Theory of Fractional Calculus

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Abstract—**This work aims to introduce a theory of fractional calculus by using a map $\kappa(x)$ instead of x in the definitions of the classical derivative and the classical integral. The applicability of the results obtained is demonstrated using fractional differential equations.**

Index Terms—**Fractional function, Fractional continuous, Fractional bounded, Fractional derivative, Fractional integral, Fractional differential equation.**

I. Introduction

THE study of fractional calculus was initiated by Leibniz in 1695, when he was asked by L'Hˆopital about the derivative of order 1/2 (see e.g., [7], [8], [19], [20]). Since then, fractional calculus has attracted the attention of mathematicians, physicists, and engineers, and is regarded as an effective and powerful tool for modeling nonlinear systems (see e.g., [2], [4], [6], [9], [10], [12], [15], [16], [17], [18], [21], [22], [23], [24], [25], [27]); hence, various definitions of the fractional derivative and the fractional integral have been introduced (to review these definitions see e.g., [5]). In this work, we propose a novel theory of fractional calculus by considering a bijective, uniformly continuous map κ(x) in the definition of the classical derivative instead of the variable x such that κ: ℝ → ℝ is defined by

$$\kappa(x) = \begin{cases} x^{\alpha}, & \text{if } x \in [0,\infty), \\ -(-x)^{\alpha}, & \text{if } x \in (-\infty,0), \end{cases} \tag{1}$$

$\alpha \in (0,1]$. Moreover, $\kappa: \mathbb{R} \to \mathbb{R}$ is continuously differentiable except at $x = 0$,

$$\kappa'(x) = \begin{cases} \alpha x^{\alpha-1}, & \text{if } x \in (0,\infty), \\ \text{does not exit}, & \text{if } x = 0, \\ \alpha(-x)^{\alpha-1}, & \text{if } x \in (-\infty,0). \end{cases} \tag{2}$$

The novelty of this theory is that it is a natural extension of the theory of classical calculus; hence, the rules of fractional calculus coincide with those of classical calculus such that there are no inconsistencies or difficulties to handle with.

For the convenience of the reader, we state below the common inconsistent and inconvenient properties in the other definitions of the fractional derivatives (see e.g., [1], [3], [11], [12], [13], [14], [28]):

1) All fractional derivatives, except Caputo's derivative, do not satisfy $D^{\alpha}(1) = 0$ if α is not a natural number.
2) All fractional derivatives do not satisfy
$$D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f).$$
3) All fractional derivatives do not satisfy
$$D^{\alpha}(f/g) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}.$$
4) All fractional derivatives do not satisfy
$$D^{\alpha}(g \circ f) = D^{\alpha}(g)\big(f(t)\big)D^{\alpha}(f).$$
5) All fractional derivatives do not satisfy $D^{\alpha_1}D^{\alpha_2}f = D^{\alpha_1+\alpha_2}f$.
6) The term $t^{1-\alpha}$ in the derivative of [14] is not essential and there is no physical interpretation of this derivative, i.e., the basis of its potential applications is not solid.

II. METHODOLOGY

In this paper, we always assume that the fractional function (or α-function) is the operator $f_{\alpha}: A \to \mathbb{R}$, $\alpha \in (0,1]$, defined by $f_{\alpha}(x) = f\big(\kappa(x)\big)$ for a real-valued function $f: B \to \mathbb{R}$ and the map $\kappa: A \to \mathbb{R}$ defined as in Eq. (1) such that $\kappa(A) \subseteq B$.

Now we use the concept of α-function, $\alpha \in (0,1]$, to define the fractional continuity, the fractional boundedness, the fractional derivative, and the fractional integral.

A. Fractional Continuity

Definition 1. A function $f: B \to \mathbb{R}$ is said to be α-continuous, $\alpha \in (0,1]$, at $x_0 \in A$, $\kappa(A) \subseteq B$, if $f_{\alpha}: A \to \mathbb{R}, \alpha \in (0,1]$, is continuous at x_0, i.e., given any $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in A$ and $\|x - x_0\| < \delta$ imply

$$\|f_{\alpha}(x) - f_{\alpha}(x_0)\| < \varepsilon \tag{3}$$

and we write it as

$$\lim_{x \to x_0} f_{\alpha}(x) = f_{\alpha}(x_0). \tag{4}$$

And $f: B \to \mathbb{R}$ is said to be α-continuous, $\alpha \in (0,1]$, on A, $\kappa(A) \subseteq B$, if $f_{\alpha}: A \to \mathbb{R}$, $\alpha \in (0,1]$, is continuous at every point in A.

Remark 2. The α-continuity, $\alpha \in (0,1]$, in Definition 1 means that f is fractionally continuous of order $\alpha, \alpha \in (0,1]$.

Proposition 3. Every continuous function is α-continuous of order $\alpha = 1$ but not every α-continuous function is continuous.

Example 4. $f(x) = \frac{1}{x-1/2}$ is not continuous at $x = \frac{1}{2}$ whenever $\alpha = 1$ but it is α-continuous whenever $\alpha \in (0,1)$.

Proposition 5. Let $f: B \to \mathbb{R}$ be an α_m-continuous function, $\alpha_m \in (0,1], m \in \mathbb{N}$, on a nonempty set A, $\kappa(A) \subseteq B$. If α_m converges to $\alpha, \alpha \in (0,1]$, then f is α-continuous, $\alpha \in (0,1]$, on A.

Theorem 6. Let $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, as a sequence of α-continuous functions, $\alpha \in (0,1]$, on a nonempty compact set

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A, $\kappa(A) \subseteq B$. If $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, converges uniformly to $f: B \to \mathbb{R}$ then f is α-continuous, $\alpha \in (0,1]$, on A.

Theorem 7 (Intermediate Value Theorem for Fractional Functions). If $f: [c, \mathrm{d}] \to \mathbb{R}, c < \mathrm{d}$, is α-continuous, $\alpha \in (0,1]$, and $z_0 \in \left(f_\alpha(a), f_\alpha(b)\right)$ $\left(\text{or } z_0 \in \left(f_\alpha(b), f_\alpha(a)\right)\right)$, $\kappa([a, \mathrm{b}]) \subseteq [c, \mathrm{d}]$, then there exists an $x_0 \in (a, b)$ such that $f_\alpha(x_0) = z_0$.

Proof. For $z_0 \in \left(f_\alpha(a), f_\alpha(b)\right)$, $\alpha \in (0,1]$, assume that there is no such x_0 in (a, b) and let the sets

$$R_1 = \{x \in [a, \mathrm{b}]: f_\alpha(x) < z_0\},$$
$$R_2 = \{x \in [a, \mathrm{b}]: f_\alpha(x) > z_0\},$$

be such that $[a, b] = R_1 \cup R_2$. If $x_0 \in R_1$, by the α-continuity of f, there exists $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $f_\alpha(x) < z_0$ and $x \in R_1$. So, $x_0 \not\in \overline{R}_2$ and $R_1 \cap \overline{R}_2 = \emptyset$. Similarly, $\overline{R}_1 \cap R_2 = \emptyset$. That is, $[a, b]$ is disconnected in $\mathbb{R}$. But $[a, b]$ is always connected in $\mathbb{R}$ and the result follows. Similarly, we can prove $f_\alpha(x_0) = z_0$ for $z_0 \in (f_\alpha(b), f_\alpha(a))$ as well.

