An Optimal Fourth Order Iterative Method for Solving Non-linear Equations

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ABSTRACT — For obtaining a simple root of nonlinear equations, we present an optimum fourth-order iterative technique. By examining certain test problems, we investigate the proposed method's convergence criteria and establish its validity and efficiency. Finally, based on numerical and graphical data, it was determined that our methods are comparable in terms of order, efficiency, and processing time to existing methods of similar kind.

Index Terms — Iterative Method, Non-linear Equation, Functional evaluations, Order of Convergence, Efficiency Index.

I. Introduction

Many mathematical modeling of any knowledge in science and engineering contains non-linear equations in the form of

$$h(t) = 0 \tag{1.1}$$

Where $h : D \subseteq R \to R$ is a scalar function on an open interval D. While there is no closed-form solution, these equations regularly emerge in real-world problems. As a result, the numerical solution of these equations is receiving a lot of interest these days. Multi-point iterations are used in the most efficient extant root solvers because they transcend the theoretical limits of one-point approaches in terms of convergence order and computational efficiency. Ostrowski [1] proposed the concept of efficiency index as a measure for comparing the efficiency of different methods. This index is described by $E = P^{1/N}$, where P is the order of convergence and N is the total number of function evaluations per iteration. Kung and Traub [7] proposed that an iteration method without memory based on N functional evaluations could achieve optimal convergence order 2^{N-1}. These iterative methods could be derived using a variety of approaches, such as Taylor series, decomposition, quadrature, and homotopy methods.

Among all the best approaches, one of the well-known approaches for obtaining the zero of "(1.1)" is the classical second-order Newton's method (NR) [6]

$$t_{n+1} = t_n - \frac{h(t_n)}{h'(t_n)} \tag{1.2}$$

We choose some existing optimal fourth-order methods given as follows:

Francisco-Cordero-Garrido-Juan [5] proposed a two-step novel optimal two-step method (FR) with fourth-order convergence

$$y_n = t_n - \frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1} = t_n - \frac{h^2(t_n) + h(t_n)h(y_n) + 2h^2(y_n)}{h(t_n)h'(t_n)} \tag{1.3}$$

Traub-Ostrowski [6] suggested an optimal two-step with fourth-order convergence algorithm (TR), which is given by

$$y_n = t_n - \frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1} = y_n - \frac{h(y_n)}{h'(t_n)}\left(\frac{h^2(t_n)}{h^2(t_n) - 2h(t_n)h(y_n)}\right) \tag{1.4}$$

Chun-Lee-Neta-Jovana [4] presented another optimal fourth-order convergence algorithm (CH)

$$y_n = t_n - \frac{2h(t_n)}{3h'(t_n)}$$

$$t_{n+1} = t_n + \frac{h'(t_n) + 3h'(y_n)}{2h'(t_n) - 6h'(y_n)}\frac{h(t_n)}{h'(t_n)} \tag{1.5}$$

Chun [3] suggested a two-step iterative technique with an ideal fourth-order convergence mechanism (KT)

$$y_n = t_n - \frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1} = y_n - \frac{h(y_n)}{h'(y_n)}\left(\frac{h(t_n) + 3h(y_n)}{h(t_n) + h(y_n)}\right) \tag{1.6}$$

Ramandeep-Cordero [11] proposed a new iterative approach (RA) for solving nonlinear equations

$$y_n = t_n - \frac{2h(t_n)}{h'(t_n)}$$

$$t_{n+1} = z_n - \frac{h(z_n)}{h'(t_n)}\left(1 + \frac{2h(z_n)}{h(t_n)}\right) \tag{1.7}$$

where $z_n = \frac{t_n + y_n}{2}$.

Rajni-Bahl [10] proposes a second optimal two-step approach (RS) with fourth-order convergence.

$$y_n = t_n - \frac{h(t_n)}{h'(t_n)}$$

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$$t_{n+1}=t_n-\left(\frac{-1}{2}+\frac{9h'(t_n)}{8h'(y_n)}+\frac{3h'(y_n)}{8h'(t_n)}\right)\frac{h(t_n)}{h'(t_n)} \quad (1.8)$$

Santiago-Francisco [12] proposes an optimal fourth-order convergence iterative approach (SA)

$$y_n=t_n-\frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1}=t_n-\frac{h^2(t_n)+h(t_n)h(y_n)+2h^2(y_n)}{h(t_n)h'(t_n)} \quad (1.9)$$

Anuradha-Jaiswal [2] has presented an efficient optimum method (AN) with fourth-order convergence

$$y_n=t_n-\frac{2h(t_n)}{3h'(t_n)}$$

$$t_{n+1}=t_n-\left(1+\frac{9}{16}\left(\frac{h'(y_n)}{h'(t_n)}-1\right)^2\right)\left(\frac{4h(t_n)}{h'(t_n)+3h'(y_n)}\right) \quad (1.10)$$

Soleymani [14] proposes an optimal fourth-order iterative approach (SO) that is free of derivatives

$$y_n=t_n-\frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1}=y_n-\left\{\frac{\frac{(h(t_n))^2}{h'(t_n)^2-2h(t_n)h(y_n)}\frac{h(y_n)}{h'(t_n)}}{\left(1+\frac{(h(y_n))^2}{(h(t_n))^2}\right)\left(1+\frac{(h(y_n))^2}{(h'(t_n))^2}\right)\left(1+\frac{(h(t_n))^2}{(h'(t_n))^2}\right)}\right\} \quad (1.11)$$

We begin with many one-step iterative methods in this study, including the classical Newton's method and a new scheme for solving nonlinear equations. To improve the presented iterative method, we use the approximants of the higher derivatives to avoid calculating the function's high-order derivatives. As a result, we can design an iterative formula without having to calculate high-order derivatives.

The remainder of this work is arranged in the following manner. We provide a new two-step optimal fourth-order iterative approach with fast convergence speed in the following part, and we show that the proposed method is at least fourth-order convergent in the following section. A comparison of our new proposed method with previous optimal schemes of similar type utilizing examples from the literature of numerical methods is shown in the penultimate part.

