The O-conditionality for Ordinal Sums of Fuzzy Implications over Overlap and Grouping Functions

Xin Guo, and Baoqing Hu

***Abstract*—Through the study of the law of O-conditionality for the fuzzy implication derived from overlap, grouping and negation functions, we first present the notion of fuzzy implications J_{R_O} and $J_{G,N}$ which are ordinal sums of fuzzy implications R_O-implications and (G,N)-implications. And then we analyze the conditions for the two fuzzy implications J_{R_O} and $J_{G,N}$ to preserve the law of O-conditionality. Finally, a new type of implications called (O,N)-implications which are derived from overlap and negation functions is given, and the law of O-conditionality for (O,N)-implications is also discussed.**

***Index Terms*—overlap functions, grouping functions, fuzzy implications, ordinal sums, O-conditionality.**

I. Introduction

THE concept of fuzzy sets was introduced by Zadeh, since then, many mathematical concepts such as numbers, group, topology, differential equation and so on, have been generalized to fuzzy sets. There are several ways to extend the propositional connectives for a set $[0,1]$, but in general these extensions do not preserve all the properties of the classical logical connectives.

Fuzzy implications [1] play a key role in fuzzy logic [2] and various applications, including approximate reasoning [3], fuzzy control [4], fuzzy relational equations [5], fuzzy mathematical morphology [6], image processing [7], and so on. Classical implications are generalized to fuzzy implications by considering truth values that vary in the unit interval $[0,1]$ rather than in the set $\{0,1\}$. Fuzzy implications are largely applied in approximate reasoning, modeling fuzzy conditions and the inference processes via the generalized Modus Ponens (GMP) and Modus Tollens (GMT) [8]. In the inference processes of fuzzy logic, many papers discuss the implementation of generalized Modus Ponens since the scheme is enabled by the laws of the T-conditionality and the U-conditionality, for t-norms and t-uninorms, respectively. The Modus Ponens $x \wedge (x \rightarrow y) \vdash y$ is generalized to the fuzzy context [9], when $\wedge$ is replaced by a t-norm, $x \rightarrow y$ is replaced by a fuzzy implication, the law of T-conditionality is stated by: $\forall x, y \in [0,1]$,

$$T(x, I(x,y)) \leq y \ (\mathbf{TC}).$$

In the literature, the studies on the T-conditionality have been done just for the three main families of fuzzy implications, namely, R-implications, (S,N)-implications and QL-implications. It is observed that **(TC)** only relates with two objects, so the associatively property of the conjunctive operator is not necessarily needed. Similarly, in some applications, fuzzy implication functions do not require the exchange principle, for example, in decision-making [10], consensus measures [11], and multi-criteria decision-making problems via similarity measures [12]. Readers can refer to the related works ([13]–[17]). In this direction, Bustince introduced overlap functions [18] and grouping functions [19], which are exceptional cases of continuous aggregate operators, given by monotonic and commutative functions, but not necessarily associative, satisfying some appropriate boundary conditions ([20]–[22]). In the sequence, the concept of fuzzy implications derived from overlap and grouping functions was introduced in [23]. Based on residual implication of general conjunctions, Dimuro et al. introduced the concept of R_O-implications which are derived from overlap functions, preserving the residual property. And they also gave the concepts of (G,N)-implications and QL-implications derived from triples (O,G,N) in [24] and [25], respectively. A generalization of **(TC)** was introduced by Dimuro et al., that is, the law of O-conditionality [26]:

$$O(x, I(x,y)) \leq y \ (\mathbf{OC}).$$

Dimuro et al. also discussed under what conditions R_O-implications, (G,N)-implications, QL-implications and D-implications satisfy (**OC**), respectively. Inspired by [26] and considering the advantages and flexibility offered by overlap and grouping functions as aggregation operators, we discuss whether some fuzzy implications generated by overlap and grouping functions satisfy the conditions of the law of O-conditionality. we first present the notion of J_{R_O}-implication and $J_{G,N}$-implication which are ordinal sums of fuzzy implications R_O-implications and (G,N)-implications. Then we study some properties of the ordinal sum, and analyze the laws of O-conditionality of the J_{R_O}-implication and the $J_{G,N}$-implication induced by the ordinal sum, respectively. Finally, we present a new fuzzy implication $I_{O,N}$ which is derived from an overlap and negation function, and discuss under what conditions it does not satisfy the law of O-conditionality.

