Parameter Estimation of Lindley Distribution under Generalized First-failure Progressive Hybrid Censoring Schemes

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Abstract—**We propose a generalized first progressive failure hybrid censoring scheme in this paper, which guarantees the time to complete the life test. We use classical and Bayesian methods to calculate the unknown parameter's point and interval estimator of Lindley distribution under the scheme. The maximum likelihood estimation is obtained by maximizing the log-likelihood function. we calculate the asymptotic confidence intervals and coverage probabilities of the parameter. Additionally, there are two bootstrap confidence intervals by bootstrap method. In Bayes theory, the Metropolis-Hastings algorithm is applied to obtain the Bayes estimator and highest posterior density credible intervals of the unknown parameter, which has a gamma prior. The waiting time data set is analysed by the mentioned method for illustration purposes, Monte Carlo simulations experiment are designed to compare the performance of the proposed point and interval estimate methods.**

Index Terms—**Lindley distribution, generalized first-failure PHC, maximum likelihood estimation, bootstrap confidence interval, Bayes estimator.**

I. Introduction

Lindley [1] first proposed the Lindley distribution (LD) in connection with Bayesian statistics, and it is fitted a counter data. As mentioned in the article, the probability density function (PDF) $f(z,a)$ and the cumulative distribution function (CDF) $F(z,a)$ of LD are:

$$f(z,a)=\frac{a^2(1+z)}{1+a}e^{-az} \tag{1}$$

$$F(z,a)=1-\frac{1+a+az}{1+a}e^{-az}, z>0, a>0, \tag{2}$$

where a is the unknown parameter of LD. Ghitany et al. [2] discussed different properties of the LD. Krishna and Kummar [3] discussed the properties and the reliability of the LD under the progressively Type-II right censored data. Al-Mutairi et al. [4] and Kumar et al. [5] studied the analysis of strength stress model reliability parameter, when stress and strength are independent Lindley random variables. Dube et al. [6] studied the reliability characteristics of the LD under progressively first failure censored sample. Gupta and Singh [7] derived the parameter estimation of the LD based on hybrid censored data. In this paper, we investigate the possibility of extending the parameter estimation of LD under more general censoring scheme settings.

Due to time, cost or other data collection constraints, Censored data are usually found in life testing. Type-I, Type-II, first-failure censoring shemes are available to apply in life test. Balakrishnan and Aggarwala [8] proposed progressive censoring, which allowed surviving units to be removed from the test before the all products fail. First-failure and progressive censoring were coupled to create the progressively first-failure system by Wu and Kus [9]. Shorter test times and resource savings are benefits of the first-failure censored sampling strategy, but its disadvantage takes a long time to complete the life test. To overcome the drawbacks, we propose a generalized first failure progressive hybrid censoring (GFFPHC) scheme, which guaranteed the life test is completed at a predetermined time. The GFFPHC scheme's life-testing can be summarized as follows: assume that N independent groups with K items in each group are put on test. Before the life-testing, we need to set:
(1) The integer $m \leq n$ and the time points $T_1 < T_2$;
(2) The GFFPHC scheme $R=(R_1, R_2, \cdots, R_m)$.
The failure number of observations up to time T_1 and T_2 is denoted by D_1 and D_2, respectively, d_1 and d_2 represent the observations of D_1 and D_2.

When the first component fails (express as $X_{1:m:N:K}$) and we remove corresponding group, R_1 groups randomly removing from the survival $N-1$ groups. Similarly, when $X_{2:m:N:K}$ occurs, we remove the corresponding group and additional R_2 groups randomly removing from remaining $N-R_1-2$ groups and so forth. For convenience, $X_{i:m:N:K}$ can be represented as $X_i, (i=1,2,...,N)$. Combined with the censoring scheme, there will be the following cases:

(1) If $X_m < T_1$, when the mth failure unit occurs, the test continue but without removing other groups. Up to T_1, the test terminates and removes all the remaining survival units. That is $R_m = R_{m+1} = \cdots = R_{d_1} = 0$;

(2) If $T_1 < X_m < T_2$, the test is terminated at X_m.

(3) If $X_m > T_2$, the test is terminated at T_2.

The GFPHC test will be completed at T_2, which indicates the longest time that researchers allow the experiment to run.

