightarrow x \diamond y = 0 \tag{5}$$

and the statements that follow are true (see [1], [13]). $\forall a, x, y, z \in \mathcal{X}$,

$$x \leq x, \tag{6}$$
$$x \leq y, y \leq z \Rightarrow x \leq z, \tag{7}$$
$$x \leq y \Rightarrow z \diamond x \leq z \diamond y, \tag{8}$$
$$x \leq y \Rightarrow y \diamond z \leq x \diamond z, \tag{9}$$
$$x \leq y \diamond x, \text{ in particular, } y \diamond z \leq x \diamond (y \diamond z), \tag{10}$$
$$y \diamond x \leq x \Leftrightarrow x = y \diamond x, \tag{11}$$
$$x \leq y \diamond y, \tag{12}$$
$$x \diamond (y \diamond z) \leq x \diamond ((a \diamond y) \diamond (a \diamond z)), \tag{13}$$
$$((a \diamond x) \diamond (a \diamond y)) \diamond z \leq (x \diamond y) \diamond z, \tag{14}$$
$$(x \diamond y) \diamond z \leq y \diamond z, \tag{15}$$
$$x \leq y \Rightarrow x \leq z \diamond y, \tag{16}$$
$$(x \diamond y) \diamond z \leq x \diamond (y \diamond z), \tag{17}$$
$$(x \diamond y) \diamond z \leq y \diamond (a \diamond z). \tag{18}$$

Iampan [1], Guntasow et al. [14], and Jun and Iampan [15], [16] introduced the concepts of UP-subalgebras, UP-ideals,

Manuscript received January 9, 2022; revised June 12, 2022.

T. Gaketem is a Lecturer of Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand (e-mail: thiti.ga@up.ac.th).

P. Khamrot is a Lecturer of Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Phitsanulok, Phitsanulok 65000, Thailand (e-mail: pk_g@rmutl.ac.th).

P. Julatha is a Lecturer of Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand (e-mail: pongpun.j@psru.ac.th).

A. Iampan is an Associate Professor of Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand (corresponding author to provide phone: +6654466666 ext. 1792; fax: +6654466664; e-mail: aiyared.ia@up.ac.th).

UP-filters, implicative UP-filters, and comparative UP-filters of UP-algebras of as the following definition.

A nonempty subset $\mathcal{S}$ of $\mathcal{X}$ is called

(1) a *UP-subalgebra* of $\mathcal{X}$ if $\forall x, y \in \mathcal{S}$,

$$x \diamond y \in \mathcal{S}, \tag{19}$$

(2) a *UP-ideal* of $\mathcal{X}$ if $\forall x, y, z \in \mathcal{X}$,

$$0 \in \mathcal{S}, \tag{20}$$
$$x \diamond (y \diamond z) \in \mathcal{S}, y \in \mathcal{S} \Rightarrow x \diamond z \in \mathcal{S}, \tag{21}$$

(3) a *UP-filter* (UPF) of $\mathcal{X}$ if (20) and $\forall x, y \in \mathcal{X}$,

$$x \in \mathcal{S}, x \diamond y \in \mathcal{S} \Rightarrow y \in \mathcal{S}, \tag{22}$$

(4) an *implicative UP-filter* (IUPF) of $\mathcal{X}$ if (20) and $\forall x, y, z \in \mathcal{X}$,

$$x \diamond (y \diamond z) \in \mathcal{S}, x \diamond y \in \mathcal{S} \Rightarrow x \diamond z \in \mathcal{S}, \tag{23}$$

(5) a *comparative UP-filter* (CUPF) of $\mathcal{X}$ if (20) and $\forall x, y, z \in \mathcal{X}$,

$$x \diamond ((y \diamond z) \diamond y) \in \mathcal{S}, x \in \mathcal{S} \Rightarrow y \in \mathcal{S}. \tag{24}$$

Jun and Iampan [16] showed that every CUPF is a UPF, but the converse is not true in general.

The concept of BFSs in a nonempty set is now reviewed.

A *bipolar fuzzy set* (BFS) [4] β in a nonempty set $\mathcal{S}$ is an object having the form

$$\beta := \{(x, \beta^-(x), \beta^+(x)) \mid x \in \mathcal{S}\},$$

where $\beta^- : \mathcal{S} \to [-1, 0]$ and $\beta^+ : \mathcal{S} \to [0, 1]$. We'll use the symbol $\beta = (\beta^-, \beta^+)$ for the BFS $\beta = \{(x, \beta^-(x), \beta^+(x)) \mid x \in \mathcal{S}\}$ for the purpose of simplicity.

Kawila et al. [8] and Gaketem et al. [12] introduced the concepts of bipolar fuzzy UP-subalgebras, bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals, and BFIUPFs of UP-algebras as the following definition.

A BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ is called

(1) a *bipolar fuzzy UP-subalgebra* of $\mathcal{X}$ if $\forall x, y \in \mathcal{X}$,

$$\beta^-(x \diamond y) \leq \max\{\beta^-(x), \beta^-(y)\}, \tag{25}$$
$$\beta^+(x \diamond y) \geq \min\{\beta^+(x), \beta^+(y)\}, \tag{26}$$

(2) a *bipolar fuzzy UP-ideal* of $\mathcal{X}$ if $\forall x, y, z \in \mathcal{X}$,

$$\beta^-(0) \leq \beta^-(x), \tag{27}$$
$$\beta^+(0) \geq \beta^+(x), \tag{28}$$
$$\beta^-(x \diamond z) \leq \max\{\beta^-(x \diamond (y \diamond z)), \beta^-(y)\}, \tag{29}$$
$$\beta^+(x \diamond z) \geq \min\{\beta^+(x \diamond (y \diamond z)), \beta^+(y)\}, \tag{30}$$

(3) a *bipolar fuzzy UP-filter* (BFUPF) of $\mathcal{X}$ if if (27), (28), and $\forall x, y \in \mathcal{X}$,

$$\beta^-(y) \leq \max\{\beta^-(x \diamond y), \beta^-(x)\}, \tag{31}$$
$$\beta^+(y) \geq \min\{\beta^+(x \diamond y), \beta^+(x)\}, \tag{32}$$

(4) a *bipolar fuzzy implicative UP-filter* (BFIUPF) of $\mathcal{X}$ if if (27), (28), and $\forall x, y \in \mathcal{X}$,

$$\beta^-(x \diamond z) \leq \max\{\beta^-(x \diamond (y \diamond z)), \beta^-(x \diamond y)\}, \tag{33}$$
$$\beta^+(x \diamond z) \geq \min\{\beta^+(x \diamond (y \diamond z)), \beta^+(x \diamond y)\}. \tag{34}$$

II. Bipolar Fuzzy Comparative UP-filters

In this section, we introduce the concept of BFCUPFs of UP-algebras and investigate some properties of BFCUPFs.

A BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ is called a *bipolar fuzzy comparative UP-filter* (BFCUPF) of $\mathcal{X}$ if (27), (28), and $\forall x, y, z \in \mathcal{X}$,

$$\beta^-(y) \leq \max\{\beta^-(x \diamond ((y \diamond z) \diamond y)), \beta^-(x)\}, \tag{35}$$
$$\beta^+(y) \geq \min\{\beta^+(x \diamond ((y \diamond z) \diamond y)), \beta^+(x)\}. \tag{36}$$

Example II.1. Consider a UP-algebra $\mathcal{X} = \{a_0, a_1, a_2, a_3, a_4\}$ with the following Cayley table:

$\diamond$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_2	a_3	a_4
a_1	a_0	a_0	a_2	a_0	a_4
a_2	a_0	a_1	a_0	a_3	a_4
a_3	a_0	a_3	a_2	a_0	a_4
a_4	a_0	a_0	a_0	a_0	a_0

Define a BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ as follows:

$\mathcal{X}$	a_0	a_1	a_2	a_3	a_4
β^-	-0.8	-0.8	-0.8	-0.8	-0.1
β^+	0.7	0.7	0.7	0.7	0.2

Then β is a BFCUPF of $\mathcal{X}$.

Example II.2. Consider a UP-algebra $\mathcal{X} = \{b_0, b_1, b_2, b_3, b_4\}$ with the following Cayley table:

$\diamond$	b_0	b_1	b_2	b_3	b_4
b_0	b_0	b_1	b_2	b_3	b_4
b_1	b_0	b_0	b_2	b_3	b_4
b_2	b_0	b_0	b_0	b_3	b_4
b_3	b_0	b_0	b_1	b_0	b_4
b_4	b_0	b_0	b_2	b_3	b_0

Define a BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ as follows:

$\mathcal{X}$	b_0	b_1	b_2	b_3	b_4
β^-	-0.9	-0.9	-0.3	-0.3	-0.3
β^+	0.7	0.7	0.5	0.5	0.5

Then β is a BFIUPF of $\mathcal{X}$, but is not a BFCUPF of $\mathcal{X}$. Indeed, $\beta^-(2) = -0.3 > -0.9 = \max\{\beta^-(0 \diamond ((2 \diamond 3) \diamond 2)), \beta^-(0)\}$.

Open problem: Prove that every BFCUPF of $\mathcal{X}$ is a BFIUPF or give a contradictory example.

The following theorem is easy to prove.