B. Fractional Boundedness

Definition 8. A function $f: B \to \mathbb{R}$ is said to be α-bounded, $\alpha \in (0,1]$, on A, $\kappa(A) \subseteq B$, if $f_\alpha: A \to \mathbb{R}$, $\alpha \in (0,1]$, is bounded on A, i.e., there exists some $M \in \mathbb{R}$ such that $\|f_\alpha(x)\| < M$ for all $x \in A$.

Remark 9. The α-boundedness, $\alpha \in (0,1]$, in Definition 8 means that f is fractionally bounded of order $\alpha, \alpha \in (0,1]$.

Proposition 10. Every bounded function is α-bounded of order $\alpha = 1$ but not every α-bounded function is bounded.

Example 11. $f(x) = \frac{1}{x^2 - 4}$ is unbounded on $[0,2]$ whenever $\alpha = 1$ but it is α-bounded whenever $\alpha \in (0,1)$.

Proposition 12. Let $f: B \to \mathbb{R}$ be an α_m-bounded function, $\alpha_m \in (0,1]$, $m \in \mathbb{N}$, on a nonempty set A, $\kappa(A) \subseteq B$. If α_m converges to α, $\alpha \in (0,1]$, then f is α-bounded, $\alpha \in (0,1]$, on A.

Theorem 13. Let $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of α-bounded functions, $\alpha \in (0,1]$, on a nonempty compact set $A, \kappa(A) \subseteq B$. If $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, converges uniformly to $f: B \to \mathbb{R}$ then f is α-bounded, $\alpha \in (0,1]$, on A.

Theorem 14. If $f: [c, \mathrm{d}] \to \mathbb{R}$ is α-continuous, $\alpha \in (0,1]$, on $[a, \mathrm{b}]$, $\kappa([a, \mathrm{b}]) \subseteq [c, \mathrm{d}]$, then f is α-bounded, $\alpha \in (0,1]$, on $[a, \mathrm{b}]$.

Proof. Assume that f is not α-bounded, $\alpha \in (0,1]$, on $[a, b]$, i.e., there exists $x_m \in [a, b]$ such that $\|f_\alpha(x_m)\| > m$ for all $m \in \mathbb{N}$. Since $[a, b]$ is compact, there is a convergent subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k} \to x_0$, as $k \to \infty$, and $x_0 \in [a, b]$ by the Bolzano-Weierstrass Theorem. Since f is α-continuous, $\alpha \in (0,1]$, at x_0, then $\lim_{k \to \infty} f_\alpha\left(x_{m_k}\right) = f_\alpha(x_0)$ which contradicts with $\left\|f_\alpha\left(x_{m_k}\right)\right\| > m_k$ for all $k \in \mathbb{N}$.

Theorem 15 (Extreme Value Theorem for Fractional Functions). If $f: [c, \mathrm{d}] \to \mathbb{R}$ is α-continuous, $\alpha \in (0,1]$, on $[a, b]$, $\kappa([a, \mathrm{b}]) \subseteq [c, \mathrm{d}]$, then f has an absolute α-maximum value and an absolute α-minimum value, $\alpha \in (0,1]$, on $[a, b]$.

Proof. Let $M = \sup_{x \in [a,b]} f_\alpha(x)$, $\alpha \in (0,1]$. Assume that $f_\alpha(x) < M$, $x \in [a, b]$. Then $g_\alpha(x) = \frac{1}{M - f_\alpha(x)}$ is α-continuous and by Theorem 14, g is α-bounded, $\alpha \in (0,1]$, and for some $K > 0$, $g_\alpha(x) \le K$, $x \in [a, b]$. Thus, $f_\alpha(x) \le M - \frac{1}{K}$, $x \in [a, b]$. Since M is the supremum of f_α, $\alpha \in (0,1]$, then $M \le M - \frac{1}{K} < M$ which is a contradiction. Thus, there is an $x_M \in [a, b]$ such that $f_\alpha(x_M) = M$, $\alpha \in (0,1]$. Similarly, we can prove that there is an $x_m \in [a, b]$ such that $f_\alpha(x_m) = \inf_{x \in [a,b]} f_\alpha(x)$, $\alpha \in (0,1]$.

Remark 16. The α-maximum value and α-minimum value, $\alpha \in (0,1]$, in Theorem 15 mean that the function f has a fractional maximum value and a fractional minimum value of order α, $\alpha \in (0,1]$.

C. Fractional Derivative

Definition 17. Let $f: J \to \mathbb{R}$ be a function, where $J \subseteq \mathbb{R}$ is an interval. We say that a real number $L_\alpha, \alpha \in (0,1]$, is an α-derivative, $\alpha \in (0,1]$, of f at $x_0 \in I$, $\kappa(I) \subseteq J$, and f is α-differentiable, $\alpha \in (0,1]$, at x_0 if $f_\alpha: I \to \mathbb{R}, \alpha \in (0,1]$, is differentiable at x_0, i.e., given any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < \|x - x_0\| < \delta$, then

$$\left\|\frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0} - L_\alpha\right\| < \varepsilon. \tag{5}$$

In other words, if we denote $D^\alpha f(x_0)$ for $L_\alpha, \alpha \in (0,1]$, we can define the α-derivative, $\alpha \in (0,1]$, of the function f at x_0 by

$$D^\alpha f(x_0) = \lim_{x \to x_0} \frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0} \tag{6}$$

provided that the limit exists.

Remark 18. The α-derivative $D^\alpha f, \alpha \in (0,1]$, in Definition 17 is the fractional derivative of f of order $\alpha, \alpha \in (0,1]$.

Proposition 19. Every differentiable function is α-differentiable of order $\alpha = 1$ but not every α-differentiable function is differentiable.

Example 20. $f(x) = \frac{x}{x-5}$ is not differentiable at $x = 5$ whenever $\alpha = 1$ but it is α-differentiable whenever $\alpha \in (0,1)$.

Proposition 21. Suppose that $f: B \to \mathbb{R}$ is α_m-continuous on a nonempty set A and α_m-differentiable, $\alpha_m \in (0,1]$, $m \in \mathbb{N}$, at $x_0 \in A$, $\kappa(A) \subseteq B$. If α_m converges to α, $\alpha \in (0,1]$, then f is α-differentiable, $\alpha \in (0,1]$, at x_0.