Furthermore, the failure samples obtained in the above three cases can be expressed as:
Case I: $X_1, X_2, \cdots, X_m, X_{m+1}, \cdots, X_{d_1}, \quad X_m < T_1, d_1 \leq m+R_m$,
Case II: $X_1, X_2, \cdots, X_m, \quad T_1 < X_m < T_2$,
Case III: $X_1, X_2, \cdots, X_{d_2}, \quad X_m > T_2, 0 < d_2 < m$.

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The likelihood function can be expressed as

$$L(\theta|x)=\begin{cases} C_{d_1}K^{d_1}\prod_{i=1}^{d_1} f(x_i)\,(1-F(x_i))^{K(R_i+1)-1} \\ (1-F(T_1))^{KR_{d_1}}, \quad \text{if } x_m<T_1, \\ C_m K^m \prod_{i=1}^{m} f(x_i)\,(1-F(x_i))^{K(R_i+1)-1}, \\ \text{if } T_1<x_m<T_2, \\ C_{d_2}K^{d_2}\prod_{i=1}^{d_2} f(x_i)\,(1-F(x_i))^{K(R_i+1)-1} \\ (1-F(T_2))^{KR^{d_2}}, \quad \text{if } x_m>T_2, \end{cases} \tag{3}$$

where $C_m = N(N-R_1-1)(N-R_1-R_2-2)\cdots(N-R_1-\cdots-R_{m-1}-m+1)$, $R_{d_1} = N-d_1-\sum_{j=1}^{m-1} R_j$, $R_{d_2} = N-d_2-\prod_{j=1}^{d_2} R_j$, C_{d_1}, C_{d_2} can be written in a similar way taking $m=d_1, m=d_2$.

Therefore, the above cases can be combined as

$$\begin{aligned} L(a|x) =& C_D K^D \prod_{i=1}^{D} f(x_i)\ (1-F(x_i))^{K(R_i+1)-1} \\ & (1-F(T_1))^{KR_{d_1'}}\ (1-F(T_2))^{KR_{d_2'}} \end{aligned} \tag{4}$$

Here, $D = d_1, R_{d_1'} = R_{d_1}, R_{d_2'} = 0$ for case I, $D = m, R_{d_1'} = 0, R_{d_2'} = 0$, for case II and $D = d_2, R_{d_1'} = 0, R_{d_2'} = R_{d_2}$, for case III.

The remainder of this paper is organized as follows. Section II includes MLE, asymptotic confidence interval (ACI) and bootstrap confidence intervals of parameter. The Bayes estimation along with its highest posterior density (HPD) credible interval of the parameter are calculated in Section III. A dataset as an example to show that the LD is a suitable distribution to fit the data in Section IV. In Section V, Monte Carlo simulation experiment are presented. Section VI presents the conclusion.

II. Maximum likelihood estimation

Let $X_1, X_2, \cdots, X_D$, be a GFFPHC sample from LD, with a censoring scheme R. Based on Equation (4), the likelihood function is

$$\begin{aligned} L(a|\mathbf{x}) =& C_D k^D \prod_{i=1}^{D} a^2 (\frac{1}{1+a})^{k(R_i+1)} (1+x_i) e^{-k(R_i+1)x_i a} \\ & (1+a+ax_i)^{k(R_i+1)-1} (\frac{1+a+aT_1}{1+a})^{kR_{d_1'}} \\ & e^{-kR_{d_1'}T_1 a} (\frac{1+a+aT_2}{1+a})^{kR_{d_2'}} e^{-kR_{d_2'}T_2 a} \end{aligned} \tag{5}$$

From Equation (5), we obtain the log-likelihood function as follows:

$$\begin{aligned} l(a|x) =& \log(C_D K^D) + \sum_{i=1}^{D} \log(1+x_i) - a\sum_{i=1}^{D} K(R_i+1)x_i \\ & + 2D\log(a) + \sum_{i=1}^{D} (K(R_i+1)-1)\log(1+a+ax_i) \\ & - \sum_{i=1}^{D} K(R_i+1)\log(1+a) + KR_{d_1'}\log(\frac{1+a+aT_1}{1+a}) \\ & - aKR_{d_1'}T_1 + KR_{d_2'}\log(\frac{1+a+aT_2}{1+a}) - aKR_{d_2'}T_2. \end{aligned} \tag{6}$$

So the log-likelihood equation is

$$\begin{aligned} \frac{\partial l}{\partial a} =& \frac{2D}{a} - \sum_{i=1}^{D} K(R_i+1)x_i + \sum_{i=1}^{D} (K(R_i+1)-1)\frac{1+x_i}{1+a+ax_i} \\ & - \sum_{i=1}^{D} K\frac{(R_i+1)}{1+a} - KR_{d_1'}T_1 + KR_{d_1'}(\frac{1+T_1}{1+a+aT_1} - \frac{1}{1+a}) \\ & + KR_{d_2'}(\frac{1+T_2}{1+a+aT_2} - \frac{1}{1+a}) - KR_{d_2'}T_2 = 0. \end{aligned} \tag{7}$$

It is difficult to calculate the exact solution of (7). Therefore, we intend to evaluate the MLE by solving the likelihood equation by Newton-Raphson method.

