The M/M/1 Repairable Queueing System with Variable Input Rates and Failure Rates

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***Abstract*—Waiting in line is an inevitable phenomenon in our daily life. The length of the queue affects whether customers enter the system. In this paper, we add variable input rates and failure rates to the classical M/M/1 queueing model. The input rate of customers is affected by the queue length in the system, it will increase or decrease when the queue length reaches the threshold m. The failure rate of an unreliable service station is different between idle and busy periods. There is a reliable repairman in the system who is responsible for repairing the faulted service station. Firstly, we construct the two-dimensional Markov chain using the quasi-birth-and-death(QBD) process theory. Then, the steady-state distribution of the system state is obtained by matrix analysis, and the steady-state performance indexes are obtained from the steady-state distribution. Finally, numerical experiments of the influence of parameters on the system performance index is provided. Numerical experiments illustrate the effectiveness of the proposed model.**

***Index Terms*—input rate, failure rate, parameter variable, repairable queueing system, matrix geometric solution.**

I. Introduction

IN real life, there are many repairable systems due to operator error or poor management and other reasons for system failure. In recent years, many of these publications only consider system failure without considering the variability of customer input rates. Xu and Xu [1] analyzed the queueing system with incomplete breakdown and delayed maintenance characteristics. Ma et al. [2] studied the queuing model with faults, N strategy and multiple work vacations. In this model, arrival intervals and service times are geometric distribution. They used the quasi-birth and death process to establish a two-dimensional Markov chain. Wu et al. [3] analyzed the machine maintenance problems of individual maintainers whose work failed. They obtained steady-state probabilities and various performance indicators. Yang and Chen [4] introduced work failure strategies in M/M/1 queueing systems with secondary alternative services. They optimized the service rate to get the minimum cost per unit of system time. Yu et al. [5] considered the queueing strategy of customers in almost visible and almost invisible incomplete fault queueing systems. For repairable queueing system, it is necessary to study the system availability. Wu et al. [6] studied a repairable system with a double threshold control policy. They applied matrix analysis to derive various performance indices such as system availability and performance indices. Other studies on models of single service station are presented in the literature [7], [8], [9], [10]. Lv et al. [11] studied a repairable queueing system with multiple service stations. Every service station may fail, and the failure rate is variable. They used the generating function method to derive the probability distributions and then obtained the steady-state mean queue length and other performance indices. Liu et al. [12] considered a warm standby system with N strategy and multiple vacations. Ji and Maxim [13] proposed repairable systems with delayed failures. They used the Laplace transform method to derive the survival probability and the mean failure time. Finally, numerical simulation is provided to verify the model. Li and Xu [14] investigated a parallel repairable system. Lyu et al. [15] studied a queueing system where two service stations could serve one customer simultaneously. They derived performance indices such as steady-state mean queue length. Lv [16] studied a queueing system with two repairmen and a limited number of repairable machines. If more than two machines fail in the system, two technicians will repair the faulty machine separately. When only one machine in the system fails, two technicians repair the faulty machine at the same time. He derived important system performance indices. Li and Li [17] obtained the steady-state conditions and probability distributions of the retrial queueing system. Ramasamy et al. [18] analyzed a model in which the service time of two service stations obeys different distributions. For this model, they derived the steady-state results in detail. Tsai et al. [19] studied open queueing networks with faults. They verified the validity of the model and the correctness of the method. In addition, they put forward reasonable suggestions for the optimization of the proposed model.

When we line up in life, the input rate may change with the queue length in the system. If the number of customers in the system is large, customers will stay for a long time after entering the system. In this case, the stay time is longer than the customer can accept. Therefore, customers choose not to enter the system to receive services, so that the input rate becomes smaller. On the other hand, when we see a lot of people lining up for a particular product in a store, we are motivated to buy it. It is like when you are faced with two restaurants you have never eaten at before, you would choose the one with more customers. Because people always

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feel that what the majority believes must be the best. This herd mentality will cause the input rate to become larger when the queue length in the system reaches the threshold.

In our life, service stations are generally not completely reliable and often fail. On the one hand, when the service station is idle, it may fail due to unfavorable factors such as aging and outside influences. On the other hand, when the service station is working, it may fail due to wear, corrosion, fatigue, etc. of the machine. In general, the failure rate of service stations is different when they are idle and when they are working. When the service station breaks down, the service provider will immediately arrange for a repairman to repair the broken service station. It is of great practical significance to study the variable breakdown rates.

In summary, the parameters of a queueing system in real life generally change with the state of the system, such as the queue length in the system, etc. Previous studies only considered different failure rates in different states, without considering the impact of the queue length on the input rate. So we study a queueing system with two input rates and breakdown rates using the matrix geometry solution [20] method.

II. Model description

We assume that the system has infinite capacity and only one service station. The service station may break down at any time. Meanwhile, a repairman is responsible for repairing it.

1) Customer arrival is a Poisson process, and the input rate varies with the queue length in the system. When the queue length is smaller than m, the input rate of customers is $\lambda_1(\lambda_1 > 0)$; when the queue length is greater than or equal to m, the input rate of customers is $\lambda_2(\lambda_2 > 0)$.

2) The service times obey the negative exponential distribution of μ.