Theorem 22. Let $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of α-continuous, α-differentiable functions, $\alpha \in (0,1]$, on a nonempty compact set A, $\kappa(A) \subseteq B$. If $\lim_{m \to \infty} f_{m,\alpha}(x_0)$ exists for some $x_0 \in A$ and $D^\alpha f_m$ converges uniformly on A, then f_m converges uniformly on A and

$$\lim_{n\to\infty} D^\alpha f_m(x) = D^\alpha\left(\lim_{n\to\infty} f_m(x)\right). \qquad (7)$$

Theorem 23. Let $f, g: J \to \mathbb{R}$ be α-differentiable functions, $\alpha \in (0,1]$, at $x_0 \in I$, where I and J are intervals such that $\kappa(I) \subseteq J$.Then

1) If f is differentiable at κ, and κ is differentiable at x_0, then $D^\alpha f(x_0) = f'_\alpha(x_0)\kappa'(x_0)$.
2) If $f(x) = c$, where c is a constant, for all $x \in I$, then $D^\alpha f(x_0) = 0$.
3) $D^\alpha(cf)(x_0) = cD^\alpha f(x_0)$, where c is a constant.
4) $D^\alpha(af + bg)(x_0) = aD^\alpha f(x_0) + bD^\alpha g(x_0)$, for all $a, b \in \mathbb{R}$.
5) $D^\alpha(fg)(x_0) = g_\alpha D^\alpha f(x_0) + f_\alpha D^\alpha g(x_0)$.
6) $D^\alpha\left(\frac{f}{g}\right)(x_0) = \frac{g_\alpha(x_0)D^\alpha f(x_0) - f_\alpha(x_0)D^\alpha g(x_0)}{g_\alpha^2(x_0)}$, if $g_\alpha(x_0) \neq 0$.

Proof. For part (1), we have

$$\begin{aligned}
D^\alpha f(x_0) &= \lim_{x\to x_0} \frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(\kappa(x)) - f(\kappa(x_0))}{x - x_0} \\
&= \lim_{\kappa(x)\to\kappa(x_0)} \frac{f(\kappa(x)) - f(\kappa(x_0))}{\kappa(x) - \kappa(x_0)} \lim_{x\to x_0} \frac{\kappa(x) - \kappa(x_0)}{x - x_0} \\
&= f'(\kappa(x_0))\, \kappa'(x_0) \\
&= f'_\alpha(x_0)\, \kappa'(x_0).
\end{aligned}$$

The proofs of parts (2),(3), and (4) follow directly from the definition 17. For (5), we have

$$\begin{aligned}
D^\alpha(fg)(x_0) &= \lim_{x\to x_0} \frac{(f_\alpha g_\alpha)(x) - (f_\alpha g_\alpha)(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{g_\alpha(\mathrm{x})\big(f_\alpha(\mathrm{x}) - f_\alpha(x_0)\big) + f_\alpha(x_0)\big(g_\alpha(x_\alpha) - g_\alpha(x_0)\big)}{x - x_0} \\
&= \lim_{x\to x_0} g_\alpha(x) \lim_{x\to x_0} \frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0} \\
&\quad + f_\alpha(x_0) \lim_{x\to x_0} \frac{g_\alpha(x_\alpha) - g_\alpha(x_0)}{x - x_0} \\
&= g_\alpha(x_0)D^\alpha f(x_0) + f_\alpha(x_0)D^\alpha g(x_0).
\end{aligned}$$

And to prove part (6),

$$\begin{aligned}
D^\alpha\left(\frac{f}{g}\right)(x_0) &= \lim_{x\to x_0} \frac{\frac{f_\alpha(x)}{g_\alpha(x)} - \frac{f_\alpha(x_0)}{g_\alpha(x_0)}}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f_\alpha(x)g_\alpha(x_0) - g_\alpha(x)f_\alpha(x_0)}{g_\alpha(x)g_\alpha(x_0)(x - x_0)} \\
&= \lim_{x\to x_0} \frac{1}{g_\alpha(x)} \lim_{x\to x_0} \frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0} \\
&\quad - \lim_{x\to x_0} \frac{g_\alpha(x) - g_\alpha(x_0)}{x - x_0} \lim_{x\to x_0} \frac{f_\alpha(x_0)}{g_\alpha(x)g_\alpha(x_0)} \\
&= \frac{g_\alpha(x_0)D^\alpha f(x_0) - f_\alpha(x_0)D^\alpha g(x_0)}{g_\alpha^2(x_0)}.
\end{aligned}$$

Theorem 24. If $f: J \to \mathbb{R}$ is α-differentiable, $\alpha \in (0,1]$, at $x_0 \in I$, where I and J are intervals such that $\kappa(I) \subseteq J$, then f is α-continuous, $\alpha \in (0,1]$, at x_0.

Proof. Since $D^\alpha f(x_0)$, $\alpha \in (0,1]$, exists and for all $x \in I, x \neq x_0$, that

$$f_\alpha(x) - f_\alpha(x_0) = \left(\frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0}\right)(x - x_0).$$

Then

$$\begin{aligned}
\lim_{x\to x_0}\big(f_\alpha(x) - f_\alpha(x_0)\big) &= \lim_{x\to x_0}\left(\frac{f_\alpha(x) - f_\alpha(x_0)}{x - x_0}\right)\lim_{x\to x_0}(x - x_0) \\
&= D^\alpha f(x_0).0 = 0.
\end{aligned}$$

Thus

$$\lim_{x\to x_0} f_\alpha(x) = f_\alpha(x_0).$$

Theorem 25 (Chain Rule for Fractional Derivative). Let $f: J_1 \to \mathbb{R}$ and $g: J_2 \to \mathbb{R}$ be such that $f_\alpha(J_1) \subseteq J_2$, where $J_1$and J_2 are intervals. If f is α-differentiable, $\alpha \in (0,1]$, at $x_0 \in I$, $\kappa(I) \subseteq J_1$, and g is differentiable at $f_\alpha(x_0)$, then

$$D^\alpha(g \circ f)(x_0) = g'\big(f_\alpha(x_0)\big)D^\alpha f(x_0). \qquad (8)$$

Proof. Since g is differentiable at $z_0 = f_\alpha(x_0)$, $\alpha \in (0,1]$, there exists $g'(z_0)$ such that

$$\lim_{\Delta z\to 0} \frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z} = g'(z_0).$$

Define h by

$$h(\Delta z) = \frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z} - g'(z_0).$$

Then

$$g(z_0 + \Delta z) - g(z_0) = \big(g'(z_0) + h(\Delta z)\big)\Delta z.$$

Taking $\Delta z = f_\alpha(x_0 + \Delta x) - f_\alpha(x_0)$ with $\Delta x \neq 0$, we get

$$\begin{aligned}
&g\big(f_\alpha(x_0 + \Delta x)\big) - g\big(f_\alpha(x_0)\big) \\
&= \Big(g'\big(f_\alpha(x_0)\big) + h(\Delta z)\Big)\big(f_\alpha(x_0 + \Delta x) - f_\alpha(x_0)\big).
\end{aligned}$$