A. Asymptotic confidence intervals estimation of paramater

The exact distribution of the parameter a's MLE cannot be determined. So, the asymptotic properties of MLE can be used to establish the ACI for a. A two-sided $100(1-\beta)\%$ confidence interval for a, which can be obtained by using the asymptotic normality distribution of MLE, is given by $\hat{a} \pm Z_{\beta/2}\sqrt{var(\hat{a})}$, where $Z_{\beta/2}$ is the upper $\frac{\beta}{2}$th percentile of standard normal distribution, and the asymptotic variance of $\hat{a}$ is $var(\hat{a}) = -I^{-1}(\hat{a})$. The $I(\hat{a})$ is the approximate Fisher information matrix, and defined as

$$I(\hat{a}) = \frac{\partial^2 l}{\partial a^2}|_{a=\hat{a}} \tag{8}$$

Using the Monte Carlo simulations obtains coverage probability (CP) of a

$$CP_a = P(|\tfrac{(\hat{a}-a)}{\sqrt{var(\hat{a})}}| \le Z_{\alpha/2})$$

The second derivatives in Equation (6) is given as follows

$$\begin{aligned} \frac{\partial^2 l}{\partial a^2} =& -\sum_{i=1}^{D} (K(R_i+1)-1)\frac{(1+x_i)^2}{(1+a+ax_i)^2} + \sum_{i=1}^{D} \frac{K(R_i+1)}{(1+a)^2} \\ & - \frac{2D}{a^2} - KR_{d_1'}[\frac{(1+T_1)^2}{(1+a+aT_1)^2} - \frac{1}{(1+a)^2}] \\ & - KR_{d_2'}[\frac{(1+T_2)^2}{(1+a+aT_2)^2} - \frac{1}{(1+a)^2}]. \end{aligned} \tag{9}$$

B. Bootstrap confidence intervals

Efron [10] introduced the percentile bootstrap (Boot-p) approach, and Hall [11] proposed the bootstrap-t method (Boot-t). Boot-p and Boot-t are the two forms of confidence intervals based on parametric bootstrap methods that are offered in this subsection. The following are the necessary steps for using the parametric bootstrap method:

Boot-p method

(1) Calculate $\hat{a}$ based on the GFFPHC sample by using Equation (7).

(2) For given $R = (R_1, R_2, \cdots, R_m), T_1$ and T_2 generate a bootstrap sample $(X_1, X_2, \cdots, X_D)$ from $f(x, \hat{a})$. Obtaining the bootstrap estimate of a saying $\hat{a}^*$, based on the bootstrap sample.

(3) Repeat (2) NP times.

(4) Assuming $\hat{a}^*$'s CDF is $G_1(x) = P(\hat{a}^* < x)$. For a fixed x value, $\hat{a}^*_{Btp}(x) = G_1^{-1}(x)$ is defined. The approximate $100(1-\beta)\%$ confidence interval of a is given by $(\hat{a}^*_{Btp}(\frac{\beta}{2}), \quad \hat{a}^*_{Btp}(1-\frac{\beta}{2}))$.

For the Bootstrap-t method, Steps (1) and (2) follow the boot-p approach that was previously mentioned.
(3) let $T^* = \frac{\hat{a}^* - \hat{a}}{\sqrt{var(\hat{a}^*)}}$.
(4) Repeat steps 2 and 3 NT times.
(5) Similar to Bootstrap-p method, assuming $\hat{T}^*$'s CDF is $G_2(y) = P(\hat{T}^* < y)$ For a fixed y value defines $\hat{a}^*_{Bt}(y) = \hat{a} + G_2^{-1}(y)\sqrt{var(\hat{a})}$. The two-sided $100(1-\beta)\%$ ACI of a is obtained by $(\hat{a}^*_{Bt}(\frac{\beta}{2}), \quad \hat{a}^*_{Bt}(1-\frac{\beta}{2}))$.