II. Fourth order convergent method

Consider t^* is an exact root of "(1.1)" where $h(t)$ is continuous and has well defined first derivatives. Let t_n be the root of n[th] approximation of "(1.1)" and is

$$t^*=t_n+\varepsilon_n \quad (2.1)$$

where ε_n is the error. Thus, we get

$$h(t^*)=0 \quad (2.2)$$

writing $h(t^*)$ by Taylor's series about t_n, we have

$$h(t^*)=h(t_n)+\varepsilon_n h'(t_n)+\frac{\varepsilon_n^2}{2!}h''(t_n)+\ldots \quad (2.3)$$

Here higher powers of ε_n are neglected that to from ε_n^3 onwards. Using "(2.2)" and "(2.3)", we have

$$\varepsilon_n=\left[-2h'(t_n)\pm\sqrt{4h'(t_n)-8h(t_n)h''(t_n)}\right]\div 2h''(t_n) \quad (2.4)$$

On Substituting t^* by t_{n+1} in "(2.1)" and from "(2.4)", we get

$$t_{n+1}=t_n-\frac{2h(t_n)}{h'(t_n)}\left(1+G(t_n)\right)^{-1} \quad (2.5)$$

Where

$$G(t_n)=(1-2\mu_n)^{\frac{1}{2}},\ \mu_n=\frac{h(t_n)h''(t_n)}{[h'(t_n)]^2},$$

$$h'(t_n)=2\,h[t_{n-1},t_n]-h'(t_{n-1}) \text{ and}$$

$$h''(t_n)=\frac{2}{t_{n-1}-t_n}\left[h'(t_{n-1})-\frac{h(t_{n-1})-h(t_n)}{t_{n-1}-t_n}\right]$$

We develop the algorithm by taking "(1.2)" as the first step and "(2.5)" as the second step.

Algorithm: The iterative scheme is computed by t_{n+1} as

$$z_n=t_n-\frac{h(t_n)}{h'(t_n)}$$

$$t_{n+1}=z_n-\frac{2h(z_n)}{h'(z_n)}\left(1+G(t_n)\right)^{-1} \quad (2.6)$$

where

$$G(t_n)=(1-2\rho_n)^{\frac{1}{2}},\ \rho_n=\frac{h(z_n)h''(z_n)}{[h'(z_n)]^2},$$

$$h'(z_n)=2\,h[t_n,z_n]-h'(t_n)$$

and $h''(z_n)=\frac{2}{t_n-z_n}\left[h'(t_n)-\frac{h(t_n)-h(z_n)}{t_n-z_n}\right]$

The method "(2.6)" is called a fourth-order convergent method (MMS), which requires two functional evaluations and one of its first derivative.

III. Convergence Criteria

Theorem: Let $t_0\in D$ be a single zero of a sufficiently differentiable function *h* for an open interval D. If t_0 is in the neighborhood of t^*. Then "(2.12)" has fourth-order convergence.

Proof: Let the single zero of (1.1) be t^* and $t^*=t_n+\varepsilon_n$ then $h(t^*)=0$

By Taylor's series, writing $h(t^*)$ about t_n, we obtain

$$h(t_n)=h'(t^*)\left(\varepsilon_n+c_2\varepsilon_n^2+c_3\varepsilon_n^3+c_4\varepsilon_n^4+\ldots\right) \quad (3.1)$$

$$h'(t_n)=h'(t^*)\left(1+2c_2\varepsilon_n+c_3\varepsilon_n^2+4c_4\varepsilon_n^3+\ldots\right) \quad (3.2)$$

Replacing "(3.1)" and "(3.2)" in the first step of "(2.6)", we get

$$z_n=t^*+c_2\varepsilon_n^2+\left(2c_3-2c_2^2\right)\varepsilon_n^3+\left(3c_4-7c_2c_3+4c_2^3\right)\varepsilon_n^4+\ldots \quad (3.3)$$

From "(3.3)", we obtain

$$h(z_n) = h(t^*)\begin{pmatrix} c_2\varepsilon_n^{\,2} + \left(2c_3 - 2c_2^{\,2}\right)\varepsilon_n^{\,3} + \\ \left(3c_4 - 7c_2c_3 + 5c_2^3\right)\varepsilon_n^{\,4} + \ldots \end{pmatrix} \quad (3.4)$$

$$h'(z_n) = h(t^*)\left(1 + \left(2c_2^2 - c_3\right)\varepsilon_n^2 + \left(6c_2c_3 - c_2^3 - 2c_4\right)\varepsilon_n^3 + \ldots\right) \quad (3.5)$$

$$h''(z_n) = h(t^*)\left(2c_2 + 4c_3\varepsilon_n + \left(2c_2c_3 + 6c_4\right)\varepsilon_n^2 + \ldots\right) \quad (3.6)$$

Putting "(3.4)" , "(3.5)" and "(3.6)" in the second step of "(2.6)", we get

$$\varepsilon_{n+1} = \left(-c_2c_3\right)\varepsilon_n^4 + o\left(\varepsilon_n^5\right)$$

Thus, we proved the convergence of this new method which is of fourth-order and its efficiency index is $\sqrt[3]{4} = 1.587$.

IV. NUMERICAL EXAMPLES

We offer numerical results on various test equations to check the performance of the fourth-order technique defined by method "(2.6)." We also compare their findings to those obtained using the NR, SO, AN, SA, RS, RA, KI, CH, TR, and FR methodologies. All numerical computations are performed using the mpmath-PYTHON package, starting with a supplied initial approximation t0. Because all of the computations are done with PYTHON (Processor Intel(R) Core(TM) i5-10210U CPU @ 2.11 GHz with 64-bit operating system), we additionally calculate the CPU execution time in seconds. We use the following stopping criteria to ensure that iterative calculation computer programs are terminated when all of the conditions are met at the same time:

i) $|t_{n+1} - t_n| < 10^{-201}$

ii) $|h(t_{n+1})| < 10^{-201}$.

Table IV(a) Test functions with their roots

Test Functions	Root, t^*
$h_1(t) = \sin(2\cos t) - 1 - t^2 + e^{\sin(t^3)}$;[8]	−0.7848959876612
$h_2(t) = \sin t + \cos t + t$;[8]	−0.4566247045676
$h_3(t) = (t+2)e^t - 1$;[8]	−0.442854010023
$h_4(t) = t^2 + \sin\left(\frac{t}{5}\right) - \frac{1}{4}$;[8]	0.40999201798913
$h_5(t) = \cos t - t$;[16]	0.73908513321516
$h_6(t) = t^3 - 10$;[16]	2.15443469003188
$h_7(t) = e^{-t} + \cos t$;[17]	1.74613953040801
$h_8(t) = e^{\sin t} - t + 1$;[17]	2.63066414792790
$h_9(t) = t^4 - 7.79075t^3 + 2.511t - 1.674$;[15]	0.27775954284172
$h_{10}(t) = \sin^2 t - t^2 + 1$;[13]	1.40449164821534

To verify the theoretical order of convergence, we calculate the computational order of convergence (p_c) using the formula [9]

$$p_c = \frac{\log\left[\left(t_{n+1} - t_n\right)/\left(t_n - t_{n-1}\right)\right]}{\log\left[\left(t_n - t_{n-1}\right)/\left(t_{n-1} - t_{n-2}\right)\right]}$$

taken into consideration the last four approximations in the iterative process.