Dividing both sides by Δx yields

$$\begin{aligned}
&\frac{g\big(f_\alpha(x_0 + \Delta x)\big) - g\big(f_\alpha(x_0)\big)}{\Delta x} \\
&= \Big(g'\big(f_\alpha(x_0)\big) + h(\Delta z)\Big)\left(\frac{f_\alpha(x_0 + \Delta x) - f_\alpha(x_0)}{\Delta x}\right).
\end{aligned}$$

Since f is α-differentiable, $\alpha \in (0,1]$, at x_0, it is α-continuous, $\alpha \in (0,1]$, at x_0. Letting $\Delta x \to 0$, then $\Delta z \to 0$, and consequently that $h(\Delta z) \to 0$. Then,

$$\begin{aligned}
&\lim_{\Delta x\to 0} \frac{g\big(f_\alpha(x_0 + \Delta x)\big) - g\big(f_\alpha(x_0)\big)}{\Delta x} \\
&= g'(f_\alpha(x_0)) \lim_{\Delta x\to 0} \frac{f_\alpha(x_0 + \Delta x) - f_\alpha(x_0)}{\Delta x}
\end{aligned}$$

implies that

$$D^\alpha(g \circ f)(x_0) = g'\big(f_\alpha(x_0)\big)\ D^\alpha f(x_0).$$

Fractional Derivative of Certain Functions of Order α, $\alpha \in (0,1]$

1) $D^\alpha(x^n) = \begin{cases} \alpha n x^{\alpha n - 1}, & \text{if } x \in (0, \infty), \\ \text{does not exist}, & \text{if } x = 0, \\ \alpha n(-1)^{n-1}(-x)^{\alpha n - 1}, & \text{if } x \in (-\infty, 0). \end{cases}$
2) $D^\alpha(e^x) = \begin{cases} \alpha x^{\alpha-1} e^{x^\alpha}, & \text{if } x \in (0, \infty), \\ \text{does not exist}, & \text{if } x = 0, \\ \alpha(-x)^{\alpha-1} e^{-(-x)^\alpha}, & \text{if } x \in (-\infty, 0). \end{cases}$
3) $D^\alpha\big(\ln(x)\big) = \begin{cases} \frac{\alpha}{x}, & \text{if } x \in (-\infty, 0) \cup (0, \infty), \\ \text{does not exist}, & \text{if } x = 0. \end{cases}$
4) $D^\alpha(a^x) = \begin{cases} \alpha\ln(a) x^{\alpha-1} a^{x^\alpha}, & \text{if } x \in (0, \infty), \\ \text{does not exist}, & \text{if } x = 0, \\ \alpha\ln(a)(-x)^{\alpha-1} a^{-(-x)^\alpha}, & \text{if } x \in (-\infty, 0). \end{cases}$

5) $D^{\alpha}(\log_a x) = \begin{cases} \frac{\alpha}{x\ln a}, & \text{if } x \in (-\infty,0)\cup(0,\infty), \\ \text{does not exist}, & \text{if } x = 0. \end{cases}$
6) $D^{\alpha}(\sin(x)) =$
$\begin{cases} \alpha x^{\alpha-1}\cos(x^{\alpha}), & \text{if } x \in (0,\infty), \\ \text{does not exist}, & \text{if } x = 0, \\ \alpha(-x)^{\alpha-1}\cos((-x)^{\alpha}), & \text{if } x \in (-\infty,0). \end{cases}$
7) $D^{\alpha}(\cos(x)) =$
$\begin{cases} -\alpha x^{\alpha-1}\sin(x^{\alpha}), & \text{if } x \in (0,\infty), \\ \text{does not exist},, & \text{if } x = 0, \\ \alpha(-x)^{\alpha-1}\sin((-x)^{\alpha}), & \text{if } x \in (-\infty,0). \end{cases}$
8) $D^{\alpha}(\sinh(x)) =$
$\begin{cases} \alpha x^{\alpha-1}\cosh(x^{\alpha}), & \text{if } x \in (0,\infty), \\ \text{does not exist}, & \text{if } x = 0, \\ \alpha(-x)^{\alpha-1}\cosh((-x)^{\alpha}), & \text{if } x \in (-\infty,0). \end{cases}$
9) $D^{\alpha}(\cosh(x)) =$
$\begin{cases} \alpha x^{\alpha-1}\sinh(x^{\alpha}), & \text{if } x \in (0,\infty), \\ \text{does not exist}, & \text{if } x = 0, \\ -\alpha(-x)^{\alpha-1}\sinh((-x)^{\alpha-1}), & \text{if } x \in (-\infty,0). \end{cases}$

Definition 26. Let $f: J \to \mathbb{R}$ be a function, where $J \subseteq \mathbb{R}$ is an interval, and let $\beta \in (n, n+1], n \in \mathbb{N}$. The β-derivative of f at $x_0 \in I$, $\kappa(I) \subseteq J$, is defined by

$$D^{\beta} f(x_0) = \lim_{x\to x_0} \frac{\left(f^{(n-1)}\right)_{\alpha}(x) - \left(f^{(n-1)}\right)_{\alpha}(x_0)}{x - x_0}, \quad (9)$$

$\alpha \in (0,1]$, provided that the limit exists, where $f^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of f.

Remark 27.
(1) The β-derivative $D^{\beta} f$ in Definition 26 is the fractional derivative of f of order β, $\beta \in (n, n+1]$, $n \in \mathbb{N}$.
(2) If $f^{(n-1)}$ is differentiable at $\kappa(x)$ and $\kappa(x)$ is differentiable at x_0, then

$$\begin{aligned} D^{\beta} f(x_0) &= \lim_{x\to x_0} \frac{\left(f^{(n-1)}\right)_{\alpha}(x) - \left(f^{(n-1)}\right)_{\alpha}(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f^{(n-1)}(\kappa(x)) - f^{(n-1)}(\kappa(x))(x_0)}{x - x_0} \\ &= \lim_{\kappa(x)\to\kappa(x_0)} \frac{f^{(n-1)}(\kappa) - f^{(n-1)}(\kappa_0)}{\kappa(x) - \kappa(x_0)} \lim_{x\to x_0} \frac{\kappa(x) - \kappa(x_0)}{x - x_0} \\ &= f_{\alpha}^{(n)}(x_0)\, \kappa'(x_0). \end{aligned}$$

Proposition 28. Every $(n+1)^{\text{th}}$ differentiable function is β-differentiable of order $\beta = n+1$, $n \in \mathbb{N}$ but not every β-differentiable function, $\beta \in (n, n+1)$, $n \in \mathbb{N}$ is $(n+1)$th differentiable.