III. Bsyesian estimation

In this section, the Bayesian estimation (BE) of unknown parameter from LD is considered based on GFFPHC sample. Assume that a has a Gamma prior:

$$a \sim \pi(a) = \frac{b^\nu a^{\nu-1} e^{-ba}}{\Gamma(\nu)}; a > 0.$$

where ν and b are non-negative known constants. Thus, The posterior density for a be given as follows

$$\begin{aligned}\pi(a|x) \propto\ & a^{2D+\nu-1} e^{-(K\sum_{i=1}^{D}(R_i+1)x_i+b+KR_{d_1'}T_1+KR_{d_2'}T_2)a} \\ & (1+a)^{-K(\sum_{i=1}^{D}(R_i+1)+R_{d_1'}+R_{d_2'})} \\ & (1+a+aT_1)^{KR_{d_1'}}(1+a+aT_2)^{KR_{d_2'}} \\ & \prod_{i=1}^{D}(1+a+ax_i)^{K(R_i+1)-1} \end{aligned} \tag{10}$$

In this paper, the BE has been obtained under the squared error (SE), the Linex and the general entropy Loss function (LF). The symmetric SELF is widely employed in the Bayes inference. The parameter's BE is the posterior mean of parameter under the SELF. Hence, the BE of a's any function $\psi(a)$ under the SELF can be obtained by

$$\hat{\psi}_{BS}(a) = E(\psi(a)|x) = \frac{\int_0^\infty \psi(a)\pi(a|x)\mathrm{d}a}{\int_0^\infty \pi(a|x)\mathrm{d}a}. \tag{11}$$

In reliability analysis and life test, the nature of the LF is not always symmetric, so the SELF is unsuitable to use in some situations. In the asymmetric case, Varian [12] introduced the asymmetric Linex LF. The Linex LF is :

$$G(\hat{\tau},\tau) = e^{c(\hat{\tau}-\tau)} - c(\hat{\tau}-\tau) - 1, c \neq 0,$$

here τ's estimator is $\hat{\tau}$. The magnitude of c indicates the asymmetry degree, and the sign of c indicates the asymmetry direction. The BE of a $\psi(a)$ under the Linex LF, is given the following

$$\begin{aligned}\hat{\psi}_{BL}(\lambda) &= -\frac{1}{c}\log[E(e^{-c\psi(\lambda)}|x)] \\ &= -\frac{1}{c}\log[\frac{\int_0^\infty e^{-c\psi(\lambda)}\pi(\lambda|x)\mathrm{d}\lambda}{\int_0^\infty \pi(\lambda|x)\mathrm{d}\lambda}]. \end{aligned} \tag{12}$$

The Linex LF applies to situations where overestimation gives rise to consequences. The Linex LF is not appropriate to estimate scale parameters,see [13,14]. Therefore, a more applicable alterative to the modified Linex LF, called the general entropy LF, was proposed by [13].

The general entropy LF is written as

$$G(\hat{\mu},\mu) = (\frac{\hat{\mu}}{\mu})^q - \log(\frac{\hat{\mu}}{\mu})^q - 1, q \neq 0.$$

Based on the general entropy LF, the BE of a $\psi(\lambda)$ can be obtained by

$$\begin{aligned}\hat{\psi}_{BGE}(a) &= [E(\psi(a)^{-q}|x)]^{-\frac{1}{q}} \\ &= [\frac{\int_0^\infty \psi(a)^{-q}\pi(a|x)\mathrm{d}a}{\int_0^\infty \pi(a|x)\mathrm{d}a}]^{-\frac{1}{q}}. \end{aligned} \tag{13}$$

It is not possible to compute Equations (11), (12) and (13) analytically. We generate samples through Metropolis-Hastings (MH) algorithm from the Equations (10) and then compute the BE. Specific steps are as follows:
Step 1. Setting initial value a_0. Let $i = 1$.
Step 2. Generate a^* from $N(a^{i-1}, V_a)$.
Step 3. Compute the acceptance probability
ρ_a=min$[1, \pi(a^*|x)/\pi(a^{i-1}|x)]$.
Step 4. Generate random number u from uniform distribution $U(0,1)$.
Step 5. If $u < \rho_a$, $a_i = a^*$, else $a_i = a_{i-1}$. Let $i = i+1$.
Step 6. Repeat step 2–step 5 Z times. Obtain the BE of $\psi = \lambda$ based on three LFs as

$$\begin{aligned}\hat{\psi}_{BS} &= \frac{1}{Z-N}\sum_{j=N+1}^{Z}\psi_j, \\ \hat{\psi}_{BL} &= \log[\frac{1}{Z-N}\sum_{j=N+1}^{Z}e^{-c\psi_j}]^{-1/c}, \\ \hat{\psi}_{BGE} &= [\frac{1}{Z–N}\sum_{j=N+1}^{Z}\psi_j^{-q}]^{-\frac{1}{q}}, \end{aligned} \tag{14}$$

where N is called burn in period, the BE under three different LFs expressed as BS, BL and BGE respectively .