3) The arrival of the fault is a Poisson process. When the service station is an idle period, the failure rate is $\alpha_1(\alpha_1 > 0)$; when the service station is in a busy period, the failure rate is $\alpha_2(\alpha_2 > 0)$. The failure rate is affected by the status of the service station.

4) When the service station fails, a reliable repairman immediately repairs the faulty service station. The repair time is an exponential distribution. The repair rate is $\eta(\eta > 0)$. The service station is repaired as well as new.

5) The service rule of the service station to customers in the system is first-come, first-served (FCFS). If there is a customer waiting in the system after the service station is repaired, the service station immediately performs the service, otherwise the service station is in idle period. If a customer arrives at the service station during the idle period, the service will be started immediately. Assuming that the arrival interval, service time, breakdown time and repair time are independent of each other.

Let $Q(t)$ be the number of customers in the system and $Y(t)$ be the number of available service stations in the system at moment t. Then $\{(Q(t), Y(t)), t \geq 0\}$ describes the instantaneous state of the system. The state space is $\Omega = \{(q, y), q = 0, 1, 2, \cdots; y = 0, 1\}$. The system states are arranged in dictionary order, and the state transfer diagram of the two-dimensional Markov chain is shown in Figure 1.

Its transfer rate matrix $\mathbf{Q}$ is the following block tridiagonal matrix.

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 & & & & & & \\ \mathbf{B} & \mathbf{A}_1 & \mathbf{C}_0 & & & & & \\ & \mathbf{B} & \mathbf{A}_1 & \mathbf{C}_0 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \mathbf{B} & \mathbf{A}_1 & \mathbf{C}_0 & & \\ & & & & \mathbf{B} & \mathbf{A} & \mathbf{C} & \\ & & & & & \mathbf{B} & \mathbf{A} & \mathbf{C} \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} -\lambda_1 - \eta & \eta \\ \alpha_1 & -\lambda_1 - \alpha_1 \end{pmatrix}, C_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\lambda_1 - \eta & \eta \\ \alpha_2 & -\lambda_1 - \mu - \alpha_2 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

$$A = \begin{pmatrix} -\lambda_2 - \eta & \eta \\ \alpha_2 & -\lambda_2 - \mu - \alpha_2 \end{pmatrix}, C = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

III. Steady-state conditions

Theorem 1. The system of equations $C + RA + R^2B = 0$ has a minimum non-negative solution

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

where

$$r_{11} = \frac{\lambda_2 (\mu + \alpha_2)}{\mu (\eta + \lambda_2)}, r_{12} = \frac{\lambda_2}{\mu}, r_{21} = \frac{\lambda_2 \alpha_2}{\mu (\eta + \lambda_2)}, r_{22} = \frac{\lambda_2}{\mu}.$$

Proof Bringing R into the system of equations $C + RA + R^2B = 0$. Then

$$\begin{cases} -(\lambda_2 + \eta) r_{11} + \alpha_2 r_{12} + \lambda_2 = 0, \\ \mu (r_{11} r_{12} + r_{12} r_{22}) + \eta r_{11} - (\lambda_2 + \mu + \alpha_2) r_{12} = 0, \\ -(\lambda_2 + \eta) r_{21} + \alpha_2 r_{22} = 0, \\ \mu (r_{12} r_{21} + r_{22}^2) + \eta r_{21} - (\lambda_2 + \mu + \alpha_2) r_{22} + \lambda_2 = 0. \end{cases} \tag{1}$$

Solving Eq. (1) to obtain the minimum non-negative solution R.

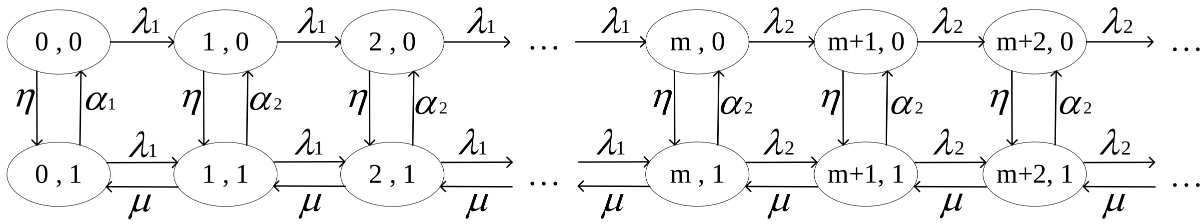

Fig. 1. State transfer diagram.