Table IV(b)Analogy of Efficiency

Methods	p_c	N	E
NR	2.00	2	1.414
SO	4.00	3	1.587
AN	4.00	3	1.587
SA	4.00	3	1.587
RS	4.00	3	1.587
RA	4.00	3	1.587
KI	4.00	3	1.587
CH	4.00	3	1.587
TR	4.00	3	1.587
FR	4.00	3	1.587
MMS	4.00	3	1.587

Where p_c is the convergence order, N is the number of functional values per iteration and E is the efficiency-index.

Table IV(c) Analogy of Different Methods

ethod	t0	n	$\lvert t_{n+1} - t_n\rvert$	$\lvert h(t_{n+1})\rvert$	NFE	CPU
h1(t)	-1					
NR		9	1.6e-201	4.0e-201	18	0.00652
SO		5	2.2e-200	4.1e-201	15	0.00602
AN		6	1.5e-201	4.1e-201	18	0.00522
SA		5	0	4.1e-201	15	0.00717
RS		6	1.8e-201	4.1e-201	18	0.00611
RA		6	5.7e-201	4.1e-201	18	0.00615
KI		6	4.1e-201	4.1e-201	18	0.00717
CH		6	1.7e-201	4.1e-201	18	0.00527
TR		5	0	4.1e-201	15	0.00731
FR		6	5.6e-201	4.1e-201	18	0.00559
MMS		5	8.1e-201	4.1e-201	15	0.00513
	-0.5					
NR		10	8.9e-201	2.4e-200	20	0.00732
SO		6	1.6e-201	4.1e-200	18	0.00745
AN		7	1.5e-201	4.1e-200	21	0.00672
SA		6	0	4.1e-200	18	0.00786
RS		7	1.7e-201	4.1e-200	21	0.00714
RA		7	9.6e-201	4.1e-200	21	0.00733
KI		7	4.1e-201	4.1e-200	21	0.00859
CH		7	1.8e-201	4.1e-200	21	0.00672
TR		6	0	4.1e-200	18	0.00805
FR		7	9.7e-200	4.1e-200	21	0.00754
MMS		6	8.1e-201	4.1e-201	18	0.00671
h2(t)	0.1					
NR		9	2.4e-201	5.3e-201	18	0.00317
SO		6	6.1e-201	1.8e-200	18	0.00302
AN		6	2.5e-201	5.3e-201	18	0.00315
SA		5	3.2e-201	5.3e-201	15	0.00355
RS		6	2.3e-201	5.3e-201	18	0.00317
RA		6	9.3e-201	5.3e-201	18	0.00325
KI		6	6.9e-201	5.3e-201	18	0.00327
CH		6	2.6e-201	5.3e-201	18	0.00307
TR		5	2.8e-201	5.3e-201	15	0.00357
FR		6	4.8e-200	5.3e-201	18	0.00298
MMS		5	1.2e-201	5.3e-201	15	0.00293
	-1					

NR		8	2.4e-201	5.3e-201	16	0.00289
SO		5	1.0e-200	5.3e-201	15	0.00271
AN		6	2.5e-201	5.3 e-20	18	0.00272
SA		5	0	5.3e-201	15	0.00299
RS		6	2.6e-201	5.3e-201	18	0.00271
RA		5	9.3e-201	5.3e-201	15	0.00279
KI		5	7.3e-201	1.8e-201	15	0.00321
CH		6	2.3e-201	5.3e-201	18	0.00270
TR		5	1.6e-201	5.3e-201	15	0.00357
FR		5	8.9e-201	5.3e-201	15	0.00295
MMS		5	1.2e-201	5.3e-201	15	0.00270
$h_3(t)$	-1.2					
NR		11	2.4e-201	4.0e-201	22	0.00251
SO			Divergent			
AN		9	2.4e-201	4.1e-201	27	0.00373
SA		7	0	4.1e-201	21	0.00560
RS		8	2.3e-201	4.1e-201	24	0.00251
RA		15	9.7e-201	4.1e-201	45	0.00343
KI			Divergent			
CH		7	2.5e-201	4.1e-201	21	0.00321
TR		6	1.1e-201	4.1e-201	18	0.00332
FR		15	9.7e-201	4.1e-201	45	0.00343
MMS		6	1.2e-201	4.1e-201	18	0.00228
	0.1					
NR		10	2.4e-201	4.0e-201	20	0.00251
SO		6	2.3e-201	4.1e-201	18	0.00270
AN		7	2.5e-201	4.1e-201	21	0.00241
SA		6	2.8e-201	4.1e-201	18	0.00238
RS		7	2.4e-200	4.1e-201	21	0.00224
RA		6	9.7e-201	4.1e-201	18	0.00294
KI		6	7.3e-201	4.1e-201	18	0.00271
CH		7	2.3e-200	4.1e-201	21	0.00253
TR		6	5.3e-201	1.1e-201	18	0.00252
FR		6	9.7e-201	4.1e-201	18	0.00275
MMS		6	1.2e-201	4.1e-201	18	0.00218
$h_4(t)$	0.2					
NR		10	2.0e-201	2.2e-201	20	0.00272
SO		6	2.4e-201	2.2e-201	18	0.00295
AN		7	2.1e-201	2.0e-201	21	0.00293
SA		7	0	2.2e-201	21	0.00336
RS		7	7.7e-201	7.7e-201	21	0.00319
RA		6	8.9e-201	2.2e-201	18	0.00317
KI		7	7.3e-201	7.7e-201	21	0.00292
CH		7	1.9e-201	2.2e-201	21	0.00305
TR		6	1.2e-201	2.2e-201	18	0.00318
FR		6	8.9e-201	2.2e-201	18	0.00299
MMS		5	8.1e-201	2.2e-201	15	0.00254
	1					
NR		10	2.0e-201	2.2e-201	20	0.00261
SO		6	2.4e-201	2.2e-201	18	0.00251
AN		7	2.1e-201	2.2e-201	21	0.00286
SA		6	0	2.2e-201	18	0.00342
RS		7	1.9e-201	2.2e-201	21	0.00275
RA		6	8.5e-201	2.2e-201	18	0.00278
KI		6	6.5e-201	2.2e-201	18	0.00322
CH		7	7.7e-201	7.7e-200	21	0.00262
TR		6	6.6e-201	7.7e-200	18	0.00329
FR		6	8.9e-201	2.2e-201	18	0.00303
MMS		5	6.1e-201	1.7e-201	15	0.00250
$h_5(t)$	1.4					
NR		9	1.6e-201	2.4e-201	18	0.00242
SO		6	1.5e-201	2.4e-200	18	0.00263
AN		6	1.7e-201	2.4e-201	18	0.00227
SA		5	2.4e-201	2.4e-201	15	0.00245
RS		6	1.6e-201	2.4e-201	18	0.00234
RA		6	5.7e-201	2.4e-201	18	0.00231
KI		6	7.3e-201	1.3e-200	18	0.00255
CH		6	1.5e-201	2.4e-201	18	0.00226
TR		5	0	2.4e-201	15	0.00274
FR		6	5.7e-201	2.4e-201	18	0.00281