A. HPD credible interval of parameter

We establish the HPD credible interval of a using MH algorithm. The order values of $a_1, a_2, \cdots, a_M$ are denoted as $a_{(1)} < a_{(2)} <, \cdots, < a_{(M)}$. The 100(1-$\beta$)% HPD credible interval for a is constructed by using the algorithm given in [15], and denotes as $(a_{(i)}, a_{(i+[(1-\beta)M])})$, $[y]$ is the integer part of y.

IV. Data analysis

This section uses a dataset as an example for illustrative purposes. It represents the waiting time (minutes) listed in Table I for 100 bank customers. Ghitany et al. [2] demonstrated that the LD fit the data set very satisfactorily. Next, we divide the data set into 50 groups with 2 items in each group, and the following first failure censored sample (FFCS) is obtained: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.3, 4.4, 4.4, 4.7, 4.7, 4.9, 5.0, 5.3, 5.5, 5.7, 6.1, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.6, 7.7, 8.0, 8.6, 8.6, 8.9, 8.9, 9.5, 9.6, 9.7,11.0,11.2,11.9,13.6,18.1,19.9,21.9.

We obtained three GFFPHC samples based on the above FFCS from the following censoring schemes (CS):
Scheme 1: The complete data set where $K = 2, m = 50, R = (0, 0, \cdots, 0)$.
Scheme 2: The GFFPHC sample, $K = 2, m = 30, R = (10, 0, \cdots, 10)$.
Scheme 3: The GFFPHC sample, $K = 2, m = 30, R = (0, 0, \cdots, 10, \cdots, 10)$.

Scheme 4: The GFFPHC sample, $K = 2, m = 30, R = (0, 0, \cdots, 20)$.

The MLEs and BEs of the parameter are calculated under complete data and GFFPHC samples. For GFFPHC samples, we set $T_1 = 8, T_2 = 15$ and $T_1 = 10, T_2 = 19$. In the BE, we use non-informative priors ($a = b = 0$) because we have no any prior information about the parameter for real data set. All intervals mentioned in the paper for the parameter are obtained under different CSs. These estimates are listed in Tables II and III. From Table II, the results of Bayesian estimate based on the censoring scheme 2 are very close to those based on the complete sample, the difference between Bayesian estimates of based on censoring scheme 4 and based on complete sample are the largest, the Bayesian estimates based on censored samples are smaller than those based on complete sample.

TABLE I
AN DATASET REPORTED IN[12]

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5

TABLE II
THE MLES AND BE OF THE PARAMETER FOR TABLE 1

				Bayesian				
				SE	Linex		GE	
T_1	T_2	Sch	MLE	q=-1	c=-1	c=1	q=-2	q=1
0	∞	1	0.1866	0.1868	0.1868	0.1867	0.1872	0.1859
8	15	2	0.1876	0.1866	0.1869	0.1864	0.1880	0.1840
		3	0.1721	0.1721	0.1722	0.1719	0.1729	0.1703
		4	0.1374	0.1373	0.1374	0.1373	0.1379	0.1362
10	19	2	0.1827	0.1829	0.1831	0.1827	0.1839	0.1810
		3	0.1721	0.1718	0.1720	0.1717	0.1728	0.1699
		4	0.1374	0.1374	0.1375	0.1373	0.1380	0.1363

V. SIMULATIONS

In this section, we report the results of Monte Carlo simulations, these results compare the performance of the MLEs and BEs under different GFFPHC samples. The Monte Carlo simulations have been performed by using different values of N, m and K, and by choosing $a = 1, T_1 = 0.8, T_2 = 1.5$ in all cases. Three different censoring schemes are used and given as follows:

Scheme I: $R_1 = R_m = (n-m)/2, R_i = 0, (i \neq 1, i \neq m)$.
Scheme II: $R_{m/2} = R_m = (n-m)/2, R_i = 0, (i \neq \frac{m}{2}, i \neq m)$.
Scheme III: $R_m = n - m, R_i = 0, (i \neq m)$.