Example 29. $f(x) = \ln\left(x - \frac{1}{3}\right)$ has no second derivative at $x = \frac{1}{3}$ but it is β-differentiable whenever $\beta \in (1,2)$.

Proposition 30. Suppose that $f: B \to \mathbb{R}$ is β_m-continuous on a nonempty set A and β_m-differentiable, $\beta_m \in (0,1]$, $m \in \mathbb{N}$, at $x_0 \in A$, $\kappa(A) \subseteq B$. If β_m converges to β, $\beta \in (0,1]$, then f is β-differentiable, $\beta \in (0,1]$, at x_0.

Theorem 31. Let $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of β-continuous, β-differentiable functions, $\beta \in (n, n+1]$, $n \in \mathbb{N}$, on a nonempty compact set A, $\kappa(A) \subseteq B$. If $\lim_{m\to\infty} f_{m,\alpha}^{(n-1)}(x_0)$ exists for some $x_0 \in A$ and $D^{\beta} f_m$ converges uniformly on A, then f_m converges uniformly on A and

$$\lim_{m\to\infty} D^{\beta} f_m(x) = D^{\beta}\left(\lim_{m\to\infty} f_m(x)\right).$$

Theorem 32 (Rolle's Theorem for Fractional Derivative). If $f: [c,d] \to \mathbb{R}$, $c < d$, is α-differentiable, $\alpha \in (0,1]$, on (a,b) and α-continuous, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, with $f_{\alpha}(a) = f_{\alpha}(b)$, then $D^{\alpha} f(x_o) = 0$ for some x_o in (a,b).

Proof. By Theorem 15 that $f_{\alpha}: [a,b] \to \mathbb{R}, \alpha \in (0,1]$, has a finite minimum value m and a finite maximum value M on $[a,b]$. If $m = M$, then f_{α} is a constant on (a,b) and $D^{\alpha} f(x) = 0$ for all $x \in (a,b)$. If $m \neq M$, then either $f_{\alpha}(x_o) = m$ or $f_{\alpha}(x_o) = M$ for some $x_o \in (a,b)$ because $f_{\alpha}(a) = f_{\alpha}(b)$. Suppose $f_{\alpha}(x_o) = m$. Then $f_{\alpha}(x_o + \varepsilon) - f_{\alpha}(x_o) \geq 0$ for all ε such that $x_o + \varepsilon \in (a,b)$. Therefore,

$$\lim_{\varepsilon\to 0^+} \frac{f_{\alpha}(x_o + \varepsilon) - f_{\alpha}(x_o)}{\varepsilon} \geq 0, \quad \text{if } \varepsilon > 0,$$

and

$$\lim_{\varepsilon\to 0^-} \frac{f_{\alpha}(x_o + \varepsilon) - f_{\alpha}(x_o)}{\varepsilon} \leq 0, \quad \text{if } \varepsilon < 0.$$

Thus, $D^{\alpha} f(x_o) = 0$.

Theorem 33 (Mean Value Theorem for Fractional Derivative). If $f: [c,d] \to \mathbb{R}$ is α-differentiable, $\alpha \in (0,1]$, on (a,b), $\kappa([a,b]) \subseteq [c,d]$, and α-continuous, $\alpha \in (0,1]$, on $[a,b]$, then

$$\left(\kappa(b) - \kappa(a)\right) D^{\alpha} f(x_o) = \left(f_{\alpha}(b) - f_{\alpha}(a)\right)\kappa'(x_o), \quad (10)$$

for $x_o \in (a,b)$.

Proof. Let $h_{\alpha}: [a,b] \to \mathbb{R}$ be defined by

$$h_{\alpha}(x) = f_{\alpha}(x) - \kappa(x)\frac{f_{\alpha}(b) - f_{\alpha}(a)}{\kappa(b) - \kappa(a)},$$

$\alpha \in (0,1]$. Then

$$D^{\alpha} h(x) = D^{\alpha} f(x) - \kappa'(x)\frac{f_{\alpha}(b) - f_{\alpha}(a)}{\kappa(b) - \kappa(a)}.$$

Note that h is α-continuous, $\alpha \in (0,1]$, on $[a,b]$ and α-differentiable, $\alpha \in (0,1]$, on (a,b) with $h_{\alpha}(a) = h_{\alpha}(b)$. By Theorem 30, there is $x_o \in (a,b)$ such that $D^{\alpha} h(x_o) = 0$ and

$$D^{\alpha} f(x_o) = \kappa'(x_o)\frac{f_{\alpha}(b) - f_{\alpha}(a)}{\kappa(b) - \kappa(a)}.$$

D. Fractional Integral

Definition 34. Let $f: [c,d] \to \mathbb{R}$ be an α-bounded function, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, and let $P = \{x_k\}_{k=1}^{n}$ be a partition of $[a,b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The α-Riemann sum, $\alpha \in (0,1]$, of f over P is defined by

$$S_{\alpha}(f,P) = \sum_{i=1}^{n} f_{\alpha}(x_i^*)\kappa'(x_i^*)(x_i - x_{i-1}), \quad (11)$$

for any selected point $x_i^* \in [x_{i-1}, x_i]$.

Definition 35. We say that $f: [c,d] \to \mathbb{R}$ is an α-Riemann integrable function, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, and a real number I_α is the α-definite integral, $\alpha \in (0,1]$, of f over $[a,b]$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every partition $P = \{x\}_{i=1}^{n}$ of $[a,b]$ with $\|P\| = \max_{1\le i\le n}\{x_i - x_{i-1}\} < \delta$ and for any $x_i^* \in [x_{i-1}, x_i]$, then

$$\|S_\alpha(f,P) - I_\alpha\| < \varepsilon \tag{12}$$

and we write it as

$$\alpha - \int_a^b f(x)dx = I_\alpha = \lim_{\|P\|\to 0} S_\alpha(f,P) = \int_a^b f_\alpha(x)\kappa'(x)dx. \tag{13}$$

And we will denote to the set of all α-Riemann integrable functions on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, by $\mathcal{R}_\alpha[a,b]$, $\alpha \in (0,1]$.

Remark 36. The α-integral $\alpha - \int_a^b f(x)dx$, $\alpha \in (0,1]$, in Definition 35 is the fractional Riemann integral of f of order $\alpha, \alpha \in (0,1]$.

Proposition 37. Every Riemann integrable function is α-Riemann integrable of order $\alpha = 1$ but not every α-Riemann integrable function is Riemann integrable.

Example 38. $f(x) = \frac{1}{x}$ is not Riemann integrable on $(0,1)$ whenever $\alpha = 1$ but it is α-Riemann integrable whenever $\alpha \in (0,1)$.

Proposition 39. Suppose that $f: [c,d] \to \mathbb{R}$ is α_m-continuous and α_m-Riemann integrable, $\alpha_m \in (0,1]$, $m \in \mathbb{N}$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$. If α_m converges to α, $\alpha \in (0,1]$, then f is α-Riemann integrable, $\alpha \in (0,1]$, on $[a,b]$.