Theorem II.3. *If $\beta = (\beta^-, \beta^+)$ is a BFS in $\mathcal{X}$ with β^- and β^+ are constant, then it is a BFCUPF of $\mathcal{X}$.*

Theorem II.4. *Every BFCUPF of $\mathcal{X}$ is a BFUPF.*

Proof: Let $\beta = (\beta^-, \beta^+)$ be a BFCUPF of $\mathcal{X}$. We are left only to show that (31) and (32) are true. Let $x, y \in \mathcal{X}$. Then

$$\begin{aligned} \beta^-(y) &\leq \max\{\beta^-(x \diamond ((y \diamond 0) \diamond y)), \beta^-(x)\} && \text{by (35)} \\ &= \max\{\beta^-(x \diamond (0 \diamond y)), \beta^-(x)\} && \text{by (3)} \\ &= \max\{\beta^-(x \diamond y), \beta^-(x)\} && \text{by (2)} \end{aligned}$$

and

$$\begin{aligned}\beta^+(y) &\geq \min\{\beta^+(x\diamond((y\diamond 0)\diamond y)),\beta^+(x)\} && \text{by (36)}\\ &= \min\{\beta^+(x\diamond(0\diamond y)),\beta^+(x)\} && \text{by (3)}\\ &= \min\{\beta^+(x\diamond y),\beta^+(x)\}. && \text{by (2)}\end{aligned}$$

Hence, β is a BFUPF of $\mathcal{X}$. ■

Example II.5. Consider a UP-algebra $\mathcal{X} = \{c_0, c_1, c_2, c_3\}$ with the following Cayley table:

$\diamond$	c_0	c_1	c_2	c_3
c_0	c_0	c_1	c_2	c_3
c_1	c_0	c_0	c_2	c_2
c_2	c_0	c_1	c_0	c_3
c_3	c_0	c_0	c_0	c_0

Define a BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ as follows:

$\mathcal{X}$	c_0	c_1	c_2	c_3
β^-	-0.6	-0.4	-0.2	-0.2
β^+	0.9	0.6	0.3	0.3

Then $\beta = (\beta^-, \beta^+)$ is a BFUPF of $\mathcal{X}$, but it is not a BFCUPF of $\mathcal{X}$. Indeed, $\beta^+(1) = 0.6 < 0.3 = \min\{\beta^+(2\diamond(1\diamond 3)), \beta^+(2)\}$.

Theorem II.6. *If $\beta = (\beta^-, \beta^+)$ is a BFCUPF of $\mathcal{X}$, then $\ker\beta = \{x \in \mathcal{X} \mid \beta^-(x) = \beta^-(0), \beta^+(x) = \beta^+(0)\}$ is a CUPF of $\mathcal{X}$.*

Proof: Clearly, $0 \in \ker\beta$. Let $x, y, z \in \mathcal{X}$ be such that $x\diamond((y\diamond z)\diamond y) \in \ker\beta$ and $x \in \ker\beta$. Then $\beta^-(x\diamond((y\diamond z)\diamond y)) = \beta^-(0) = \beta^-(x)$ and $\beta^+(x\diamond((y\diamond z)\diamond y)) = \beta^+(0) = \beta^+(x)$. Thus

$$\begin{aligned}\beta^-(0) &\leq \beta^-(y) && \text{by (27)}\\ &\leq \max\{\beta^-(x\diamond((y\diamond z)\diamond y)),\beta^-(x)\} && \text{by (35)}\\ &= \max\{\beta^-(0),\beta^-(0)\}\\ &= \beta^-(0)\end{aligned}$$

and

$$\begin{aligned}\beta^+(0) &\geq \beta^+(y) && \text{by (28)}\\ &\geq \min\{\beta^+(x\diamond((y\diamond z)\diamond y)),\beta^+(x)\} && \text{by (36)}\\ &= \min\{\beta^+(0),\beta^+(0)\}\\ &= \beta^+(0).\end{aligned}$$

That is, $\beta^-(y) = \beta^-(0)$ and $\beta^+(y) = \beta^+(0)$, so $y \in \ker\beta$. Hence, $\ker\beta$ is a CUPF of $\mathcal{X}$. ■

Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. For $r^- \in [-1, 0]$ and $r^+ \in [0, 1]$, the sets

$$L_N(\beta; r^-) = \{x \in \mathcal{X} \mid \beta^-(x) \leq r^-\},$$

$$U_N(\beta; r^-) = \{x \in \mathcal{X} \mid \beta^-(x) \geq r^-\},$$

$$L_P(\beta; r^+) = \{x \in \mathcal{X} \mid \beta^+(x) \leq r^+\},$$

$$U_P(\beta; r^+) = \{x \in \mathcal{X} \mid \beta^+(x) \geq r^+\}$$

are called the *negative lower r^--cut*, the *negative upper r^--cut*, the *positive lower r^+-cut* and the *positive upper r^+-cut* of β, respectively. The set $C(\beta; (r^-, r^+)) = L_N(\beta; r^-) \cap U_P(\beta; r^+)$ is called the (r^-, r^+)-cut of β. For any $r \in [0, 1]$, we denote the set $C(\beta; r) = C(\beta; (-r, r))$ is called the r-cut of β.