We obtain BEs under the BS, BL and BGE. In the BE, we have chosen the hyper-parameters by moment method, that is the prior distribution means are exactly equal to the true value of corresponding parameters ($\nu = b = 1$). Computing the BE and 95% HPD credible intervals using the MH technique, based on 5000 MCMC samples and discard the first 1000 values as ”burn-in”. For comparision purpose, we calculate MLEs and the 95% ACIs, and calculate two 95% Boot confidence intervals using 1000 re-sampling. We simulate the whole process 1000 times and calculate biases and mean squares errors (MSEs) of different estimates. we get 95% the average of interval lengths and the CPs of the parameter by numerical simulation test. In the BE, the CPs and average credible lengths are independent of the LFs. Tables IV-V list the results of numerical simulation. From Tables IV-V, we can draw the following conclusions:

(1) From the results of the report, we observe that the biases and MSEs of the parameter decrease as the values of N and m increase, and the average lengths of the ACI/HPD credible intervals decrease when the values of N and m increase. Also, the biases and MSEs for all estimates based on GFFPHC scheme with $K = 5$ are smaller than those for GFPHC scheme with $K = 3$ in most cases. As the K increases, the average length of interval estimations narrows down.

(2) In simulations experiment, the MLEs outperform the BEs in respect of the biases. For point estimation, the change bias and MSEs of MLEs and BEs for a under three censoring schemes are no significantly different. Tables IV-V show the BEs of the parameter is sensitive to the values of c and q based on the asymmetric LSs.

(3) We can observe that CPs for a are always close to the level of 95% in interval estimation. Boot-t confidence intervals are be superior to Boot-p confidence intervals and ACI in term of average interval lengths. For three censoring schemes, we can easily notice that scheme III gives the smallest the average interval, and the scheme I gives the greatest the interval lengths. Among all the interval estimates, the performances of the HPD credible intervals is optimal.

VI. CONCLUSIONS

In this paper, we design a new life test scheme called GFPHCS. The classical estimates and BEs the unknown parameter of LD have been obtained based on the GFFPHCS. We compute BE of the unknown parameter under square error, Linex and general entropy LFs. The MLEs cannot be obtained in analytical expression form, but can be derived by the numerical method. The parameter's ACI and the corresponding CPs are derived by using the observed Fisher's information matrix. We construct two bootstrap confidence intervals, and the corresponding CPs are computed. Also, BEs and the corresponding HPD credible interval for the parameter a are computed using Metropolis-Hastings method. From our study, we find that MLE for a is better in term of biases. And the MLE calculation is simple, so we recommend ML method for a. For interval estimate of a, HPD credible interval is recommended.

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TABLE III
THE CORRESPONDING 95% INTERVAL ESTIMATIONS OF a FOR TABLE 1

T_1	T_2	CS	MLE	Boot-p	Boot-t	HPD
0	∞	1	(0.1605, 0.2126)	(0.1625, 0.2170)	(0.1589, 0.2127)	(0.1630, 0.2123)
8	15	2	(0.1446, 0.2306)	(0.1763, 0.2935)	(0.1757, 0.2521)	(0.1434, 0.2353)
		3	(0.1379, 0.2062)	(0.1360, 0.1869)	(0.1293, 0.1845)	(0.2130, 0.1357)
		4	(0.1128, 0.1619)	(0.1087, 0.1456)	(0.1038, 0.1438)	(0.1088, 0.1686)
10	19	2	(0.1460, 0.2194)	(0.1293, 0.1853)	(0.1079, 0.1852)	(0.1433, 0.2280)
		3	(0.1379, 0.2062)	(0.1238, 0.1832)	(0.1063, 0.1832)	(0.1340, 0.2116)
		4	(0.1128, 0.1619)	(0.09752, 0.1367)	(0.08126, 0.1367)	(0.1085, 0.1681)

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TABLE IV
BIASES AND MSEs (IN THE PARENTHESE) OF a'S BE FOR DIFFERENT LF