Theorem 2. A sufficient necessary condition for the normal return of the QBD process $\{(Q(t) = q, Y(t) = y), t \geq 0\}$ is

$$\rho = \frac{\lambda_2(\alpha_2 + \eta)}{\mu\eta} < 1.$$

Proof From the structure of Q matrix, we know that $\{(Q(t) = q, Y(t) = y), t \geq 0\}$ is a QBD process. Let $H = B + A + C$, then

$$H = \begin{pmatrix} -\eta & \eta \\ \alpha_2 & -\alpha_2 \end{pmatrix}.$$

Obviously, H is a finite integrable matrix. Its steady-state probability vector $P = \begin{pmatrix} p_0 & p_1 \end{pmatrix}$ satisfies

$$\begin{pmatrix} p_0 & p_1 \end{pmatrix} H = 0,\ p_0 + p_1 = 1.$$

The solution is

$$p_0 = \frac{\alpha_2}{\eta + \alpha_2},\ p_1 = \frac{\eta}{\eta + \alpha_2}.$$

A sufficient necessary condition for the normal return of the QBD processes $\{(Q(t), Y(t)), t \geq 0\}$ is $PCe < PBe$, where e is a two-dimensional column vector whose elements are all equal to 1. Then

$$\begin{aligned} &\begin{pmatrix} p_0 & p_1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &< \begin{pmatrix} p_0 & p_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

By simple operation, this condition is equivalent to

$$\rho = \frac{\lambda_2(\alpha_2 + \eta)}{\mu\eta} < 1.$$

IV. Steady-state probability

When the system satisfies the condition of steady-state distribution, the Markov process

$$\{(Q(t) = q, Y(t) = y), t \geq 0\}$$

returns normally and the steady-state distribution of the system state exists. Define the steady-state probability:

$$\prod = (\pi_0, \pi_1, \pi_2, \cdots),$$

where,
the steady-state probability

$$\pi_q = (\pi_{q,0}, \pi_{q,1}),\ q = 0, 1, 2, \cdots,$$

the steady-state distribution

$$\pi_{q,y} = \lim_{t \to \infty} P\{\mathrm{Q}(t) = q, Y(t) = y\}, (q, y) \in \Omega.$$

Under the sufficient necessary conditions for the normal return of the QBD process $\{(Q(t), Y(t)), t \geq 0\}$ in Theorem 2, the m-dimensional random array

$$B[R] = \begin{pmatrix} A_0 & C_0 & & & & \\ B & A_1 & C_0 & & & \\ & B & A_1 & C_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B & A_1 & C_0 \\ & & & & B & RB + A \end{pmatrix},$$

and the steady-state distribution satisfies the system of equations

$$\begin{cases} (\pi_0, \pi_1, \cdots, \pi_m) B[R] = 0, \\ \sum\limits_{q=0}^{m-1} \pi_q e + \pi_m (I - R)^{-1} e = 1, \\ \pi_q = \pi_m R^{q-m}, q \geq m, \end{cases} \tag{2}$$

where I is a 2-dimensional unit matrix, e is a column vector of dimension 2 and all elements are 1.

Theorem 3. Let $D_0 = A_0$, $D_i = A_1 - BD_{i-1}^{-1} C_0 (m > 0,\ 1 \leq i \leq m - 1)$, then D_0 and D_i are both invertible matrices.

Proof Because

$$D_0 = A_0 = \begin{pmatrix} -\lambda_1 - \eta & \eta \\ \alpha_1 & -\lambda_1 - \alpha_1 \end{pmatrix},$$

the determinant of D_0 is

$$\begin{aligned} |D_0| &= |A_0| \\ &= \begin{vmatrix} -\lambda_1 - \eta & \eta \\ \alpha_1 & -\lambda_1 - \alpha_1 \end{vmatrix} \\ &= \lambda_1 (\lambda_1 + \alpha_1 + \eta) \\ &\neq 0, \end{aligned}$$

so D_0 is invertible. And because

$$\begin{aligned} D_1 &= A_1 - BD_0^{-1} C_0 \\ &= \begin{pmatrix} -\eta - \lambda_1 & \eta \\ \alpha_2 + \frac{\mu\alpha_1}{\eta + \alpha_1 + \lambda_1} & \frac{\mu\alpha_1}{\eta + \alpha_1 + \lambda_1} - \alpha_2 - \lambda_1 \end{pmatrix}, \end{aligned}$$

the determinant of D_1 is

$$\begin{aligned} |D_1| &= \begin{vmatrix} -\eta - \lambda_1 & \eta \\ \alpha_2 + \frac{\mu\alpha_1}{\eta + \alpha_1 + \lambda_1} & \frac{\mu\alpha_1}{\eta + \alpha_1 + \lambda_1} - \alpha_2 - \lambda_1 \end{vmatrix} \\ &= \frac{\lambda_1 \left((\eta + \lambda_1)(\eta + \alpha_2 + \lambda_1) + \alpha_1 (\eta + \mu + \alpha_2 + \lambda_1) \right)}{\eta + \alpha_1 + \lambda_1} \\ &\neq 0. \end{aligned}$$

Therefore, D_1 is also an invertible matrix. By recursion, we can get $D_i = A_1 - BD_{i-1}^{-1} C_0 (m > 0,\ 1 \leq i \leq m - 1)$ as an invertible matrix.