The paper is organized as follows. Section II presents basic concepts that are needed to develop the paper, including the concepts of R_O-implications and (G,N)-implications, and the law of O-conditionality for any fuzzy implication, especially the law of O-conditionality for R_O-implications and (G,N)-implications. Section III discusses some properties of J_{R_O} and $J_{G,N}$, and studies the law of O-conditionality for J_{R_O} and $J_{G,N}$. In section IV, we give a new fuzzy implication (O,N)-implication, and discuss the law of O-conditionality for it. Section V is the conclusion, with our final remarks and future work.

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Xin Guo (corresponding author) is a Lecturer of School of Mathematics and Statistics, Anyang Normal University, Anyang, China (e-mail: guox97@126.com).

Baoqing Hu is a Professor of School of Mathematics and Statistics, Wuhan University, Wuhan, China (e-mail: bqhu@whu.edu.cn).

II. Preliminaries

In this section, we recall some fundamental concepts related to the theory of R_O-implications and (G,N)-implications which shall be needed in the sequel.

A. t-norms, t-conorms, overlap and grouping functions

Definition 2.1: [27] A bivariate function $T:[0,1]^2 \to [0,1]$ is said to be a t-norm if it satisfies the following conditions: for all $x,y,z \in [0,1]$,

(T1) Commutativity: $T(x,y)=T(y,x)$;
(T2) Associativity: $T(T(x,y),z)=T(x,T(y,z))$;
(T3) Monotonicity: $T(x,y)\le T(x,z)$ whenever $y\le z$;
(T4) Boundary condition: $T(x,1)=x$.

Definition 2.2: [27] A bivariate function $S:[0,1]^2\to[0,1]$ is said to be a t-conorm if, for all $x,y,z\in[0,1]$, it satisfies the following conditions:

(S1) Commutativity: $S(x,y)=S(y,x)$;
(S2) Associativity: $S(S(x,y),z)=S(x,S(y,z))$;
(S3) Monotonicity: $S(x,y)\le S(x,z)$ whenever $y\le z$;
(S4) Boundary condition: $S(x,0)=x$.

Definition 2.3: [28] A function $N:[0,1]\to[0,1]$ is said to be a fuzzy negation, if the following conditions hold:

(N1) N satisfies the boundary conditions:
$$N(0)=1, N(1)=0;$$
(N2) N is decreasing: if $x\le y$, then $N(y)\le N(x)$;
(N3) N is strictly decreasing:
$$\text{if } x<y, \text{ then } N(y)<N(x);$$
(N4) N is continuous;
(N5) N is involutive: $\forall x\in[0,1]$,
$$N(N(x))=x;$$
(N6) N is frontier:
$$N(x)\in\{0,1\} \text{ if and only if } x=0 \text{ or } x=1;$$
(N7) $N(x)=1$ if and only if $x=0$.

The standard fuzzy negation $N_Z:[0,1]\to[0,1]$, is defined by

$$N_Z(x)=1-x.$$

The least fuzzy negation $N_\perp:[0,1]\to[0,1]$, is defined by

$$N_\perp(x)=\begin{cases}1, & x=0,\\ 0, & x\in]0,1].\end{cases}$$

The greatest fuzzy negation $N_\top:[0,1]\to[0,1]$, is defined by

$$N_\top(x)=\begin{cases}0, & x=1,\\ 1, & x\in[0,1[.\end{cases}$$

In the following, we introduce the concepts of overlap and grouping functions ([18], [19]).

Definition 2.4: [18] A bivariate function $O:[0,1]^2\to[0,1]$ is said to be an overlap function if it satisfies the following conditions: for any $x,y,z\in[0,1]$,

(O1) O is commutative: $O(x,y)=O(y,x)$;
(O2) $O(x,y)=0$ iff $xy=0$;
(O3) $O(x,y)=1$ iff $xy=1$;
(O4) O is increasing: if $x\le y$, then $O(x,z)\le O(y,z)$;
(O5) O is continuous.

Moreover, an overlap function O is said to satisfy (O6) the property of 1-section deflation: $\forall x\in[0,1]$, $O(x,1)\le x$; O satisfies (O7) the property of 1-section inflation: $\forall x\in[0,1]$, $O(x,1)\ge x$.