					SE	Linex		GE	
K	N	m	Sch	MLE	q=-1	c=-1	c=1	q=-2	q=1
3	30	20	I	0.026938	0.026969	0.039936	0.014450	0.039015	0.002776
				(0.02568)	(0.02451)	(0.02666)	(0.02281)	(0.02589)	(0.02264)
			II	0.022749	0.022816	0.035515	0.010555	0.034648	-0.000932
				(0.02698)	(0.02574)	(0.02782)	(0.0241)	(0.02703)	(0.02403)
			III	0.023509	0.023546	0.034841	0.012591	0.034189	0.002184
				(0.02334)	(0.02241)	(0.02394)	(0.0212)	(0.02337)	(0.02118)
	40	30	I	0.022459	0.022603	0.031390	0.014026	0.030890	0.005985
				(0.01899)	(0.01842)	(0.01959)	(0.01744)	(0.01921)	(0.01726)
			II	0.021355	0.021345	0.030027	0.012867	0.029556	0.004875
				(0.01743)	(0.01692)	(0.018)	(0.01604)	(0.01766)	(0.01587)
			III	0.0147866	0.0149056	0.0230814	0.0069102	0.0227341	-0.0007948
				(0.01562)	(0.01521)	(0.016)	(0.01458)	(0.01572)	(0.01456)
	50	30	I	0.0159875	0.0160960	0.0242745	0.0080975	0.0239204	0.0004028
				(0.01546)	(0.01504)	(0.01584)	(0.0144)	(0.01556)	(0.01436)
			II	0.017567	0.017598	0.025581	0.009789	0.025233	0.002291
				(0.01494)	(0.01454)	(0.01532)	(0.01506)	(0.01506)	(0.01385)
			III	0.009553	0.009837	0.016473	0.003320	0.016273	-0.003066
				(0.01374)	(0.0134)	(0.01387)	(0.01303)	(0.01368)	(0.0131)
		40	I	0.013496	0.013877	0.020373	0.007494	0.020112	0.001375
				(0.0135)	(0.01322)	(0.01382)	(0.01272)	(0.01363)	(0.01264)
			II	0.015764	0.015936	0.022426	0.009562	0.022150	0.003481
				(0.01379)	(0.01349)	(0.01413)	(0.01296)	(0.01393)	(0.01286)
			III	0.0119690	0.0120232	0.0183151	0.0058373	0.0180886	-0.0001386
				(0.01274)	(0.01244)	(0.01296)	(0.01201)	(0.01278)	(0.01197)
	60	40	I	0.013137	0.013194	0.019504	0.006992	0.019272	0.001015
				(0.01248)	(0.01222)	(0.01275)	(0.01179)	(0.01258)	(0.01174)
			II	0.014591	0.014741	0.020975	0.008613	0.020740	0.002720
				(0.01235)	(0.01211)	(0.01264)	(0.01168)	(0.01248)	(0.01161)
			III	0.003474	0.003772	0.009218	-0.001593	0.009104	-0.006913
				(0.01079)	(0.01059)	(0.01086)	(0.01038)	(0.01074)	(0.01047)
		50	I	0.008461	0.008714	0.013864	0.003638	0.013707	-0.001288
				(0.01122)	(0.01104)	(0.01141)	(0.01073)	(0.01128)	(0.01071)
			II	0.007297	0.007326	0.012414	0.002308	0.012272	-0.002585
				(0.01018)	(0.01001)	(0.01034)	(0.009737)	(0.01022)	(0.009726)
			III	0.0096056	0.0098143	0.0148723	0.0048263	0.0147183	-0.0000106
				(0.01017)	(0.01003)	(0.01038)	(0.009744)	(0.01027)	(0.009705)
5	30	20	I	0.0004591	0.0008665	0.0116306	-0.00958	0.0112243	-0.0199264
				(0.01804)	(0.01739)	(0.01831)	(0.01675)	(0.01788)	(0.01708)
			II	-0.00428	-0.004047	0.006447	-0.014237	0.006102	-0.024419
				(0.01763)	(0.01708)	(0.01787)	(0.01656)	(0.01746)	(0.01695)
			III	0.006695	0.007059	0.016155	-0.001814	0.015804	-0.010493
				(0.01694)	(0.01643)	(0.01724)	(0.01581)	(0.0169)	(0.01594)
	40	30	I	0.0149385	0.0150058	0.022894	0.0072861	0.02254	-0.0001079
				(0.01647)	(0.01603)	(0.01686)	(0.01535)	(0.01656)	(0.0153)
			II	0.009850	0.009853	0.017561	0.002305	0.017268	-0.005022
				(0.01458)	(0.01417)	(0.01485)	(0.01364)	(0.0146)	(0.01366)
			III	0.0003546	0.0008566	0.0075864	-0.00575	0.0074245	-0.0123113
				(0.01189)	(0.01165)	(0.01202)	(0.01139)	(0.01185)	(0.01152)
	50	30	I	0.010050	0.010403	0.017273	0.003661	0.017033	-0.002891
				(0.01379)	(0.01348)	(0.01404)	(0.01304)	(0.01383)	(0.01305)
			II	0.0133692	0.0133359	0.0201137	0.0066832	0.0198529	0.0002686
				(0.01375)	(0.01349)	(0.0141)	(0.01301)	(0.01389)	(0.01297)
			III	0.006607	0.006676	0.012114	0.001317	0.011969	-0.003934
				(0.0113)	(0.01111)	(0.01145)	(0.01083)	(0.01132)	(0.01086)
		40	I	0.0116822	0.0117486	0.0177941	0.0058050	0.0175916	0.0000448
				(0.01091)	(0.01067)	(0.01112)	(0.01031)	(0.01098)	(0.01028)
			II	0.010371	0.010567	0.016536	0.004695	0.016339	-0.001003
				(0.01201)	(0.01173)	(0.01218)	(0.01137)	(0.01203)	(0.01134)
			III	0.002267	0.002474	0.007853	-0.002827	0.007736	-0.008074
				(0.01053)	(0.01035)	(0.01062)	(0.01014)	(0.0105)	(0.0102)
	60	40	I	0.002266	0.002769	0.008154	-0.002535	0.008032	-0.007772
				(0.01076)	(0.01059)	(0.01087)	(0.01037)	(0.01075)	(0.01043)
			II	0.006696	0.006850	0.012178	0.001600	0.012038	-0.003546
				(0.01028)	(0.0101)	(0.01041)	(0.009844)	(0.0103)	(0.009858)
			III	0.007411	0.007680	0.012202	0.003214	00.012093	-0.001159
				(0.009227)	(0.009087)	(0.009336)	(0.008884)	(0.009249)	(0.008882)
		50	I	0.0095361	0.0097822	0.014927	0.0047103	0.014769	-0.0002073
				(0.01066)	(0.01047)	(0.01083)	(0.01016)	(0.01071)	(0.01013)
			II	0.007687	0.007674	0.012512	0.002900	0.012387	-0.001766
				(0.00898)	(0.00885)	(0.009126)	(0.008626)	(0.009035)	(0.008614)
			III	0.0082063	0.0083094	0.0128412	0.0038332	0.0127279	-0.0005438
				(0.009275)	(0.009178)	(0.00944)	(0.008963)	(0.009349)	(0.009349)