Using Eq. (2) we get

$$\begin{aligned} \pi_i &= -\pi_{i+1} BD_i^{-1} (0 \leq i \leq m - 1, m > 0), \\ \pi_m &= -\pi_{m-1} C_0 (RB + A)^{-1}, \end{aligned}$$

then

$$\begin{aligned} \pi_0 &= -\pi_1 BD_0^{-1} = (-1)^2 \pi_2 BD_1^{-1} BD_0^{-1} \\ &= \cdots = (-1)^m \pi_m \prod_{y=0}^{m-1} BD_y^{-1}, \\ \pi_1 &= -\pi_2 BD_1^{-1} = (-1)^2 \pi_3 BD_2^{-1} BD_1^{-1} \\ &= \cdots = (-1)^{m-1} \pi_m \prod_{y=1}^{m-1} BD_y^{-1}, \\ &\vdots \end{aligned}$$

The recurrence leads to

$$\pi_i = (-1)^{m-i} \pi_m \prod_{y=i}^{m-1} BD_y^{-1} (m > 0,\ 0 \leq i \leq m - 1).$$

Then the steady-state boundary probability vector of the system is

$$\pi_i = \pi_m F_i (m > 0,\ 0 \le i \le m-1),$$

where

$$F_i = (-1)^{m-i} \prod_{y=i}^{m-1} BD_y^{-1},$$

π_m satisfies

$$\begin{cases} \pi_m (\sum\limits_{i=0}^{m-1} F_i + (I-R)^{-1})e = 1, \\ \pi_m (F_{m-1} C_0 + A + RB) = 0. \end{cases}$$

In particular, when $m = 0$, it means that the input rate of customers is not affected by the number of customers and is constant at λ_2. At this time, the block matrix in the first row and first column of the state transfer rate matrix of the system changes. The matrix $B[R]$ becomes

$$B[R] = \begin{pmatrix} A_{00} & C \\ B & RB + A \end{pmatrix},$$

where

$$A_{00} = \begin{pmatrix} -\lambda_2 - \eta & \eta \\ \alpha_1 & -\lambda_2 - \alpha_1 \end{pmatrix}.$$

When the QBD process returns normally, the steady-state distribution satisfies the system of equations

$$\begin{cases} (\pi_0, \pi_1) B[R] = 0, \\ \pi_0 e + \pi_1 (I-R)^{-1} e = 1, \\ \pi_q = \pi_1 R^{q-1}, q \ge 1. \end{cases} \tag{3}$$

Theorem 4. Let $\varphi_1 = \eta\mu - \lambda_2\eta - \lambda_2\alpha_2$, when $m = 0$, the elements of the system steady-state boundary probability vector $\begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix}$ are as follows:

$$\pi_{00} = \frac{\alpha_1 (\eta\mu - \eta\lambda_2 - \lambda_2\alpha_2)}{(\eta + \lambda_2)\mu(\eta + \alpha_1)}, \tag{4}$$

$$\pi_{01} = \frac{\eta\mu - \eta\lambda_2 - \lambda_2\alpha_2}{\mu(\eta + \alpha_1)}, \tag{5}$$

$$\pi_{10} = \frac{\varphi_1 \left(\lambda_2\mu\alpha_1 + \eta\lambda_2\alpha_2 + \lambda_2^2\alpha_2 + \lambda_2\alpha_1\alpha_2\right)}{(\eta + \lambda_2)^2 \mu^2 (\eta + \alpha_1)}, \tag{6}$$

$$\pi_{11} = \frac{\varphi_1 \left(\eta\lambda_2 + \lambda_2^2 + \lambda_2\alpha_1\right)}{(\eta + \lambda_2)\mu^2(\eta + \alpha_1)}. \tag{7}$$

Proof Using Eq. (3), we have

$$\begin{cases} \pi_0 = -\pi_1 B A_{00}^{-1}, \\ \pi_0 e + \pi_1 (I-R)^{-1} e = 1, \\ \pi_0 C + \pi_1 (RB + A) = 0. \end{cases} \tag{8}$$

Bringing A_{00}, C, B, A, R into Eq. (8) yields

$$\begin{cases} \pi_{00} + \pi_{01} - \dfrac{(\pi_{10}\eta + \pi_{11}\eta + \pi_{10}\lambda_2)\mu}{\eta\lambda_2 - \eta\mu + \lambda_2\alpha_2} = 1, \\ \dfrac{\pi_{11}\mu\alpha_1}{\alpha_1\lambda_2 + \eta\lambda_2 + \lambda_2^2} = \pi_{00}, \\ \dfrac{\pi_{11}\mu(\eta + \lambda_2)}{\alpha_1\lambda_2 + \eta\lambda_2 + \lambda_2^2} = \pi_{01}, \\ \pi_{10}(-\eta - \lambda_2) + \pi_{11}\alpha_2 + \pi_{00}\lambda_2 = 0, \\ \pi_{10}(\eta + \lambda_2) - \pi_{11}(\alpha_2 + \mu) + \pi_{01}\lambda_2 = 0. \end{cases} \tag{9}$$

The result in Theorem 4 is obtained by solving Eq. (9).

V. SYSTEM STEADY-STATE PERFORMANCE INDICES

According to the steady-state probability expressions obtained from the matrix analysis, we obtain the steady-state performance indices of the queueing system with two input rates and breakdown rates.

A. *When $m > 0$, input rate and failure rate will change*

1) Steady-state queue length distribution of the system

$$P(Q = q) = \begin{cases} \pi_m F_q \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & 0 \le q \le m-1, \\ \pi_m R^{q-m} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & q \ge m, \end{cases}$$

2) The availability of the service station

$$\begin{aligned} A =& P(Y = 1) \\ =& \sum_{i=0}^{m-1} \pi_m F_i e_1 + \sum_{i=m}^{\infty} \pi_i e_1 \\ =& \sum_{i=0}^{m-1} \pi_m F_i e_1 \\ &+ (\pi_m + \pi_m R + \pi_m R^2 + \cdots) e_1 \\ =& \sum_{i=0}^{m-1} \pi_m F_i e_1 + \pi_m (I-R)^{-1} e_1, \end{aligned} \tag{10}$$

where $e_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^\top$.