Lemma 2.5: [26] Let $O:[0,1]^2\to[0,1]$ be an overlap function. If O satisfies (O6), then $O(x,y)\le x$, for all $x\in[0,1]$.

Proof: Assume that O satisfies (O6). Since O is increasing, for all $x,y\in[0,1]$, it holds that

$$O(x,y)\le O(x,1)\le x.$$

■

Definition 2.6: [19] A bivariate function $G:[0,1]^2\to[0,1]$ is said to be a grouping function if it satisfies the following conditions: for any $x,y,z\in[0,1]$,

(G1) G is commutative: $G(x,y)=G(y,x)$;
(G2) $G(x,y)=0$ iff $x=y=0$;
(G3) $G(x,y)=1$ iff $x=1$ or $y=1$;
(G4) G is increasing: if $x\le y$, then $G(x,z)\le G(y,z)$;
(G5) G is continuous.

Moreover, a grouping function G is said to satisfy (G6) the property of 0-section deflation: $\forall y\in[0,1]$, $G(0,y)\le y$; G satisfies (G7) the property of 0-section inflation: $\forall y\in[0,1]$, $G(0,y)\ge y$.

B. Fuzzy implications

Definition 2.7: [1] A bivariate function $I:[0,1]^2\to[0,1]$ is called a fuzzy implication, if for any $x,y,z\in[0,1]$, it holds that:

(I1) First place non-increasing:
$$\text{if } x\le y, \text{ then } I(x,z)\ge I(y,z);$$
(I2) Second place non-decreasing:
$$\text{if } y\le z, \text{ then } I(x,y)\le I(x,z);$$
(I3) Boundary condition:
$$I(0,0)=1, I(1,1)=1, I(1,0)=0.$$

In the following, we present some properties that are used in this paper.

Definition 2.8: [1] A fuzzy implication $I:[0,1]^2\to[0,1]$ satisfies:

(LBC) The left boundary condition: $\forall y\in[0,1]$,
$$I(0,y)=1;$$
(OP) The ordering property: $\forall x,y\in[0,1]$,
$$x\le y\Leftrightarrow I(x,y)=1;$$
(LOP) The left ordering property: $\forall x,y\in[0,1]$,
$$x\le y\Rightarrow I(x,y)=1;$$
(EP) The exchange principle: $\forall x,y,z\in[0,1]$,
$$I(x,I(y,z))=I(y,I(x,z));$$
(IP) The identity principle: $\forall x\in[0,1]$,
$$I(x,x)=1;$$
(CAB) The conditional antecedent boundary condition: $\forall x,y\in[0,1]$,
$$x>y\Rightarrow I(x,y)\le y.$$

C. The residual implication R_O-implication

Residual implications (R-implications, for short) consist in the fuzzy implications obtained by the generalization of Boolean implications. That is, for a universe set U,

$$A^C\cup B=(A-B)^C=\cup\{C\subseteq U|(A\cap C)\subseteq B\},$$

where $A,B\subseteq U$.

This class of implications is related to a residual concept from the intuitionistic logic, to use an overlap function O to replace the conjunctive $\cap$ in [29], and give the definition of R_O-implications.

Definition 2.9: [29] Let $O : [0,1]^2 \to [0,1]$ be an overlap function. The function $I_O : [0,1]^2 \to [0,1]$ is given by

$$I_O(x,y) = \max\{z \in [0,1] | O(x,z) \le y\}$$

for all $x, y \in [0,1]$. Then I_O is called a residual implication derived from the overlap function O, for short,we call it a R_O-implication.

Proposition 2.10: [23] Let $I_O : [0,1]^2 \to [0,1]$ be a R_O-implication, then O and I_O satisfy the residual property: $\forall x, y, u \in [0,1]$,

$$O(x,u) \le y \Leftrightarrow I_O(x,y) \ge u.$$

Proposition 2.11: [23] Let $O : [0,1]^2 \to [0,1]$ be an overlap function, and I_O be the residual implication derived from the overlap function O. Then it holds that:

(1) the R_O-implication I_O satisfies (LOP) if and only if O satisfies (O6);
(2) the R_O-implication I_O satisfies (OP) if and only if O satisfies (O6) and (O7).