Theorem II.7. *A BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ is a BFCUPF of $\mathcal{X}$ if and only if the followings are valid:*

1. *for all $r^- \in [-1, 0]$, $L_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$ if it is nonempty,*
2. *for all $r^+ \in [0, 1]$, $U_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$ if it is nonempty.*

Proof: Suppose that β is a BFUPF of $\mathcal{X}$. Let $r^- \in [-1, 0]$ be such that $L_N(\beta; r^-) \neq \emptyset$ and let $a \in L_N(\beta; r^-)$. Then $\beta^-(a) \leq r^-$. By (27), we have $\beta^-(0) \leq \beta^-(a) \leq r^-$. Thus $0 \in L_N(\beta; r^-)$.

Let $x, y, z \in \mathcal{X}$ be such that $x\diamond((y\diamond z)\diamond y) \in L_N(\beta; r^-)$ and $x \in L_N(\beta; r^-)$. Then $\beta^-(x\diamond((y\diamond z)\diamond y)) \leq r^-$ and $\beta(x) \leq r^-$. By (35), we have $\beta^-(y) \leq \max\{\beta^-(x\diamond((y\diamond z)\diamond y)), \beta^-(x)\} \leq r^-$. Thus $y \in L_N(\beta; r^-)$. Hence, $L_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$.

Let $r^+ \in [0, 1]$ be such that $U_P(\beta; r^+) \neq \emptyset$ and let $a \in U_P(\beta; r^+)$. Then $\beta^+(a) \geq r^+$. By (28), we have $\beta^+(0) \geq \beta^+(a) \geq r^+$. Thus $0 \in U_P(\beta; r^+)$.

Let $x, y, z \in \mathcal{X}$ be such that $x\diamond((y\diamond z)\diamond y) \in U_P(\beta; r^+)$ and $x \in U_P(\beta; r^+)$. Then $\beta^+(x\diamond((y\diamond z)\diamond y)) \geq r^+$ and $\beta^+(x) \geq r^+$. By (36), we have $\beta^+(y) \geq \min\{\beta^+(x\diamond((y\diamond z)\diamond y)), \beta^+(x)\} \geq r^+$. Thus $y \in U_P(\beta; r^+)$. Hence, $U_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$.

For the converse, suppose that $r^- \in [0, 1], L_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$ if it is nonempty and $r^+ \in [0, 1], U_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$ if it is nonempty.

Let $x \in \mathcal{X}$. Then $\beta^-(x) \in [-1, 0]$. Choose $r^- = \beta^-(x)$. Then $\beta^-(x) \leq r^-$. Thus $x \in L_N(\beta; r^-) \neq \emptyset$, so $L_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$. By (20), we have $0 \in L_N(\beta; r^-)$. Thus $\beta^-(0) \leq r^- = \beta^-(x)$.

Let $x, y, z \in \mathcal{X}$. Then $\beta^-(x\diamond((y\diamond z)\diamond y)), \beta^-(x) \in [-1, 0]$. Choose $r^- = \max\{\beta^-(x\diamond((y\diamond z)\diamond y)), \beta^-(x)\}$. Then $\beta^-(x\diamond((y\diamond z)\diamond y)) \leq r^-$ and $\beta^-(x) \leq r^-$. Thus $x\diamond((y\diamond z)\diamond y), x \in L_N(\beta; r^-) \neq \emptyset$, so $L_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$. By (24), we have $y \in L_N(\beta; r^-)$. Thus $\beta^-(y) \leq r^- = \max\{\beta^-(x\diamond((y\diamond z)\diamond y)), \beta^-(x)\}$.

Let $x \in \mathcal{X}$. Then $\beta^+(x) \in [0, 1]$. Choose $r^+ = \beta^+(x)$. Then $\beta^+(x) \geq r^+$. Thus $x \in U_P(\beta; r^+) \neq \emptyset$, so $U_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$. By (20), we have $0 \in U_P(\beta; r^+)$. Thus $\beta^+(0) \geq r^+ = \beta^+(x)$.

Let $x, y, z \in \mathcal{X}$. Then $\beta^+(x\diamond((y\diamond z)\diamond y)), \beta^+(x) \in [0, 1]$. Choose $r^+ = \min\{\beta^+(x\diamond((y\diamond z)\diamond y)), \beta^+(x)\}$. Then $\beta^+(x\diamond((y\diamond z)\diamond y)) \geq r^+$ and $\beta^+(x) \geq r^+$. Thus $(x\diamond((y\diamond z)\diamond y)), x \in U_P(\beta; r^+) \neq \emptyset$, so $U_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$. By (24), we have $y \in U_P(\beta; r^+)$. Thus $\beta^+(y) \geq r^+ = \min\{\beta^+(x\diamond((y\diamond z)\diamond y)), \beta^+(x)\}$.