Theorem 40. Let $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of α-continuous, α-Riemann integrable functions, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$. If $f_m: B \to \mathbb{R}$, $m \in \mathbb{N}$, converges uniformly to $f: [c,d] \to \mathbb{R}$ then f is α-Riemann integrable, $\alpha \in (0,1]$, on $[a,b]$.

Theorem 41. If $f \in \mathcal{R}_\alpha[a,b]$, $\alpha \in (0,1]$, then the α-Riemann integral, $\alpha \in (0,1]$, of f is unique.

Proof. Assume that $I_{\alpha,1}$ and $I_{\alpha,2}$ are α-Riemann integrals, $\alpha \in (0,1]$, of f and let $\varepsilon > 0$ be given. Then for $i = 1,2$, there exists $\delta_i = \delta_i(\varepsilon/2) > 0$ such that

$$\|P\| < \delta_i \implies \|S_\alpha(f,P) - I_{\alpha,i}\| < \frac{\varepsilon}{2}.$$

where P is any partition of $[a,b]$. Letting $\delta = \min\{\delta_1, \delta_2\}$, we get

$$\begin{aligned} 0 \le \|I_{\alpha,1} - I_{\alpha,2}\| &= \|I_{\alpha,1} - S_\alpha(f,P) + S_\alpha(f,P) - I_{\alpha,2}\| \\ &\le \|I_{\alpha,1} - S_\alpha(f,P)\| + \|S_\alpha(f,P) - I_{\alpha,2}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, then

$$0 \le \|I_{\alpha,1} - I_{\alpha,2}\| < \varepsilon$$

holds for all $\varepsilon > 0$. Thus, $\|I_{\alpha,1} - I_{\alpha,2}\| = 0$ and $I_{\alpha,1} = I_{\alpha,2}$.

Proposition 42. The α-definite integral, $\alpha \in (0,1]$, in (13) can be written as follows

$$\alpha - \int_a^b f(x)dx = \int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau. \tag{14}$$

Proof. The proof is directly from the Theorem (Change of Variables for Continuous Integrands) (see e.g., [26]).

Now we use the formula (14) of the fractional integral to introduce the following theorems.

Theorem 43. Let $f, g \in \mathcal{R}_\alpha[a,b]$, $\alpha \in (0,1]$. Then

1) $\int_{\kappa(b)}^{\kappa(a)} f(\tau)d\tau = -\int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau.$
2) $\int_{\kappa(a)}^{\kappa(a)} f(\tau)d\tau = 0.$
3) $\int_{\kappa(a)}^{\kappa(b)} cf(\tau)d\tau = c\int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau, c \in \mathbb{R}.$
4) $\int_{\kappa(a)}^{\kappa(b)} \big(f(\tau) \pm g(\tau)\big)d\tau = \int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau \pm \int_{\kappa(a)}^{\kappa(b)} g(\tau)d\tau.$
5) $\int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau + \int_{\kappa(b)}^{\kappa(c)} f(\tau)d\tau = \int_{\kappa(a)}^{\kappa(c)} f(\tau)d\tau.$
6) If f is α-continuous on $[a,b]$, then
$$\min_x f_\alpha(x) \le \frac{1}{\kappa(b) - \kappa(a)} \int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau \le \max_x f_\alpha(x). \tag{15}$$
7) If $f_\alpha(x) \le g_\alpha(x)$ on $[a,b]$, then
$$\int_{\kappa(a)}^{\kappa(b)} f(\tau)d\tau \le \int_{\kappa(a)}^{\kappa(b)} g(\tau)d\tau. \tag{16}$$

Theorem 44 (Mean Value Theorem for Fractional Definite Integrals). If $f: [c,d] \to \mathbb{R}$ is α-continuous $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, then

$$f_\alpha(x_o) = \frac{1}{\kappa(b) - \kappa(a)} \int_{\kappa(a)}^{\kappa(b)} f_\alpha(\tau)d\tau \tag{17}$$

for some point $x_o \in [a,b]$.

Proof. The proof is directly from (15) and by applying Theorem 7.

Theorem 45 (Fundamental Theorem of Fractional Calculus).

1) If $f: [c,d] \to \mathbb{R}$ is α-continuous, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, and κ is continuously differentiable on $[a,b]$, then
$$\frac{d}{d\kappa(x)}\left(\int_{\kappa(a)}^{\kappa(x)} f(\tau)d\tau\right) = f_\alpha(x), \tag{18}$$
for each $x \in [a,b]$.
2) If $f: [c,d] \to \mathbb{R}$ is α-differentiable, $\alpha \in (0,1]$, on $[a,b]$, $\kappa([a,b]) \subseteq [c,d]$, f' is α-continuous, $\alpha \in (0,1]$, on $[a,b]$ and κ is continuously differentiable on $[a,b]$, then
$$\int_{\kappa(a)}^{\kappa(b)} f'(\tau)d\tau = f_\alpha(b) - f_\alpha(a). \tag{19}$$

Proof. To prove part (1), for any $x, x + \Delta x \in [a,b]$, assume that $\kappa(x + \Delta x) = \kappa(x) + \Delta\kappa(x)$ and let

$$F_\alpha(x) = \int_{\kappa(a)}^{\kappa(x)} f(\tau)d\tau$$

and

$$F_\alpha(x + \Delta x) = \int_{\kappa(a)}^{\kappa(x)+\Delta\kappa(x)} f(\tau)d\tau.$$

for $f_\alpha:[a,b]\to\mathbb{R}$, $\alpha\in(0,1]$. Subtracting the last two equalities gives

$$F_\alpha(x+\Delta x)-F_\alpha(x)=\int_{\kappa(a)}^{\kappa(x)+\Delta\kappa(x)} f(\tau)d\tau-\int_{\kappa(a)}^{\kappa(x)} f(\tau)d\tau$$
$$=\int_{\kappa(x)}^{\kappa(x)+\Delta\kappa(x)} f(\tau)d\tau.$$

By Theorem 44, we have

$$F_\alpha(x+\Delta x)-F_\alpha(x)=f_\alpha(x_o)\Delta\kappa(x).$$

for some point $x_o\in[a,b]$. Dividing both sides by $\Delta\kappa(x)$ gives

$$\frac{F_\alpha(x+\Delta x)-F_\alpha(x)}{\Delta\kappa(x)}=f_\alpha(x_o).$$

Letting $\Delta\kappa(x)\to 0$ ($\Delta x\to 0$) on both sides of the equation, we get

$$\frac{d}{d\kappa(x)}F\big(\kappa(x)\big)=\lim_{\Delta\kappa(x)\to 0}\frac{F_\alpha(x+\Delta x)-F_\alpha(x)}{\Delta\kappa(x)}$$
$$=\lim_{\Delta\kappa(x)\to 0} f_\alpha(x_o)=f_\alpha(x),$$

where $\lim\limits_{\Delta x\to 0} x_o=x$ by the Squeeze Theorem.