D. The (G,N)-implication $I_{G,N}$

The class of fuzzy implications called (G,N)-implications derived from grouping functions and negation functions, were introduced by Dimuro in [23]. A (G,N)-implication is a generalization to $[0,1]$ of the Boolean material implication defined as

$$p \to q \equiv \neg p \vee q.$$

If $\vee$ and $\neg$ are replaced by a grouping function and a fuzzy negation, respectively, we can get the definition of (G,N)-implications.

Definition 2.12: [23] Let $G : [0,1]^2 \to [0,1]$ be a grouping function and $N : [0,1] \to [0,1]$ be a fuzzy negation. $I_{G,N} : [0,1]^2 \to [0,1]$ is given by

$$I_{G,N}(x,y) = G(N(x), y),$$

for all $x, y \in [0,1]$, then $I_{G,N}$ is called a fuzzy implication, denoted by (G,N)-implication.

Proposition 2.13: [23] Let $G : [0,1]^2 \to [0,1]$ be a grouping function, $N : [0,1] \to [0,1]$ be a fuzzy negation, and $I_{G,N}$ be the (G,N)-implication derived from G and N. Then

(i) $I_{G,N}$ does not satisfy (OP);
(ii) $I_{G,N}$ satisfies (OP) if and only if $N = N_\top$.

E. The law of O-conditionality

In this section, we recall the law of O-conditionality for some fuzzy implications, and discuss their several properties.

Definition 2.14: [26] A fuzzy implication I satisfies the law of O-conditionality for an overlap function O if and only if, for all $x, y \in [0,1]$, it holds that:

$$O(x, I(x,y)) \le y \text{ (OC)}.$$

In fact, (OC) means $x *_O (x \to y) \le y$, and is equivalent to (TC) whenever T is a positive (without zero divisors) and continuous t-norm. All two laws are generalized Modus Ponens (GMP). But the associativity and exchange principle is no needed for (OC), hence it is more flexible and more general.

Example 2.15: Let the overlap function O be defined by

$$O(x,y) = (xy)^p,$$

where $p > 1$. Let I be an implication function such that $I(x,y) \le y$. Then $O(x, I(x,y)) = x^p I(x,y)^p \le x^p y^p \le y$.

Proposition 2.16: [26] If a fuzzy implication I satisfies (CAB) and (LOP), then I satisfies (OC) for any overlap function O satisfying (O6).

Next we recall some results about the law of O-conditionality for R_O-implications and (G,N)-implications.

Theorem 2.17: [26] Let $O : [0,1]^2 \to [0,1]$ be an overlap function and I_O be a R_O-implication. Then I_O satisfies (CAB) if and only if $O \ge \min$.

Theorem 2.18: [26] Any R_O-implication $I_O : [0,1]^2 \to [0,1]$ derived from the overlap function $O : [0,1]^2 \to [0,1]$, satisfies (OC) for O.

In the following, we recall the law of O-conditionality for (G,N)-implications and state under what conditions a (G,N)-implication $I_{G,N}$ satisfies (OC).

Theorem 2.19: [26] Let $G : [0,1]^2 \to [0,1]$ be a grouping function, $N : [0,1] \to [0,1]$ be a fuzzy negation, and $I_{G,N}$ be a (G,N)-implication. Then $I_{G,N}$ satisfies (CAB), and G satisfies (G6).

Theorem 2.20: [26] Let $O : [0,1]^2 \to [0,1]$ be an overlap function, $G : [0,1]^2 \to [0,1]$ be a grouping function and $N : [0,1] \to [0,1]$ be a fuzzy negation. If O satisfies (O6), G satisfies (G6) and $N = N_\perp$, then the (G,N)-implication $I_{G,N} : [0,1]^2 \to [0,1]$ satisfies (OC) for O.

III. The O-conditionality for J_{R_O}-implications and $J_{G,N}$-implications

In this section, we first present the notion of ordinal sums of implications, and analyze the conditions for J_{R_O}-implications and $J_{G,N}$-implications to satisfy the law of O-conditionality.

Definition 3.1: [32] Let $\{J_i\}_{i\in I}$ be a family of implications and $]a_i, b_i[_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, such that $a_i > 0$ for each $i \in I$. Then the function $J_I : [0,1]^2 \to [0,1]$ given by

$$J_I(x,y) = \begin{cases} a_i + (b_i - a_i) J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & x, y \in]a_i, b_i[, \\ I_{RS}(x,y), & \text{otherwise}, \end{cases}$$

is an implication, which is called an ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$.

The ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$ is a method of constructing new fuzzy implications, and it can preserve many good properties. Inspired by the idea, we use the ordinal sum of R_O-implications and (G,N)-implications to get two new fuzzy implications, and discuss the law of O-conditionality whether it can be preserved for the two new implications.

If $\{J_i\}_{i\in I}$, J_i are replaced by R_O-implications, then we get the following definition.

Definition 3.2: Let $\{J_i\}_{i\in I}$ be a family of R_O-implications and $]a_i, b_i[_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, such that $a_i > 0$ for each $i \in I$. Then the function $J_{R_O} : [0,1]^2 \to [0,1]$ given by

$$J_{R_O}(x,y) = \begin{cases} a_i + (b_i - a_i) J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & x, y \in]a_i, b_i[, \\ I_{RS}(x,y), & \text{otherwise}, \end{cases}$$

is an implication, which is called an ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$, denoted by J_{R_O}-implication.

On the other hand, in the ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$, if J_i are replaced by (G,N)-implications, then we can get the following definition.

Definition 3.3: Let $\{J_i\}_{i\in I}$ be a family of (G,N)-implications and $]a_i, b_i[_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, such that $a_i > 0$ for each $i \in I$. Then the function $J_{G,N} : [0,1]^2 \to [0,1]$ given by

$$J_{G,N}(x,y) = \begin{cases} a_i + (b_i - a_i)J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & x, y \in]a_i, b_i[, \\ I_{RS}(x,y), & \text{otherwise}, \end{cases}$$

is an implication, which is called an ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$, denoted by $J_{G,N}$-implication.

Similar to the ordinal sum of the summands $(a_i, b_i, J_i)_{i\in I}$ in Definition 3.1, the above two new implications are given, then we discuss under what conditions the two new implications satisfy the law of O-conditionality.

Theorem 3.4: Let $J_{R_O} : [0,1]^2 \to [0,1]$ be an implication given by Definition 3.2. If O satisfied (O6), then J_{R_O} satisfies (OC).

Proof: Let $\{J_i\}_{i\in I}$ be a family of R_O-implications. By Theorem 2.17, any R_O-implication satisfies (OC), hence each J_i satisfies (OC).

(1) Let $x, y, u \in [a_i, b_i]$. Then

$$\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}, \frac{u-a_i}{b_i-a_i} \in [0,1].$$

By Proposition 2.10, any R_O-implication $J_i : [0,1]^2 \to [0,1]$ satisfies the residuation property:

$$\begin{aligned} &O(\frac{x-a_i}{b_i-a_i}, \frac{u-a_i}{b_i-a_i}) \le \frac{y-a_i}{b_i-a_i} \\ \Leftrightarrow &J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}) \ge \frac{u-a_i}{b_i-a_i}, \end{aligned}$$

$$\begin{aligned} J_{R_O}(x,y) &= a_i + (b_i - a_i)J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}) \\ &\ge a_i + (b_i - a_i)\frac{u-a_i}{b_i-a_i} \\ &= u. \end{aligned}$$

Consider $u = J_{R_O}(x,y)$, we can get

$$O(x, J_{R_O}(x,y)) \le y.$$

(2) Let $x, y \in [0,1]/[a_i, b_i]$. We consider the following two cases.

If $x \le y$, then

$$\begin{aligned} J_{R_O}(x,y) = I_{RS}(x,y) = 1, \\ O(x, J_{R_O}(x,y)) = O(x,1) \le x \le y. \text{ (by (O6))} \end{aligned}$$

If $x \ge y$, then

$$\begin{aligned} J_{R_O}(x,y) = I_{RS}(x,y) = 0, \\ O(x, J_{R_O}(x,y)) = O(x,0) = 0 \le y. \end{aligned}$$

Hence, we can get the ordinal sum of the R_O-implications: J_{R_O} satisfies (OC) for O satisfies (O6). ■

Theorem 3.5: Let $J_{G,N} : [0,1]^2 \to [0,1]$ be an implication given by Definition 3.3. If O satisfies (O6), G satisfies (G6) and $N = N_\perp$, then $J_{G,N}$ satisfies (OC).