3) The probability that the service station is in a fault state

$$P(Y = 0) = \sum_{i=0}^{m-1} \pi_m F_i e_2 + \pi_m (I-R)^{-1} e_2,$$

where $e_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$.

4) The mean queueing length in the steady-state system

$$\begin{aligned} E(L) =& \sum_{i=0}^{m-1} i\pi_m F_i e + \sum_{i=m}^{\infty} i\pi_i e \\ =& \left[m\pi_m + (m+1)\pi_m R + (m+2)\pi_m R^2 + \cdots\right] e \\ &+ \sum_{i=0}^{m-1} i\pi_m F_i e \\ =& \sum_{i=0}^{m-1} i\pi_m F_i e + \pi_m (R(I-R)^{-2} + m(I-R)^{-1}) e. \end{aligned} \tag{11}$$

5) The mean waiting queue length in the steady-state system

$$\begin{aligned} E(L_q) &= \sum_{i=1}^{\infty} (i-1)\pi_i e \\ &= \sum_{i=1}^{\infty} i\pi_i e - \sum_{i=1}^{\infty} \pi_i e \\ &= E(L) - (1 - \pi_0 e). \end{aligned}$$

6) The steady-state mean sojourn time of the system

$$W = (E(L) + 1)\frac{1}{\mu}.$$

B. When $m = 0$, the input rate will be fixed at λ_2

1) Steady-state queue length distribution of the system

$$P(Q=q)=\begin{cases}\pi_0\begin{pmatrix}1\\1\end{pmatrix}, & q=0,\\ \pi_1 R^{q-1}\begin{pmatrix}1\\1\end{pmatrix}, & q\geq 1.\end{cases}$$

2) The availability of the service station

$$A=P(Y=1)=\sum_{i=0}^{\infty}\pi_i e_1=\pi_0 e_1+\pi_1(I-R)^{-1}e_1. \tag{12}$$

3) The probability that the service station is in a fault state

$$P(Y=0)=\pi_0 e_2+\pi_1(I-R)^{-1}e_2.$$

4) The mean queueing length in steady-state system

$$\begin{aligned}E(L)&=\sum_{i=0}^{\infty}i\pi_i e\\&=(\pi_1+2\pi_2+3\pi_3+\cdots)\,e\\&=\left[\pi_1\left(I+2R+3R^2+\cdots\right)\right]e\\&=\pi_1(I-R)^{-2}e.\end{aligned} \tag{13}$$

5) The mean waiting queue length in the steady-state system

$$E(L_q)=E(L)-1+\pi_0 e.$$

6) The mean sojourn time in the steady-state system

$$W=(E(L)+1)\frac{1}{\mu}.$$

VI. Special cases

This section gives the formulas for the availability and the mean queue length in the steady-state system at the threshold $m = 0$ and $m = 1$.

A. $m = 0$

1) When $m = 0$, the input rate of customers is fixed at λ_2. Using Eq. (4), Eq. (5), Eq. (6), Eq. (7) and Eq. (12), we obtain the availability of the service station

$$\begin{aligned}A&=\pi_0 e_1+\pi_1(I-R)^{-1}e_1\\&=\frac{\eta\mu+\lambda_2\alpha_1-\lambda_2\alpha_2}{\eta\mu+\mu\alpha_1}.\end{aligned}$$

2) Using Eq. (6), Eq. (7) and Eq. (11), we obtain the mean queueing length in steady-state system

$$\begin{aligned}E(L)=&\pi_1(I-R)^{-2}e\\=&\frac{\lambda_2\alpha_1(\lambda_2-\mu)^2+\lambda_2\alpha_2\left(\eta\mu+\lambda_2\mu+\mu\alpha_1-\lambda_2^2\right)}{\mu\left(\eta+\alpha_1\right)\left(\mu\eta-\lambda_2\eta-\lambda_2\alpha_2\right)}\\&+\frac{\lambda_2^3-\left(\eta+\lambda_2\right)\mu\lambda_2}{(\eta\lambda_2-\eta\mu+\lambda_2\alpha_2)\mu}-\frac{\lambda_2^2}{(\eta+\alpha_1)\mu}\\=&\frac{\lambda_2^2-\eta\mu-\lambda_2\mu}{\lambda_2\eta-\mu\eta+\lambda_2\alpha_2}-\frac{\lambda_2}{\eta+\alpha_1}-1.\end{aligned}$$