MMS		5	8.1e-201	2.4e-201	15	0.00232
	0.5					
NR		9	8.1e-201	1.3e-200	18	0.00231
SO		5	1.6e-201	2.4e-201	15	0.00221
AN		6	8.0e-201	1.3e-200	18	0.00226
SA		5	0	2.4e-201	15	0.00244
RS		6	1.6e-201	2.4e-201	18	0.00223
RA		5	7.3e-201	1.3e-200	15	0.00214
KI		5	4.8e-201	2.4e-201	15	0.00264
CH		6	1.7e-201	2.4e-201	18	0.00289
TR		5	0	2.4e-201	15	0.00268
FR		5	7.3e-201	1.3e-201	15	0.00273
MMS		5	8.1e-201	2.4e-201	15	0.00220
$h_6(t)$	1.9					
NR		9	1.6e-200	2.0e-199	18	0.00159
SO		6	1.3e-200	2.0e-199	18	0.00166
AN		6	1.7e-200	2.0e-199	18	0.00149
SA		6	0	2.0e-199	18	0.00158
RS		6	1.5e-200	2.0e-199	18	0.00173
RA		6	7.8e-200	1.2e-198	18	0.00152
KI		8	4.5e-200	2.0e-198	24	0.00160
CH		6	1.6e-201	2.0e-199	18	0.00149
TR		5	0	2.0e-199	15	0.00157
FR		6	7.5e-200	1.5e-199	18	0.00177
MMS		5	6.5e-201	2.0e-201	15	0.00150
	2.5					
NR		9	1.6e-200	2.0e-189	18	0.00150
SO		6	1.3e-200	2.0e-199	18	0.00155
AN		6	1.5e-200	2.0e-199	18	0.00157
SA		5	7.1e-200	1.2e-198	15	0.00169
RS		6	1.7e-200	2.0e-199	18	0.00160
RA		6	6.2e-200	2.0e-199	18	0.00159
KI		6	4.5e-200	2.0e-198	18	0.00158
CH		6	1.6e-201	2.0e-199	18	0.00148
TR		5	0	2.0e-199	15	0.00181
FR		6	7.5e-200	2.0e-199	18	0.00181
MMS		5	6.5e-201	2.0e-199	15	0.00149
$h_7(t)$	1					
NR		8	4.8e-201	6.5e-201	16	0.00331
SO		6	6.5e-201	6.5e-201	18	0.00304
AN		6	4.7e-201	6.5e-201	18	0.00310
SA		5	0	6.5e-201	15	0.00339
RS		6	4.9e-201	6.5e-201	18	0.00310
RA		5	2.2e-200	6.5e-201	15	0.00311
KI		5	1.6e-200	6.5e-201	15	0.00364
CH		6	4.8e-200	6.5e-201	18	0.00319
TR		5	0	6.5e-201	15	0.00417
FR		5	2.2e-200	6.5e-201	15	0.00489
MMS		5	3.2e-201	6.5e-201	15	0.00305
	1.7					
NR		8	4.8e-201	6.5e-201	16	0.00341
SO		5	6.5e-201	6.5e-201	15	0.00310
AN		6	4.7e-201	6.5e-201	18	0.00315
SA		5	2.1e-200	6.5e-201	15	0.00345
RS		6	9.4e-200	1.0e-199	18	0.00437
RA		5	2.1e-201	6.5e-201	15	0.00365
KI		5	1.6e-200	6.5e-201	15	0.00383
CH		6	4.9e-201	6.5e-201	18	0.00318
TR		5	0	6.5e-201	15	0.00427
FR		5	2.2e-202	6.5e-201	15	0.00396
MMS		5	3.2e-201	6.5e-201	15	0.00311
$h_8(t)$	2.4					
NR		8	3.5e-200	8.8e-200	16	0.00394
SO		5	1.9e-200	8.8e-200	15	0.00436
AN		6	6.2e-200	1.4e-199	18	0.00392
SA		5	3.9e-201	8.8e-201	15	0.00410
RS		6	6.2e-200	1.4e-199	18	0.00432
RA		5	6.8e-201	1.4e-199	15	0.00392
KI			Divergent			
CH		6	3.6e-200	8.8e-200	18	0.00498