Hence, β is a BFCUPF of $\mathcal{X}$. ■

Corollary II.8. *If $\beta = (\beta^-, \beta^+)$ is a BFCUPF of $\mathcal{X}$, then for all $r \in [0, 1]$, $C(\beta; r)$ is a CUPF of $\mathcal{X}$ if it is nonempty.*

Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. The BFS $\beta_c = (\beta_c^-, \beta_c^+)$ defined by for all $x \in \mathcal{X}$, $\beta_c^-(x) = -1 - \beta^+(x)$ and $\beta_c^+(x) = 1 - \beta^+(x)$ is called the *complement* [17] of β in $\mathcal{X}$.

Lemma II.9. *[12] Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. For all $r^- \in [-1, 0]$ and $r^+ \in [0, 1]$, the followings are true:*

1. $L_N(\beta_c; r^-) = U_N(\beta; -1 - r^-)$,
2. $U_N(\beta_c; r^-) = L_N(\beta; -1 - r^-)$,
3. $L_P(\beta_c; r^+) = U_P(\beta; 1 - r^+)$,
4. $U_P(\beta_c; r^+) = L_P(\beta; 1 - r^+)$.

Theorem II.10. *Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. Then $\beta_c = (\beta_c^-, \beta_c^+)$ is a BFCUPF of $\mathcal{X}$ if and only if the followings are true:*

(1) *for all $r^- \in [-1, 0]$, $U_N(\beta; r^-)$ is a CUPF of $\mathcal{X}$ if it is nonempty,*

(2) *for all $r^+ \in [0, 1]$, $L_P(\beta; r^+)$ is a CUPF of $\mathcal{X}$ if it is nonempty.*

Proof: It follows from Theorem II.7 and Lemma II.9. ■

A neutrosophic set (NS) N in a nonempty set $\mathcal{S}$ as a structure of the form $N = \{(x, f(x), g(x), h(x)) \mid x \in \mathcal{S}\}$ when the fuzzy sets $f : \mathcal{S} \to [0, 1]$, $g : \mathcal{S} \to [0, 1]$ and $h : \mathcal{S} \to [0, 1]$ are called a truth membership function, an indeterminate membership function, and a false membership function, respectively. For the sake of simplicity, we shall denote the symbol $\langle f, g, h\rangle$ of the NS $\{(x, f(x), g(x), h(x)) \mid x \in \mathcal{S}\}$. Songsaeng and Iampan [18] and Songsaeng et al. [19] applied neutrosophic set theory to UP-algebras, introduced many concepts of neutrosophic UP-substructures and investigated their properties. The concept of neutrosophic comparative UP-filters of UP-algebras is one of important concepts of neutrosophic UP-substructures and defined the following definition.

Definition II.11. [19] A NS $\langle f, g, h\rangle$ in $\mathcal{X}$ is called a *neutrosophic comparative UP-filter* (NCUPF) of $\mathcal{X}$ if the following conditions hold: for all $x, y, z \in \mathcal{X}$,

$$f(0) \geq f(x), \tag{37}$$

$$g(0) \leq g(x), \tag{38}$$

$$h(0) \geq h(x), \tag{39}$$

$$f(y) \geq \min\{f(x \diamond ((y \diamond z) \diamond y)), f(x)\}, \tag{40}$$

$$g(y) \leq \max\{g(x \diamond ((y \diamond z) \diamond y)), g(x)\}, \tag{41}$$

$$h(y) \geq \min\{h(x \diamond ((y \diamond z) \diamond y)), h(x)\}. \tag{42}$$

Example II.12. Consider the UP-algebra $\mathcal{X} = \{a_0, a_1, a_2, a_3, a_4\}$ defined in Example II.1. Define a NS $\langle f, g, h\rangle$ in $\mathcal{X}$ as follows:

$\mathcal{X}$	a_0	a_1	a_2	a_3	a_4
f	0.6	0.6	0.6	0.6	0.2
g	0.4	0.4	0.4	0.4	0.6
h	0.9	0.9	0.9	0.9	0.1

Then $\langle f, g, h\rangle$ is a NFCUPF of $\mathcal{X}$.

Definition II.13. A fuzzy set f in $\mathcal{X}$ is called a *fuzzy comparative UP-filter* (FCUPF) of $\mathcal{X}$ if it satisfies the conditions (37) and (40).

Definition II.14. A fuzzy set g in $\mathcal{X}$ is called an *anti-fuzzy comparative UP-filter* (AFCUPF) of $\mathcal{X}$ if it satisfies the conditions (38) and (41).

Example II.15. Consider the UP-algebra $\mathcal{X} = \{a_0, a_1, a_2, a_3, a_4\}$ defined in Example II.1. Define fuzzy sets f and g in $\mathcal{X}$ as follows:

$\mathcal{X}$	a_0	a_1	a_2	a_3	a_4
f	0.3	0.3	0.3	0.3	0.2
g	0.5	0.5	0.5	0.5	1

Then f is a FCUPF and g is an AFCUPF of $\mathcal{X}$.