For part (2), let

$$g_\alpha(x)=\int_{\kappa(a)}^{\kappa(x)} f'(\tau)d\tau,$$

for $f_\alpha:[a,b]\to\mathbb{R},\alpha\in(0,1]$. By part (1), we have that $g'_\alpha(x)=f'_\alpha(x)$. Thus, there is a constant ν such that $g_\alpha(x)-f_\alpha(x)=\nu$ for all $x\in[a,b]$. Since

$$g_\alpha(a)=\int_{\kappa(a)}^{\kappa(a)} f'(\tau)d\tau=0$$

and $g_\alpha(a)-f_\alpha(a)=\nu$, then $\nu=-f_\alpha(a)$ and $g_\alpha(x)-f_\alpha(x)=-f_\alpha(a)$ for all $x\in[a,b]$. Moreover, since $g_\alpha(b)-f_\alpha(b)=-f_\alpha(a)$, then $g_\alpha(b)=f_\alpha(b)-f_\alpha(a)$.

Fractional Integral of Certain Functions of Order $\alpha,\alpha\in(0,1]$

1) $\alpha-\int_a^b x^n dx=\frac{1}{n+1}\big(\kappa^{n+1}(b)-\kappa^{n+1}(a)\big)$.

2) $\alpha-\int_a^b e^x dx=e^{\kappa(b)}-e^{\kappa(a)}$.

3) $\alpha-\int_a^b \frac{1}{x}dx=\ln\big(\kappa(b)\big)-\ln\big(\kappa(a)\big)$.

4) $\alpha-\int_a^b \sin(x)dx=\cos\big(\kappa(a)\big)-\cos\big(\kappa(b)\big)$.

5) $\alpha-\int_a^b \cos(x)dx=\sin\big(\kappa(b)\big)-\sin\big(\kappa(a)\big)$.

6) $\alpha-\int_a^b \sinh(x)dx=\cosh\big(\kappa(b)\big)-\cosh\big(\kappa(a)\big)$.

7) $\alpha-\int_a^b \cosh(x)dx=\sinh\big(\kappa(b)\big)-\sinh\big(\kappa(a)\big)$.

V. APPLICATIONS

Now we apply our results to solve some interesting fractional differential equations.

Example 46 [14]. The equation $y^{(1/2)}+y=t^2+2t^{3/2}$ is converted to

$$y'_{1/2}+2\kappa y_{1/2}=2\kappa^3+4\kappa^{5/2}$$

because $y^{(1/2)}=\frac{1}{2}\kappa^{-1}y'_{1/2}$. By the method of integral factor, the general solution is

$$y_{1/2}(t)=e^{-\kappa^2}\left(\int\left(2\kappa^3+4\kappa^{5/2}\right)e^{\kappa^2}d\kappa+c\right),$$

where c is a constant.

Example 47 [14]. The equation $y^{(1/2)}+\sqrt{t}y=te^{-t}$ becomes

$$y'_{1/2}+2\kappa^{3/2}y_{1/2}=2\kappa^2e^{-\kappa}$$

where $y^{(1/2)}=\frac{1}{2}\kappa^{-1}y'_{1/2}$. Using the method of integral factor, the general solution is

$$y_{1/2}(t)=e^{-\frac{4}{5}\kappa^{5/2}}\left(2\int\kappa^2\exp\left(\frac{4}{5}\kappa^{5/2}-\kappa\right)d\kappa+c\right),$$

where c is a constant.

Example 48. The equation $D^{1/4}y+\frac{2t}{1+t^2}y=3t$ with $y_{1/4}(1)=4$ is converted to

$$y'_{1/4}+\frac{8\kappa^4}{1+\kappa^2}y_{1/4}=12\kappa^4,$$

where $D^{1/4}y=\frac{1}{4}\kappa^{-3}y'_{1/4}$. The method of integral factor gives the general solution

$$y_{1/4}(t)=\exp\left(\frac{-8\kappa^3}{3}+8\kappa-8\tan^{-1}(\kappa)\right)$$
$$\cdot\left(\int_1^\kappa 12\exp\left(\frac{8s^3}{3}-8s+8\tan^{-1}(s)\right)s^4ds+c\right),$$

where c is a constant. Plugging the initial condition into the general solution, we get

$$y_{1/4}(t)=\exp\left(\frac{-8\kappa^3}{3}+8\kappa-8\tan^{-1}(\kappa)\right)$$
$$\cdot\left(\int_1^\kappa 12\exp\left(\frac{8s^3}{3}-8s+8\tan^{-1}(s)\right)s^4ds\right.$$
$$\left.+4\exp\left(2\pi-\frac{16}{3}\right)\right).$$

Example 49. The equation $D^{0.7}y=\frac{3t^2+4t+2}{2(y-1)}$ has the converted form

$$y'_{0.7}=\frac{\kappa^{3/7}(3\kappa^2+4\kappa+2)}{1.4(y_{0.7}-1)}$$

because $D^{0.7}y=0.7\kappa^{-3/7}y'_{0.7}$. Using the separation of variables method, the general solution is

$$y_{0.7}(t)=1-\sqrt{1-2(-c-0.00735294(85\kappa^2+160\kappa+136)\kappa^{10/7})}$$
$$=1-\sqrt{1-2(-c-0.00735294(85t^{1.4}+160t^{0.7}+136)t)}$$

or

$$y_{0.7}(t)=1+\sqrt{1-2(-c-0.00735294(85\kappa^2+160\kappa+136)\kappa^{10/7})}$$
$$=1+\sqrt{1-2(-c-0.00735294(85t^{1.4}+160t^{0.7}+136)t)},$$

where c is a constant.

Example 50. The equation $4D^{3/2}y - 8D^{1/2}y + 3t^{-1}y = 0$ is transferred to the equation

$$y''_{1/2} - 2y'_{1/2} + \tfrac{3}{2}y_{1/2} = 0.$$

where $D^{3/2}y = \frac{1}{2}\kappa^{-1}y''_{1/2}$ and $D^{1/2}y = \frac{1}{2}\kappa^{-1}y'_{1/2}$.
Hence, the general solution is

$$y_{1/2}(t) = c_1e^{\kappa}\cos\left(\frac{\kappa}{\sqrt{2}}\right) + c_2e^{\kappa}\sin\left(\frac{\kappa}{\sqrt{2}}\right)$$
$$= c_1e^{x^{\frac{1}{2}}}\cos\left(\frac{t^{1/2}}{\sqrt{2}}\right) + c_2e^{x^{\frac{1}{2}}}\sin\left(\frac{t^{1/2}}{\sqrt{2}}\right),$$

where c_1 and c_2 are constants.