B. $m = 1$

Customers' input rates will change. The system steady-state boundary probability vectors π_0 and π_1 are

$$\begin{aligned}\pi_{00}&=\frac{\alpha_1\varphi_1}{\varphi_2+\varphi_3},\\ \pi_{01}&=\frac{(\eta+\lambda_1)\varphi_1}{\varphi_2+\varphi_3},\\ \pi_{10}&=\frac{\lambda_1\left(\alpha_1\left(\mu+\alpha_2\right)+\alpha_2\left(\eta+\lambda_1\right)\right)\varphi_1}{\mu\left(\eta+\lambda_2\right)\left(\varphi_2+\varphi_3\right)}, & (14)\\ \pi_{11}&=\frac{\lambda_1\left(\eta+\alpha_1+\lambda_1\right)\varphi_1}{\mu(\varphi_2+\varphi_3)}, & (15)\end{aligned}$$

where

$$\begin{aligned}\varphi_1&=\eta\mu-\lambda_2\eta-\lambda_2\alpha_2,\\ \varphi_2&=(\eta+\lambda_1)\left(\eta\mu+(\eta+\alpha_2)\left(\lambda_1-\lambda_2\right)\right),\\ \varphi_3&=\alpha_1\left(\eta\mu+\lambda_1\eta+\lambda_1\mu+\lambda_1\alpha_2-\lambda_2\eta-\lambda_2\alpha_2\right).\end{aligned}$$

The calculation method is the same as Theorem 4.

1) Using Eq. (10), Eq. (14) and Eq. (15), we obtain the availability of the service station

$$\begin{aligned}A=&\sum_{i=0}^{m-1}\pi_m F_i e_1+\pi_m(I-R)^{-1}e_1\\=&\pi_1F_0e_1+\pi_1(I-R)^{-1}e_1\\=&-\pi_1BD_0^{-1}e_1+\pi_1(I-R)^{-1}e_1\\=&\frac{\lambda_1\left(\eta\left(\eta+\lambda_1\right)+\alpha_1\left(\eta+\lambda_2\right)\right)\varphi_1}{\left(\eta\mu-\left(\eta+\alpha_2\right)\lambda_2\right)\left(\varphi_2+\varphi_3\right)}\\&+\frac{\left(\eta+\lambda_1\right)\varphi_1}{\varphi_2+\varphi_3}.\end{aligned}$$

2) Using Eq. (11), Eq. (14) and Eq. (15), we obtain the mean queueing length in steady-state system

$$\begin{aligned}E(L)=&\frac{\mu\lambda_1\alpha_1\left(\eta+\mu+\alpha_2-\lambda_2\right)\left(\eta+\lambda_2\right)}{\varphi_1\left(\varphi_2+\varphi_3\right)}\\&+\frac{\mu\lambda_1\left(\eta+\lambda_1\right)\left(\eta^2+\alpha_2\eta+\alpha_2\lambda_2\right)}{\varphi_1\left(\varphi_2+\varphi_3\right)}.\end{aligned}$$

VII. Numerical experiments

This section analyzes the effect of different threshold values m and variations of each parameter on the availability of service station, the steady-state mean queue length and mean sojourn time of the system through numerical experiments.

It is necessary to study the availability of the service station if the service station may fail. Assuming that $\lambda_1 = 2.5$, $\mu = 4$, $\eta = 0.8$, $\alpha_1 = 0.01$, m takes values of 0 and 1, λ_2 takes values of 1.5 and 3, and α_2 varies in the range of 0.05 to 0.1. From Figure 2, when the threshold value m and the second arrival rate λ_2 are constant, A decreases with the increase of the second failure rate α_2. As α_2 increases, the possibility of service station failure is increasing, that is, the availability in the steady-state system is decreasing. When λ_2= 1.5 $< \lambda_1$, the larger the threshold m, the smaller the availability of service station A. Because as the threshold m increases, the steady-state mean queueing length increases, resulting in the decrease of A. In contrast, when λ_2= 3 $>\lambda_1$, as the threshold m increases, the availability A increases. This is consistent with the actual situation.

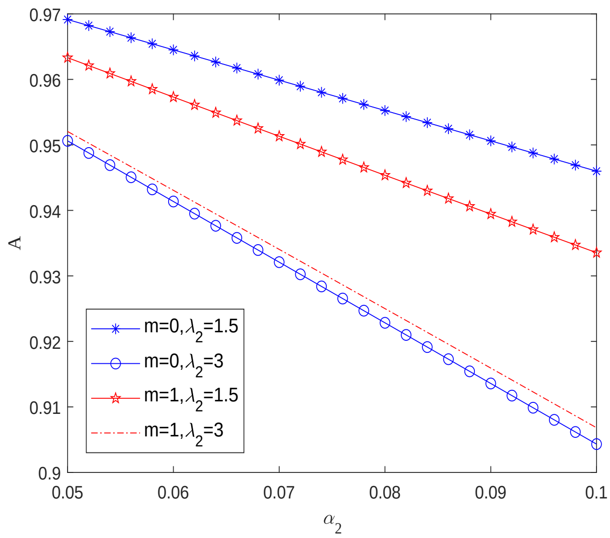

Fig. 2. The trend of availability versus α_2 for m takes different values ($\lambda_1 = 2.5$, $\mu = 4$, $\alpha_1 = 0.01$ and $\eta = 0.8$).