TR		5	0	8.8e-201	15	0.00458
FR		5	6.8e-200	1.4e-198	15	0.00404
MMS		5	4.2e-200	1.4e-199	15	0.00393
	3.3					
NR		9	6.2e-200	1.4e-199	18	0.00422
SO		6	3.5e-200	8.8e-200	18	0.00440
AN		6	3.5e-200	8.8e-200	18	0.00441
SA		6	6.8e-200	1.4e-199	18	0.00471
RS		6	3.5e-201	8.8e-201	18	0.00450
RA		6	6.7e-200	1.4e-199	18	0.00436
KI			Divergent			
CH		6	2.5e-200	8.8e-200	18	0.00491
TR		6	2.6e-200	8.8e-200	18	0.00583
FR		6	6.9e-200	1.4e-199	18	0.00483
MMS		5	4.2e-200	1.4e-199	15	0.00383
$h_9(t)$	0.1					
NR		10	8.1e-201	8.1e-201	20	0.00385
SO		4	6.5e-201	2.4e-200	12	0.00384
AN		5	8.9e-201	4.1e-201	15	0.00385
SA		8	3.2e-201	2.4e-201	24	0.00391
RS		8	4.2e-201	4.1e-201	24	0.00392
RA		5	2.4e-201	4.1e-201	15	0.00392
KI		3	4.2e-200	4.1e-201	9	0.00393
CH		5	8.9e-201	4.1e-201	15	0.00388
TR		8	3.2e-201	2.4e-201	24	0.00420
FR		8	4.0e-201	4.1e-201	24	0.00407
MMS		3	4.1e-201	4.1e-201	9	0.00383
	1.2					
NR		10	8.1e-201	8.1e-201	20	0.00365
SO		4	6.5e-201	2.4e-200	12	0.00361
AN		6	4.8e-201	4.8e-200	18	0.00387
SA		9	4.1e-201	4.1e-201	27	0.00383
RS		9	4.2e-201	4.1e-201	27	0.00379
RA		4	4.2e-200	4.1e-201	12	0.00365
KI		4	4.1e-200	4.1e-201	12	0.00451
CH		5	8.9e-201	4.1e-201	15	0.00365
TR		8	3.2e-201	2.4e-201	24	0.00432
FR		8	4.0e-200	4.1e-201	24	0.00435
MMS		3	4.0e-201	4.1e-201	9	0.00362
$h_{10}(t)$	0.7					
NR		12	3.1e-200	7.6e-200	24	0.00678
SO			Divergent			
AN		9	3.0e-200	7.6e-200	27	0.00603
SA		8	0	7.6e-200	24	0.00679
RS		9	3.2e-200	7.6e-200	27	0.00813
RA		11	6.6e-200	1.7e-199	33	0.00819
KI		10	9.3e-200	7.6e-200	30	0.00691
CH		8	6.8e-200	1.7e-199	24	0.00558
TR		7	5.2e-201	1.7e-199	21	0.00609
FR		11	6.5e-200	1.7e-199	33	0.00783
MMS		5	4.4e-200	7.6e-200	15	0.00546
	1.6					
NR		9	3.1e-200	7.6e-200	18	0.00484
SO		5	4.8e-200	1.7e-199	15	0.00676
AN		6	6.7e-200	1.7e-199	18	0.00391
SA		5	3.5e-200	7.6e-200	15	0.00441
RS		6	6.8e-200	1.7e-199	18	0.00570
RA		11	6.5e-200	1.7e-199	33	0.00761
KI		6	8.2e-200	1.6e-199	18	0.00485
CH		6	6.9e-200	1.7e-199	18	0.00391
TR		5	1.7e-200	7.6e-200	15	0.00536
FR		10	6.5e-200	1.7e-199	30	0.00393
MMS		4	6.4e-200	7.6e-200	12	0.00389

Where t_0 is the initial approximation, n is the number of iterations and *NFE* is number of function evaluations.

The graphical behavior is reflected in "Fig. 1" to "Fig. 20". We use Origin Pro software for graphical comparisons.