Example 51. The equation $4y^{\left(\frac{8}{5}\right)} + 4y' + 0.6t^{-\frac{2}{3}}y = \frac{e^{-\frac{t}{2}}}{t^2}$, $t > 0$ is converted to the equation

$$y''_{0.6} + y'_{0.6} + \tfrac{1}{4}y_{0.6} = \frac{e^{-\kappa/2}}{2.4\kappa^{4/3}}$$

because $y^{(8/5)} = 0.6\kappa^{-2/3}y''_{0.6}$. Using the variation of parameters method, the homogeneous equation

$$y''_{0.6} + y'_{0.6} + \tfrac{1}{4}y_{0.6} = 0$$

has the complementary solution

$$y_{0.6,c}(t) = (v_1 + v_2\kappa)e^{-\kappa/2} = (v_1 + v_2t^{0.6})e^{-t^{0.6}/2}.$$

where v_1, v_2 are constants. Then, the fundamental solutions are

$$y_{0.6,1}(t) = e^{-\kappa/2} = e^{-t^{0.6}/2},$$
$$y_{0.6,2}(t) = \kappa e^{-\kappa/2} = t^{0.6}e^{-t^{0.6}/2}$$

and the Wronskian is

$$W(y_{0.6,1}, y_{0.6,2}) = y_{0.6,1}y'_{0.6,2} - y'_{0.6,1}y_{0.6,2} = e^{-\kappa}.$$

Since $g_{0.6}(t) = \frac{e^{-\kappa/2}}{2.4\kappa^{4/3}}$, then

$$u_1(t) = -\int \frac{y_{0.6,2}(t)g_{0.6}(t)}{W(y_{0.6,1}, y_{0.6,2})}d\kappa = -0.625\kappa^{\frac{2}{3}} = -0.625t^{0.4},$$
$$u_2(t) = \int \frac{y_{0.6,1}(t)g_{0.6}(t)}{W(y_{0.6,1}, y_{0.6,2})}d\kappa = -1.25\kappa^{-\frac{1}{3}} = -1.25t^{-0.2}.$$

Thus, the particular solution is

$$Y_{0.6}(t) = u_1(t)y_{0.6,1}(t) + u_2(t)y_{0.6,2}(t)$$
$$= -1.875\kappa^{2/3}e^{-\kappa/2}$$
$$= -1.875t^{0.4}e^{-t^{0.6}/2}.$$

and the general solution is

$$y_{0.6}(t) = y_{0.6,c}(t) + Y_{0.6}(t)$$
$$= (v_1 + v_2\kappa)e^{-\kappa/2} - 1.875\kappa^{2/3}e^{-\kappa/2}$$
$$= (v_1 + v_2t^{0.6})e^{-t^{0.6}/2} - 1.875t^{0.4}e^{-t^{0.6}/2}.$$

Example 52. The equation $x^{\frac{13}{3}}D^{1.3}y + x^{\frac{10}{3}}D^{0.3}y - 0.3y = 0$ has a transferred Euler differential equation

$$\kappa^2y''_{0.3} + \kappa y'_{0.3} - y_{0.3} = 0$$

where $D^{1.3}y = 0.3\kappa^{-7/3}y''_{0.3}$ and $D^{0.3}y = 0.3\kappa^{-7/3}y'_{0.3}$.
Assume that $y_{0.3} = \kappa^r$, where r is a constant. Then

$$0 = \kappa^2y''_{0.3} + \kappa y'_{0.3} - y_{0.3}$$
$$= \kappa^2[\kappa^r]'' + \kappa[\kappa^r]' - [\kappa^r]$$
$$= \kappa^2[r(r-1)\kappa^{r-2}] + \kappa[r\kappa^{r-1}] - [\kappa^r]$$
$$= [r^2 - 1]\kappa^r.$$

Since $\kappa \neq 0$, then $r_1 = -1$ and $r_2 = 1$. Thus, the general solution is

$$y_{0.3}(x) = c_1\kappa^{-1} + c_2\kappa = c_1x^{-0.3} + c_2x^{0.3},$$

where c_1 and c_2 are constants.

Example 53. The equation $D^{5/4}y - \kappa y = 0$ becomes

$$y''_{1/4} - 4\kappa^4y_{1/4} = 0$$

because $D^{5/4}y = \frac{1}{4}\kappa^{-3}y''_{1/4}$. To find the series solution, assume that the general solution has the form

$$y_{1/4}(x) = \sum_{n=0}^{\infty} a_n\kappa^n.$$

Then

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\kappa^n - \sum_{n=4}^{\infty}4a_{n-4}\kappa^n = 0$$

gives

$$2a_2 + 6a_3\kappa + 12a_4\kappa^2 + 20a_5\kappa^3$$
$$+ \sum_{n=4}^{\infty}(n+1)(n+2)a_{n+2}\kappa^n$$
$$= \sum_{n=4}^{\infty}4a_{n-4}\kappa^n.$$

and

$$a_2 = a_3 = a_4 = a_5 = 0,$$
$$a_{n+2} = \frac{4}{(n+1)(n+2)}a_{n-4}, \qquad n = 4,5,6,\ldots$$

produces the formulas

$$a_{6n} = \frac{4^na_0}{5.6.11.12.\ldots(6n-1)(6n)},$$
$$a_{6n+1} = \frac{4^na_1}{6.7.12.13.\ldots(6n)(6n+1)}.$$

Thus, the general solution is

$$y_{1/4}(x) = a_0\left[1 + \frac{4\kappa^6}{5.6} + \frac{4^2\kappa^{12}}{5.6.11.12} + \cdots + \frac{4^n\kappa^{6n}}{5.6.\ldots(6n-1)(6n)} + \cdots\right]$$
$$+ a_1\left[\kappa + \frac{4\kappa^7}{6.7} + \frac{4^2\kappa^{13}}{6.7.12.13} + \cdots + \frac{4^n\kappa^{6n+1}}{6.7\ldots(6n)(6n+1)} + \cdots\right]$$
$$= a_0\left[1 + \frac{4x^{\frac{3}{2}}}{5.6} + \frac{4^2x^3}{5.6.11.12} + \cdots + \frac{4^nx^{\frac{6n}{4}}}{5.6.\ldots(6n-1)(6n)} + \cdots\right]$$
$$+ a_1\left[x^{\frac{1}{4}} + \frac{4x^{\frac{7}{4}}}{6.7} + \frac{4^2x^{\frac{13}{4}}}{6.7.12.13} + \cdots + \frac{4^nx^{\frac{6n+1}{4}}}{6.7\ldots(6n)(6n+1)} + \cdots\right].$$

VI. CONCLUSION

The theory of fractional calculus has been studied as the natural extension of the theory of classical calculus. The fractional derivative and the fractional integral have been defined using the map $\kappa(x)$ instead of x in the definitions of the classical derivative and the classical integral respectively. Moreover, the fractional differential equations have been solved using the solution methods of ordinary differential equations.

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