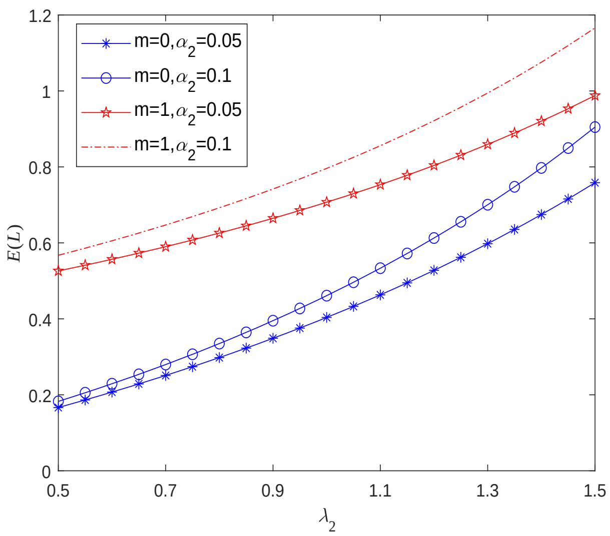

Fig. 3. The trend of mean queue length versus λ_2 ($\lambda_1 = 2.5$, $\mu = 4$, $\alpha_1 = 0.01$ and $\eta = 0.8$).

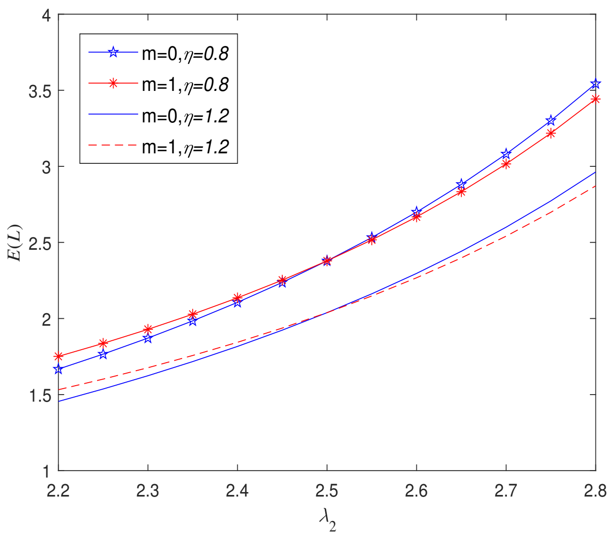

Fig. 4. The trend of mean queue length versus λ_2 ($\lambda_1 = 2.5$, $\mu = 4$, $\alpha_1 = 0.01$ and $\alpha_2 = 0.05$).

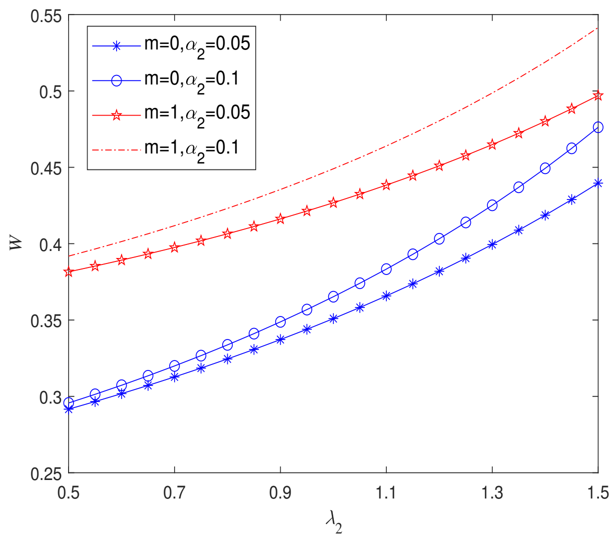

Fig. 5. The trend of mean sojourn time versus λ_2 ($\lambda_1 = 2.5$, $\mu = 4$, $\alpha_1 = 0.01$ and $\eta = 0.8$).

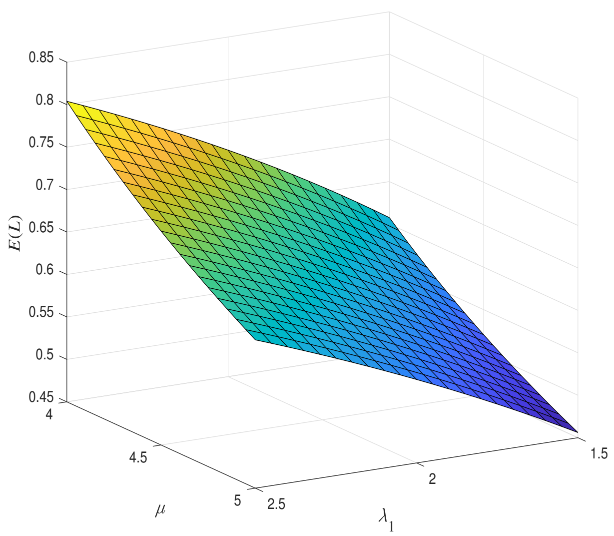

Fig. 6. The trend of mean queue length versus μ and λ_1 ($m = 1$, $\lambda_2 = 1.2$, $\alpha_1 = 0.01$, $\alpha_2 = 0.05$ and $\eta = 0.8$).