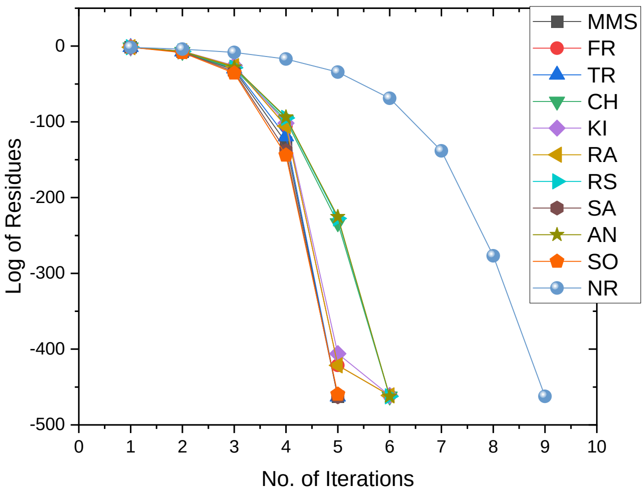

Fig. 1. $h_1(t)=0$ at $t_0=1$

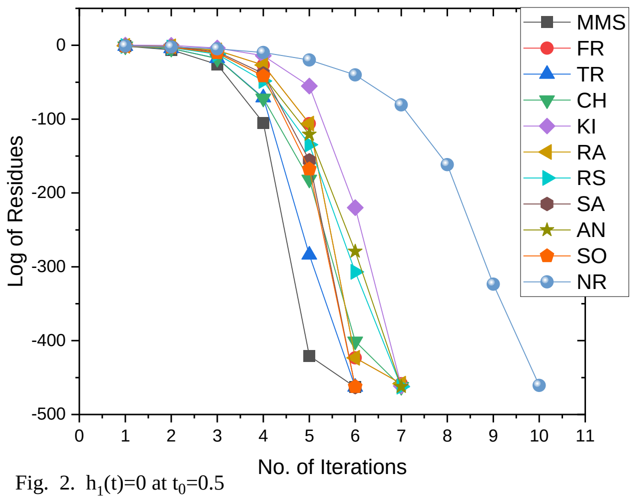

Fig. 2. $h_1(t)=0$ at $t_0=0.5$

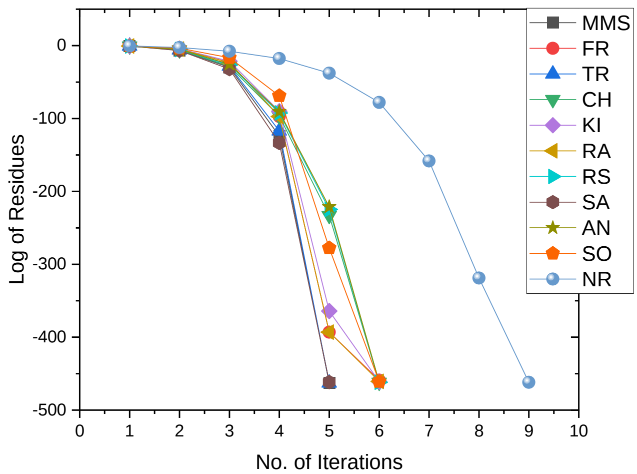

Fig. 3. $h_2(t)=0$ at $t_0=0.1$

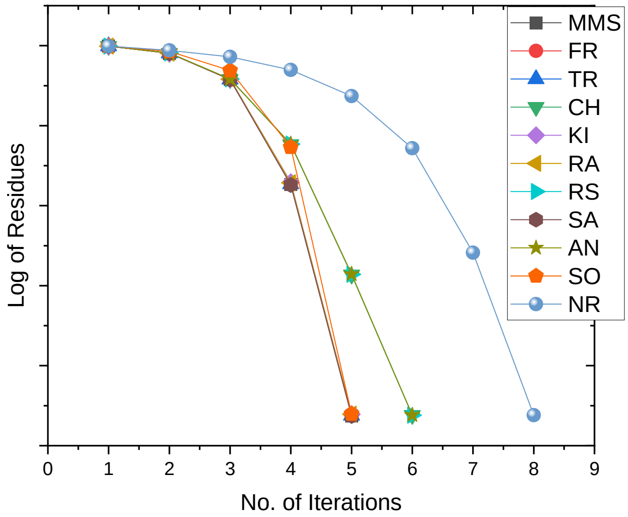

Fig. 4. $h_3(t)=0$ at $t_0=-1$

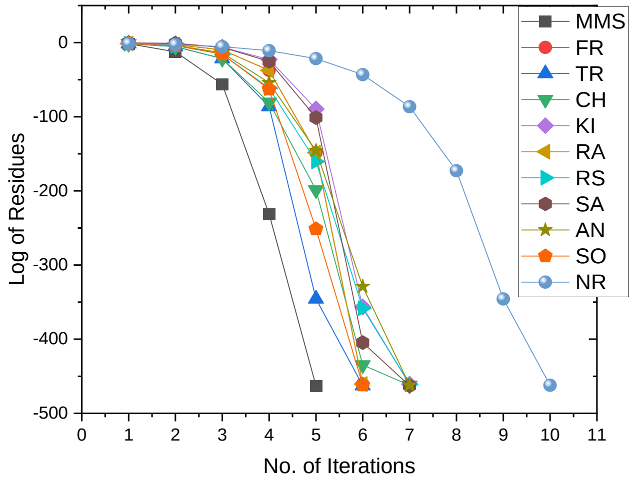

Fig. 7. $h_4(t)=0$ at $t_0=0.2$

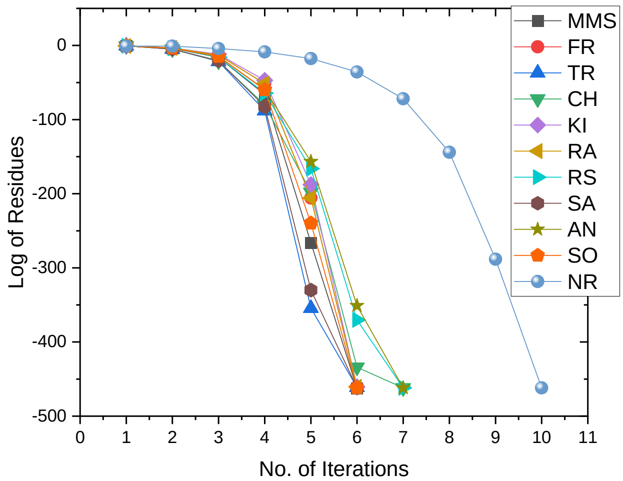

Fig. 5. $h_3(t)=0$ at $t_0=-1.2$

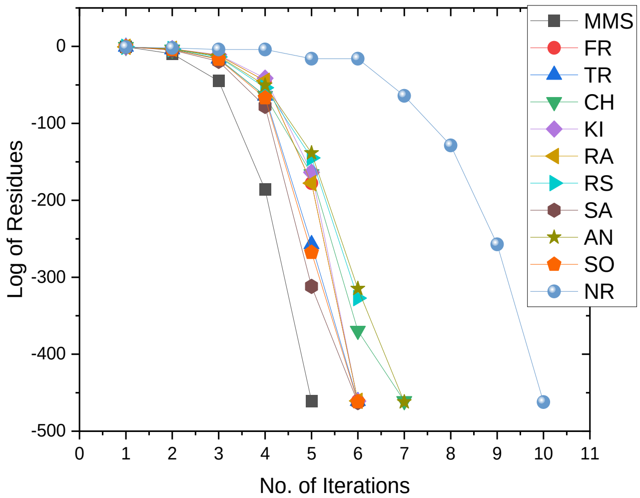

Fig. 8. $h_4(t)=0$ at $t_0=1$

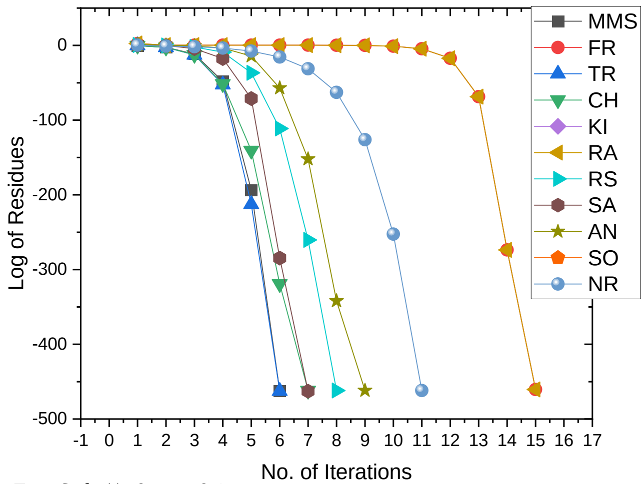

Fig. 6. $h_3(t)=0$ at $t_0=0.1$

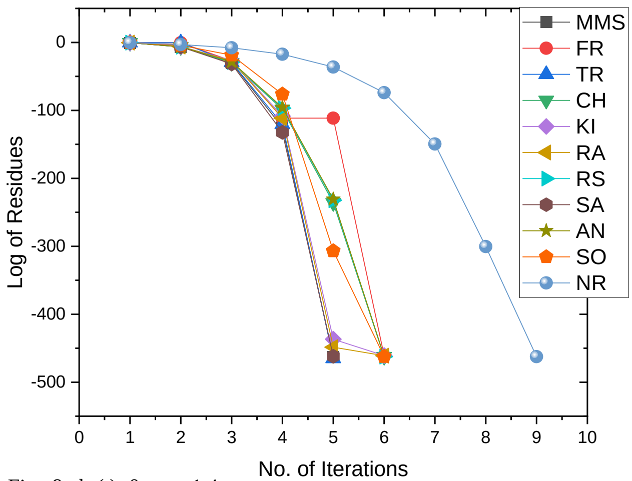

Fig. 9. $h_5(t)=0$ at $t_0=1.4$

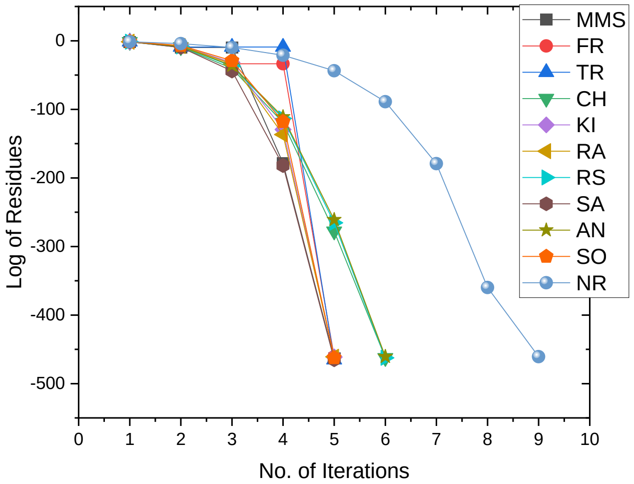

Fig. 10. $h_5(t)=0$ at $t_0=0.5$

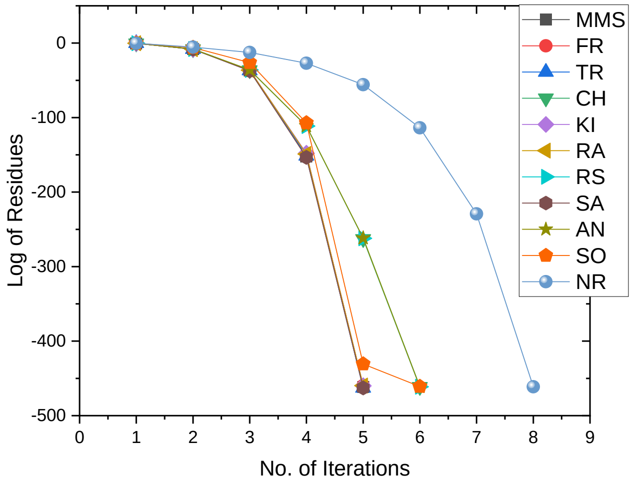

Fig. 13. $h_7(t)=0$ at $t_0=1$

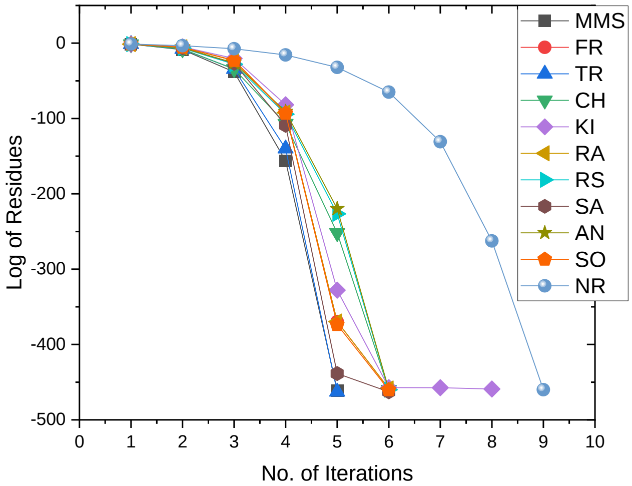

Fig. 11. $h_6(t)=0$ at $t_0=1.9$

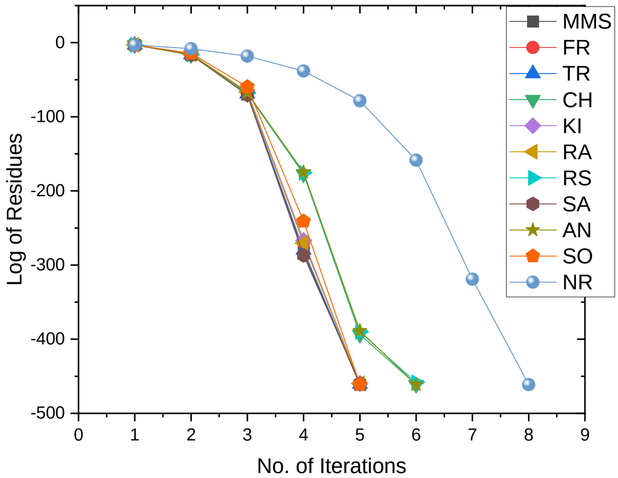

Fig. 14. $h_7(t)=0$ at $t_0=1.7$