Proposition II.16. *A NS $\langle f, g, h\rangle$ in $\mathcal{X}$ is a NCUPF of $\mathcal{X}$ if and only if f and h are FCUPFs of $\mathcal{X}$, and g is an AFCUPF of $\mathcal{X}$.*

Proof: It is clear. ■

For a function f from a nonempty set $\mathcal{S}$ into the set of all real numbers R, the functions $-f$, $1-f$, $1+f$ and $f-1$ are defined as follows:

$$-f : \mathcal{S} \to R, x \mapsto -f(x), \tag{43}$$

$$1-f : \mathcal{S} \to R, x \mapsto 1-f(x), \tag{44}$$

$$1+f : \mathcal{S} \to R, x \mapsto 1+f(x), \tag{45}$$

$$f-1 : \mathcal{S} \to R, x \mapsto f(x)-1. \tag{46}$$

Then $f = 1-(1-f) = -(-f) = 1+(f-1)$ and $1+f = 1-(-f)$.

Lemma II.17. *Let f be a fuzzy set in $\mathcal{X}$. The following are true.*

(1) *f is a FCUPF of $\mathcal{X}$ if and only if $1-f$ is an AFCUPF of $\mathcal{X}$.*

(2) *f is an AFCUPF of $\mathcal{X}$ if and only if $1-f$ is a FCUPF of $\mathcal{X}$.*

Proof: (1) Assume that f is a FCUPF of $\mathcal{X}$. Then for all $x, y, z \in \mathcal{X}$, we have

$$(1-f)(0) = 1-f(0) \leq 1-f(x) = (1-f)(x)$$

and

$$\begin{aligned}(1-f)(y) &= 1-f(y)\\ &\leq 1-\min\{f(x \diamond ((y \diamond z) \diamond y)), f(x)\}\\ &= \max\{1-f(x \diamond ((y \diamond z) \diamond y)), 1-f(x)\}\\ &= \max\{(1-f)(x \diamond ((y \diamond z) \diamond y)), (1-f)(x)\}.\end{aligned}$$

Hence, $1-f$ is an AFCUPF of $\mathcal{X}$.

Conversely, assume that $1-f$ is an AFCUPF of $\mathcal{X}$. Since $f = 1-(1-f)$, we get

$$f(0) = 1-(1-f)(0) \geq 1-(1-f)(x) = f(x)$$

and

$$\begin{aligned}f(y) &= 1-(1-f)(y)\\ &\geq 1-\max\{(1-f)(x \diamond ((y \diamond z) \diamond y)), (1-f)(x)\}\\ &= \min\{1-(1-f)(x \diamond ((y \diamond z) \diamond y)), 1-(1-f)(x)\}\\ &= \min\{f(x \diamond ((y \diamond z) \diamond y)), f(x)\}\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Hence, f is a FCUPF of $\mathcal{X}$.

(2) It follows from (1) and $f = 1-(1-f)$. ■

Theorem II.18. *Let f be a fuzzy set and $\beta = (\beta^-, \beta^+)$ be the BFS in $\mathcal{X}$ such that $\beta^- = -f$ and $\beta^+ = f$. Then f is a FCUPF of $\mathcal{X}$ if and only if β is a BFCUPF of $\mathcal{X}$.*

Proof: Let f be a FCUPF of $\mathcal{X}$. Then β satisfies the conditions (28) and (36). Next, let $x, y, z \in \mathcal{X}$. Then

$$\beta^-(0) = -f(0) \leq -f(x) = \beta^-(x)$$

and

$$\begin{aligned}\beta^-(y) &= -f(y)\\ &\leq -\min\{f(x \diamond ((y \diamond z) \diamond y)), f(x)\}\\ &= \max\{-f(x \diamond ((y \diamond z) \diamond y)), -f(x)\}\\ &= \max\{\beta^-(x \diamond ((y \diamond z) \diamond y)), \beta^-(x)\}.\end{aligned}$$

Thus β satisfies the conditions (27) and (35). Hence, β is a BFCUPF of $\mathcal{X}$.

Conversely, let β be a BFCUPF of $\mathcal{X}$. Then the conditions (28) and (36) hold. Thus $f = \beta^+$ is a FCUPF of $\mathcal{X}$. ■

Theorem II.19. *A BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ is a BFCUPF of $\mathcal{X}$ if and only if β^+ and $-\beta^-$ are FCUPFs of $\mathcal{X}$.*

Proof: Assume that β is a BFCUPF of $\mathcal{X}$. Then β^+ satisfies the conditions (28) and (36) which imply that β^+ is a FCUPF of $\mathcal{X}$. Since β satisfies the conditions (27) and (35), we have $-\beta^-(0) \geq -\beta^-(x)$ and

$$\begin{aligned}-\beta^-(y) &\geq -\max\{\beta^-(x \diamond ((y \diamond z) \diamond y)), \beta^-(x)\} \\ &= \min\{-\beta^-(x \diamond ((y \diamond z) \diamond y)), -\beta^-(x)\}\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Thus $-\beta^-$ is a FCUPF of $\mathcal{X}$.