Proof: (1) If $x, y \in [a_i, b_i]$, then

$$\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \in [0,1].$$

If $x > a_i$, then

$$\frac{x-a_i}{b_i-a_i} > 0, N_\perp(\frac{x-a_i}{b_i-a_i}) = 0,$$

and

$$\begin{aligned} &O(x, J_{G,N_\perp}(x,y)) \\ &= O(x, a_i + (b_i - a_i)J_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i})) \\ &= O(x, a_i + (b_i - a_i)G_i(N_\perp(\frac{x-a_i}{b_i-a_i}), \frac{y-a_i}{b_i-a_i})) \\ &= O(x, a_i + (b_i - a_i)G_i(0, \frac{y-a_i}{b_i-a_i})) \\ &\le O(x, a_i + (b_i - a_i)\frac{y-a_i}{b_i-a_i}) \qquad \text{(by (G6))} \\ &= O(x,y) \le y \qquad \text{(by Lemma 2.5)}. \end{aligned}$$

If $x = a_i$, then

$$\frac{x-a_i}{b_i-a_i} = 0, N_\perp(0) = 1,$$

and

$$\begin{aligned} &O(x, J_{G,N_\perp}(x,y)) \\ &= O(a_i, a_i + (b_i - a_i)J_i(0, \frac{y-a_i}{b_i-a_i})) \\ &= O(a_i, a_i + (b_i - a_i)G_i(N_\perp(0), \frac{y-a_i}{b_i-a_i})) \\ &= O(a_i, a_i + (b_i - a_i)G_i(1, \frac{y-a_i}{b_i-a_i})) \\ &= O(a_i, a_i + (b_i - a_i)) \\ &= O(a_i, b_i) \le a_i \qquad \text{(by Lemma 2.5)} \\ &\le y. \end{aligned}$$

(2) Let $x, y \in [0,1]/[a_i, b_i]$. We consider the following two cases.

If $x \le y$, by (O6) we get that

$$\begin{aligned} O(x, J_{G,N_\perp}(x,y)) &= O(x, I_{RS}(x,y)) \\ &= O(x,1) \le x \le y. \end{aligned}$$

If $x \ge y$, then

$$\begin{aligned} O(x, J_{G,N_\perp}(x,y)) &= O(x, I_{RS}(x,y)) \\ &= O(x,0) = 0 \le y. \end{aligned}$$

■

IV. THE (O,N)-IMPLICATION AND ITS O-CONDITIONALITY

In [26], Dimuro et al. mainly discussed the law of O-conditionality for R_O-implications, (G,N)-implications, QL-implications and D-implications, but QL-implications and D-implications do not satisfy the law of O-conditionality. In the sequence, we introduce a new fuzzy implication which does not satisfy the law of O-conditionality.

Theorem 4.1: Let $O : [0,1]^2 \to [0,1]$ be an overlap function and $N : [0,1] \to [0,1]$ be a fuzzy negation. Then the function $I_{O,N} : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$ by

$$I_{O,N}(x,y) = N(O(x, N(y)))$$

is a fuzzy implication, denoted by a (O,N)-implication $I_{O,N}$.

Proof: If $x \leq y$, then

$$\begin{array}{ll} O(x, N(z)) \leq O(y, N(z)) & \text{(by (O4))}, \\ N(O(x, N(z))) \geq N(O(y, N(z))) & \text{(by (N2))}, \end{array}$$

that is, $I_{O,N}(x, y) \geq I_{O,N}(x, z)$, $I_{O,N}$ satisfies (I1).

If $y \leq z$, then

$$\begin{array}{ll} O(x, N(y)) \geq O(x, N(z)) & \text{(by (O4))}, \\ N(O(x, N(y))) \leq N(O(x, N(z))) & \text{(by (N2))}, \end{array}$$

that is,

$$I_{O,N}(x, y) \leq I_{O,N}(x, z),$$

and so $I_{O,N}$ satisfies (I2).

$$\begin{aligned} I_{O,N}(0, 0) &= N(O(0, N(0))) \\ &= N(O(0, 1)) \\ &= N(0) \\ &= 1. \end{aligned}$$

$$\begin{aligned} I_{O,N}(1, 1) &= N(O(1, N(1))) \\ &= N(O(1, 0)) \\ &= N(0) \\ &= 1. \end{aligned}$$

$$\begin{aligned} I_{O,N}(1, 0) &= N(O(1, N(0))) \\ &= N(O(1, 1)) \\ &= N(1) \\ &= 0. \end{aligned}$$

That is, $I_{O,N}$ satisfies (I3).