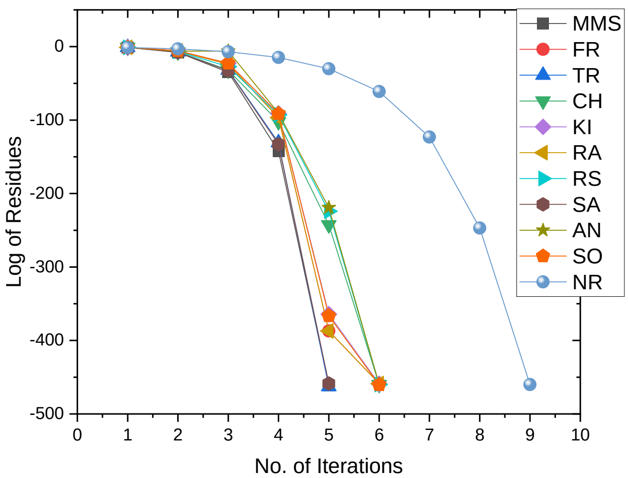

Fig. 12. $h_6(t)=0$ at $t_0=2.5$

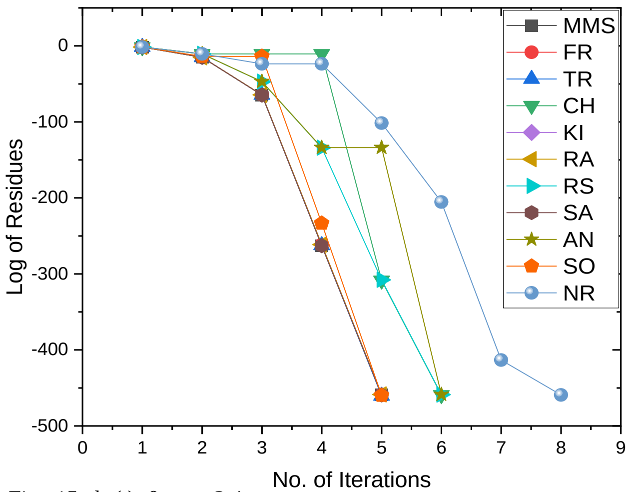

Fig. 15. $h_8(t)=0$ at $t_0=2.4$

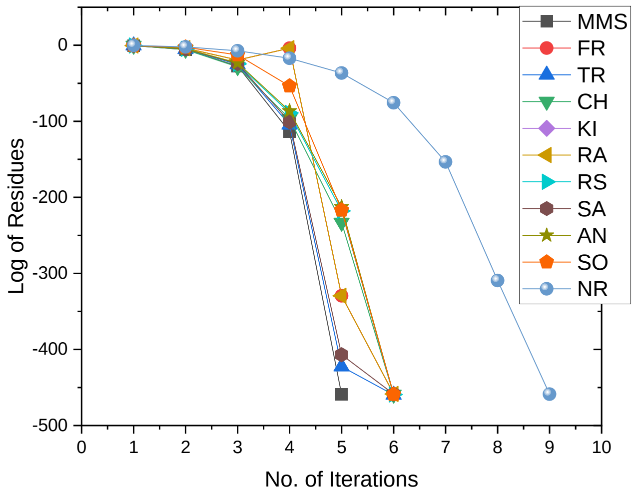

Fig. 16. $h_8(t)=0$ at $t_0=3.3$

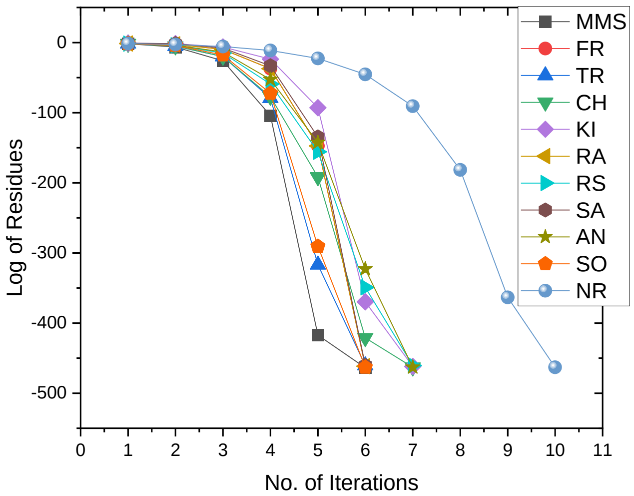

Fig. 17. $h_9(t)=0$ at $t_0=0.1$

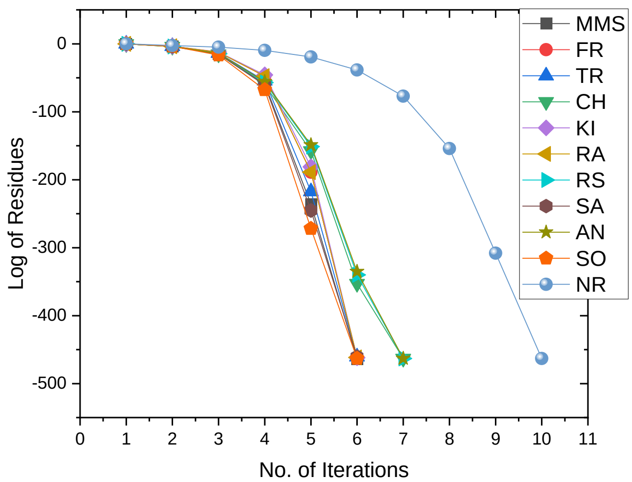

Fig. 18. $h_9(t)=0$ at $t_0=1.2$

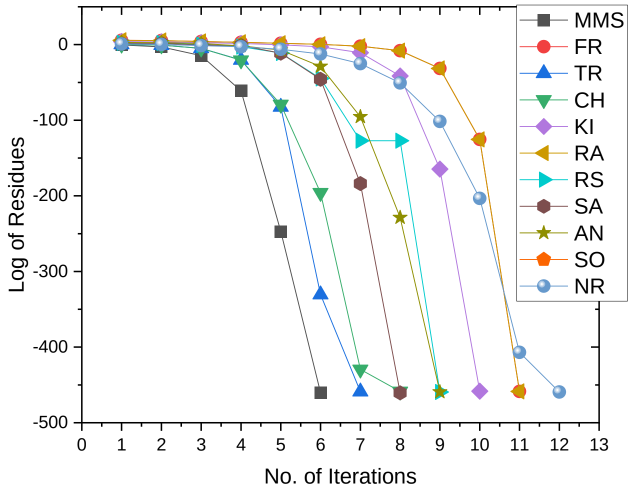

Fig. 19. $h_{10}(t)=0$ at $t_0=0.7$

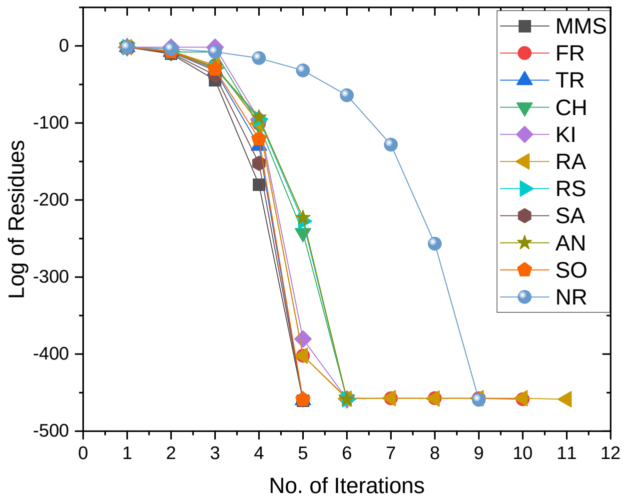

Fig. 20. $h_{10}(t)=0$ at $t_0=1.6$

Figures 1-20 show the residual fall of iterative methods NR, SO, AN, SA, RS, RA, KI, CH, TR, FR and MMS. for simple roots for a nonlinear function h_1,-h_{10} respectively.

V. CONCLUSIONS

We modified the proposed iterative technique by employing approximants of the second derivative to avoid calculating the higher derivatives of the function. As a result, we have a modified iterative approach that is free of the function's higher derivatives. The order of convergence of the method "(2.6)" has been proven to be four. With an efficiency score of 1.587, this method introduced the novel optimal fourth-order convergent iterative method. Two functional evaluations and one of the first derivatives are required. The efficiency of various approaches is compared in Table IV(b). The computational findings in table IV(c) and the graphical results in "Fig. 1" to "Fig. 20" show that the current approach MMS outperforms earlier methods in terms of CPU time for similar tasks. Other optimal fourth-order iterative methods were competitive with the current iterative strategy. As a result, the findings of the study make a significant contribution to the field of computational sciences.

Acknowledgments

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Applications in Mathematical Sciences, 2021, 20, pp. 1633-1643. The current research interests are to develop new iterative methods for the solution of nonlinear, systems of linear and nonlinear equations.