Conversely, assume that β^+ and $-\beta^-$ are FCUPFs of $\mathcal{X}$. Then β satisfies the conditions (28) and (36). Since $\beta^- = -(-\beta^-)$, we get that β satisfies the conditions (27) and (35). Hence, β is a BFCUPF of $\mathcal{X}$. ■

Theorem II.20. *Let f be a fuzzy set and $\beta = (\beta^-, \beta^+)$ be the BFS in $\mathcal{X}$ such that $\beta^- = f - 1$ and $\beta^+ = 1 - f$. Then f is an AFCUPF of $\mathcal{X}$ if and only if β is a BFCUPF of $\mathcal{X}$.*

Proof: Assume that f is an AFCUPF of $\mathcal{X}$. By Lemma II.17(2), we get that $\beta^+ = 1 - f$ is a FCUPF of $\mathcal{X}$. Thus β satisfies the conditions (28) and (36). By the assumption and $\beta^- = f - 1$, we get

$$\beta^-(0) = f(0) - 1 \leq f(x) - 1 = \beta^-(x)$$

and

$$\begin{aligned}\beta^-(y) &= f(y) - 1 \\ &\leq \max\{f(x \diamond ((y \diamond z) \diamond y)), f(x)\} - 1 \\ &= \max\{f(x \diamond ((y \diamond z) \diamond y)) - 1, f(x) - 1\} \\ &= \max\{\beta^-(x \diamond ((y \diamond z) \diamond y)), \beta^-(x)\}\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Hence, β satisfies the conditions (27) and (35). Therefore, β is a BFCUPF of $\mathcal{X}$.

Conversely, it follows from Lemma II.17(2). ■

Theorem II.21. *A BFS $\beta = (\beta^-, \beta^+)$ in $\mathcal{X}$ is a BFCUPF of $\mathcal{X}$ if and only if $1 - \beta^+$ and $1 + \beta^-$ are AFCUPFs of $\mathcal{X}$.*

Proof: It follows from Lemma II.17(1), Theorem II.19 and $1 + \beta^- = 1 - (-\beta^-)$. ■

Theorem II.22. *Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. The following are equivalent.*

(1) *β is a BFCUPF of $\mathcal{X}$.*
(2) *β^+ is a FCUPF and $1 + \beta^-$ is an AFCUPF of $\mathcal{X}$.*
(3) *$1 - \beta^+$ is an AFCUPF and $-\beta^-$ is a FCUPF of $\mathcal{X}$.*

Proof: It follows from Lemma II.17(1), Theorems II.19, II.21 and $1 + \beta^- = 1 - (-\beta^-)$. ■

For fuzzy sets f and g in a nonempty set $\mathcal{S}$, we denote $(f - 1, g)$ for the BFS $\{(x, (f - 1)(x), g(x)) \mid x \in \mathcal{S}\}$.

Theorem II.23. *A NS $\langle f, g, h\rangle$ in $\mathcal{X}$ is a NCUPF of $\mathcal{X}$ if and only if the BFSs $(g - 1, f)$ and $(g - 1, h)$ are BFCUPFs of $\mathcal{X}$.*

Proof: Assume that $\langle f, g, h\rangle$ in $\mathcal{X}$ is a NCUPF of $\mathcal{X}$. By Proposition II.16, we have f and h are FCUPFs and g is an AFCUPF of $\mathcal{X}$. Then $(g - 1, f)$ and $(g - 1, h)$ satisfy the conditions (28) and (36). We will show that $(g - 1, f)$ and $(g - 1, h)$ satisfy the conditions (27) and (35). Let $x, y, z \in \mathcal{X}$. Thus

$$(g - 1)(0) = g(0) - 1 \leq g(x) - 1 = (g - 1)(x)$$

and

$$\begin{aligned}(g - 1)(y) &= g(y) - 1 \\ &\leq \max\{g(x \diamond ((y \diamond z) \diamond y)), g(x)\} - 1 \\ &= \max\{g(x \diamond ((y \diamond z) \diamond y)) - 1, g(x) - 1\} \\ &= \max\{(g - 1)(x \diamond ((y \diamond z) \diamond y)), (g - 1)(x)\}.\end{aligned}$$

This is shown that $(g - 1, f)$ and $(g - 1, h)$ satisfy the conditions (27) and (35). Therefore, $(g - 1, f)$ and $(g - 1, h)$ are BFCUPFs of $\mathcal{X}$.