Hence, $I_{O,N}(x, y) = N(O(x, N(y)))$ is a fuzzy implication. ■

By Theorem 2.17, we discuss weather $I_{O,N}$ satisfies (CAB) and (LOP).

Theorem 4.2: Let O be an overlap function and N be a fuzzy negation. Then $I_{O,N}$ satisfies (LOP) if and only if $N = N_\perp$.

Proof: ($\Leftarrow$) If $x \leq y$, then $N(x) \geq N(y)$,

$$\begin{array}{ll} O(x, N(y)) \leq O(x, N(x)) & \text{(by (O4))}, \\ N(O(x, N(y))) \geq N(O(x, N(x))) & \text{(by (N2))}, \end{array}$$

If $x = 0$, then

$$O(0, N_\perp(0)) = 0.$$

If $x > 0$, then

$$O(x, N_\perp(x)) = O(x, 0) = 0.$$

Hence

$$N(O(x, N(y))) \geq N(O(x, N(x))) = N(0) = 1,$$

that is,

$$I_{O,N}(x, y) = 1.$$

($\Rightarrow$)

$$\begin{array}{ll} I_{O,N}(x, y) = 1 = N(O(x, N(y))) & \\ \Leftrightarrow O(x, N(y)) = 0 & \text{(by (N7))} \\ \Leftrightarrow x = 0 \vee N(y) = 0 & \text{(by (O2))} \end{array}$$

Suppose that $N \neq N_\perp$. Then there exist $x, y \in]0, 1[$, $0 < x < y$, such that

$$0 < N(y) < 1, 0 < N(x) < 1,$$

one has that $I_{O,N}(x, y) \neq 1$, since $x > 0$ and $N(y) > 0$. Thus, $I_{O,N}$ does not satisfy (LOP). While if $I_{O,N}(x, y)$ satisfies (LOP), then $N = N_\perp$. ■

Theorem 4.3: Let $O : [0, 1]^2 \to [0, 1]$ be an overlap function and $N_\perp : [0, 1]^2 \to [0, 1]$ be the least fuzzy negation. Then $I_{O,N_\perp}$ does not satisfy (CAB).

Proof: Take $x, y \in]0, 1]$ such that $1 \geq x > y > 0$. Then

$$N_\perp(y) = 0$$

and

$$I_{O,N_\perp}(x, y) = N_\perp(O(x, 0)) = N_\perp(0) = 1 > y.$$

Thus $I_{O,N_\perp}$ does not satisfy (CAB). ■

Theorem 4.4: Let $O : [0, 1]^2 \to [0, 1]$ be an overlap function and $N_\perp : [0, 1]^2 \to [0, 1]$ be the least fuzzy negation. Then $I_{O,N_\perp}$ does not satisfy (OC).

Proof: Take $x = 1$ and $0 < y < 1$, then $N_\perp(y) = 0$. Hence

$$\begin{aligned} I_{O,N_\perp}(x, y) &= N_\perp(O(x, N_\perp(y))) \\ &= N_\perp(O(1, 0))) \\ &= N_\perp(0) \\ &= 1, \end{aligned}$$

and so,

$$O(x, I_{O,N_\perp}(x, y)) = O(1, 1) = 1 > y.$$

Therefore, $I_{O,N\perp}$ does not satisfy (OC). ■

V. Conclusions

The overlap and grouping functions are a special class of binary aggregation operators, while the associativity of these functions is generally not required in application problems. When considering fuzzy implications derived from overlap and grouping functions, some properties may not be verified, such as the commutative principle or the left-neutrality principle, but only weaker versions of these properties. In this paper, based on R_O-implications and (G, N)-implications derived from overlap, grouping and negation functions, we discuss the law of O-conditionality for the order sum of R_O-implications and (G, N)-implications, and also study the law of O-conditionality of (O, N)-implications. Future theoretical work is concerned with the investigation of the law of O-conditionality in the interval-valued setting, as in ([15], [30], [31], [33], [34]). These results can be used for performing inferences, decision making and in the fuzzy control of agents intentions.

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