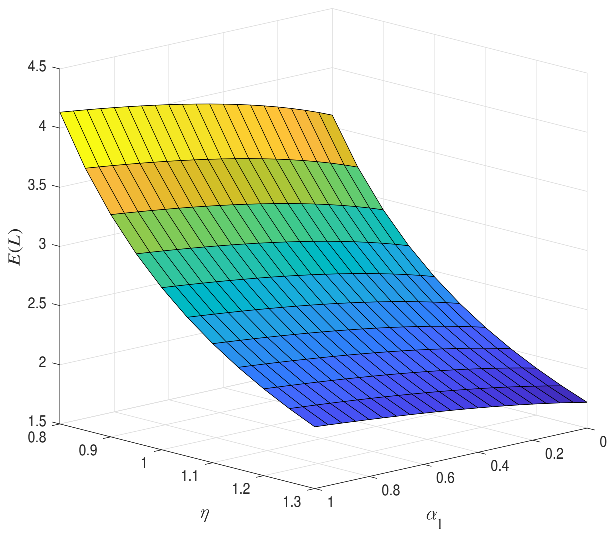

Fig. 7. The trend of mean queue length versus α_1 and η ($m = 1$, $\lambda_1 = 2.5$, $\lambda_2 = 1.2$, $\mu = 5$ and $\alpha_2 = 1.2$).

Assuming that $\lambda_1 = 2.5$, $\mu = 4$, $\eta = 0.8$, $\alpha_1 = 0.01$, m takes values of 0 and 1, α_2 takes values of 0.05 and 0.1, and λ_2 varies in the range of 0.5 to 1.5. Figure 3 depicts the effects of the second input rate λ_2, threshold m and busy period failure rate α_2 on $E(L)$ in the system. When the threshold m and the busy period failure rate α_2 are constant, $E(L)$ increases with the increase of λ_2, and the rising trend is increasing. This is because as λ_2 increases, more and more customers enter the system, which increases the mean queue length. When the second input rate λ_2 and the threshold m are constant, as α_2 increases, $E(L)$ is increasing. This result is consistent with real-life situations. When the second input rate λ_2 and the busy period failure rate α_2 are constant, as m increases, $E(L)$ is increasing. The larger the value of λ_2, the closer $E(L)$ is of the two values of m. This is because the closer λ_2 is to λ_1, the less $E(L)$ is affected by the threshold m. It must be noted here that due to λ_1=2.5 $> \lambda_2$, then the mean queue length in steady state increases as m increases.

Assuming that $\lambda_1 = 2.5$, $\mu = 4$, $\alpha_1 = 0.01$, $\alpha_2 = 0.05$, m takes values of 0 and 1, η takes values of 0.8 and 1.2, and λ_2 varies in the range of 2.2 to 2.8. Figure 4 depicts the effect of the second input rate λ_2, the threshold m, and the repair rate η on $E(L)$ in the steady-state system. In Figure 4, on the basis of satisfying the steady-state conditions, when $\lambda_1 = \lambda_2$, $E(L)$ is equal and no longer affected by the threshold m. If $\lambda_1 < \lambda_2$, the steady-state mean queue length decreases as m increases. The case of $\lambda_1 < \lambda_2$ is also usual in real life. In addition, from Figure 4, $E(L)$ decreases with the increase of the repair rate η.

Assuming that $\lambda_1 = 2.5$, $\mu = 4$, $\eta = 0.8$, $\alpha_1 = 0.01$, m takes values of 0 and 1, α_2 takes values of 0.05 and 0.1, and λ_2 varies in the range of 0.5 to 1.5. Figure 5 depicts the effect of the second input rate λ_2, the threshold m and the busy period failure rate α_2 on the system mean sojourn time W. The trend of the curves in Figure 5 is the same as that in Figure 3, so the trend of W and $E(L)$ is the same.

When $m = 1$, Figure 6 depicts the influences of the first input rate λ_1 and the service rate μ on $E(L)$ of the system at steady state. In Figure 6, when μ is constant, $E(L)$ increases with the increase of λ_1. When λ_1 is constant, $E(L)$ decreases with increasing μ. Figure 7 depicts the influences of α_1 and η on the mean queue length $E(L)$. It can be seen that when η is constant, $E(L)$ increases with the increase of α_1. When α_1 is constant, the trend of $E(L)$ with η is the same as Figure 4. This is consistent with our intuition.

VIII. CONCLUSION

We study the queueing system with variable input rates and failure rates. In this mode, the service station may break down at any time and a repairman can fix it immediately. Customer arrival is a Poisson process, and customer service time is an exponential distribution. We construct a two-dimensional Markov chain using the quasi-birth-and-death processes theory. The constructed balance equation is deduced in detail. Meanwhile, we give the display results of the mean queue length and mean sojourn time in steady state for the number of customers thresholds $m = 0$ and $m = 1$. The effects of different thresholds m and the change of each parameter on the performance indexes are analyzed by numerical examples. The results show that different thresholds and parameters have significant effects on the mean queue length and mean sojourn time in the steady-state system. Comparing the model with the immutable input rate, the model features in this paper are of strong general relevance significance. In practical application, the service organization can appropriately adjust the threshold m according to the actual situation in order to maximize the benefits. Based on the research in this paper, the model with variable input rates and failure rates can be extended to the M/M/N(N>1) queueing system in the future.

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