Conversely, assume that $(g - 1, f)$ and $(g - 1, h)$ are BFCUPFs of $\mathcal{X}$. By Theorem II.22, we have f and h are FCUPFs and $1 + (g - 1)$ is an AFCUPF of $\mathcal{X}$. Since $g = 1 + (g - 1)$, we obtain that g is an AFCUPF of $\mathcal{X}$. It follows from Proposition II.16 that $\langle f, g, h\rangle$ is a NCUPF of $\mathcal{X}$. ■

Theorem II.24. *Let $\beta = (\beta^-, \beta^+)$ be a BFS in $\mathcal{X}$. Then β is a BFSUPF of $\mathcal{X}$ if and only if the NS $\langle \beta^+, 1 + \beta^-, -\beta^-\rangle$ is a NCUPF of $\mathcal{X}$.*

Proof: It follows from Proposition II.16 and Theorem II.22. ■

III. Conclusion

In this paper, we have introduced the concept of BFCUPFs of UP-algebras and provided some properties of BFCUPFs. The results showed that BFCUPFs and CUPFs are related as described by their cut. In this study we have an open problem to prove that every BFCUPF of $\mathcal{X}$ is a BFIUPF or give a contradictory example.

In the future, we will extend go to hesitant fuzzy comparative UP-filters in UP-algebras and provide some properties.

References

[1] A. Iampan, "A new branch of the logical algebra: UP-algebras," *J. Algebra Relat. Top.*, vol. 5, no. 1, pp. 35–54, 2017.

[2] C. Prabpayak and U. Leerawat, "On ideals and congruences in KU-algebras," *Sci. Magna*, vol. 5, no. 1, pp. 54–57, 2009.

[3] L. A. Zadeh, "Fuzzy sets," *Inf. Cont.*, vol. 8, no. 3, pp. 338–353, 1965.

[4] W.-R. Zhang, "Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis," in *NAFIPS/IFIS/NASA '94. Proc. First Int. Joint Conf. N. Am. Fuzzy Inf. Process. Soc. Biannu. Conf. Ind. Fuzzy Control Intell.*, 1994, pp. 305–309.

[5] Y. B. Jun and S. Z. Song, "Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets," *Sci. Math. Jpn.*, vol. 68, no. 2, pp. 287–297, 2008.

[6] K. J. Lee and Y. B. Jun, "Bipolar fuzzy a-ideals of BCI-algebras," *Commun. Korean Math. Soc.*, vol. 26, no. 4, pp. 531–542, 2011.

[7] Y. B. Jun, K. J. Lee, and E. H. Roh, "Ideals and filters in CI-algebras based on bipolar-valued fuzzy sets," *Ann. Fuzzy Math. Inform.*, vol. 4, no. 1, pp. 109–121, 2012.

[8] K. Kawila, C. Udomsetchai, and A. Iampan, "Bipolar fuzzy UP-algebras," *Math. Comput. Appl.*, vol. 23, no. 4, p. 69, 2018.

[9] A. Al-Masarwah and A. G. Ahmad, "Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras," *Int. J. Innov. Comput. Inf. Control*, vol. 14, no. 6, pp. 2025–2041, 2018.

[10] G. Muhiuddin and D. Al-Kadi, "Bipolar fuzzy implicative ideals of BCK-algebras," *J. Math.*, vol. 2021, pp. Article ID 6 623 907, 9 pages, 2021.

[11] T. Gaketem and P. Khamrot, "On some semigroups characterized in terms of bipolar fuzzy weakly interior ideals," *IAENG Int. J. Comput. Sci.*, vol. 48, no. 2, pp. 250–256, 2021.
[12] T. Gaketem, P. Khamrot, P. Julatha, R. Prasertpong, and A. Iampan, "A novel concept of bipolar fuzzy sets in UP-algebras: bipolar fuzzy implicative UP-filters," December 2021, submitted.
[13] A. Iampan, "Introducing fully UP-semigroups," *Discuss. Math., Gen. Algebra Appl.*, vol. 38, no. 2, pp. 297–306, 2018.
[14] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, "Fuzzy translations of a fuzzy set in UP-algebras," *J. Indones. Math. Soc.*, vol. 23, no. 2, pp. 1–19, 2017.
[15] Y. B. Jun and A. Iampan, "Implicative UP-filters," *Afr. Mat.*, vol. 30, no. 7-8, pp. 1093–1101, 2019.
[16] ——, "Comparative and allied UP-filters," *Lobachevskii J. Math.*, vol. 40, no. 1, pp. 60–66, 2019.
[17] N. Udten, N. Songseang, and A. Iampan, "Translation and density of a bipolar-valued fuzzy set in UP-algebras," *Ital. J. Pure Appl. Math.*, vol. 41, pp. 469–496, 2019.
[18] M. Songsaeng and A. Iampan, "Neutrosophic set theory applied to UP-algebras," *Eur. J. Pure Appl. Math.*, vol. 12, no. 4, pp. 1382–1409, 2019.
[19] M. Songsaeng, K. P. Shum, R. Chinram, and A. Iampan, "Neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras," *Neutrosophic Sets Syst.*, vol. 47, pp. 620–643, 2021.