TABLE V
THE ALS AND CPS (IN THE PARENTHESE) OF a FOR DIFFERENT METHODS

K	N	m	Sch	ACI	Boot-p	Boop-t	HPD
3	30	20	I	0.6267 (0.959)	0.6332 (0.947)	0.615 (0.956)	0.6093 (0.962)
			II	0.6188 (0.95)	0.6229 (0.945)	0.605 (0.941)	0.6033 (0.949)
			III	0.5871 (0.95)	0.592 (0.941)	0.5829 (0.944)	0.572 (0.949)
	40	30	I	0.5157 (0.949)	0.5303 (0.937)	0.5119 (0.952)	0.5056 (0.947)
			II	0.5129 (0.965)	0.5256 (0.956)	0.5082 (0.955)	0.5029 (0.962)
			III	0.4992 (0.965)	0.5005 (0.952)	0.4945 (0.961)	0.4895 (0.958)
	50	30	I	0.4993 (0.96)	0.5008 (0.959)	0.4952 (0.954)	0.4899 (0.959)
			II	0.4935 (0.959)	0.4959 (0.962)	0.4889 (0.961)	0.4843 (0.954)
			III	0.4505 (0.943)	0.4572 (0.948)	0.4515 (0.941)	0.4431 (0.951)
		40	I	0.444 (0.953)	0.4572 (0.945)	0.4422 (0.949)	0.4357 (0.948)
			II	0.4435 (0.95)	0.4573 (0.941)	0.4422 (0.944)	0.4351 (0.948)
			III	0.4382 (0.943)	0.4408 (0.947)	0.4345 (0.938)	0.4296 (0.944)
	60	40	I	0.4385 (0.945)	0.4409 (0.946)	0.4345 (0.938)	0.4313 (0.946)
			II	0.4357 (0.949)	0.4379 (0.949)	0.4323 (0.948)	0.4287 (0.942)
			III	0.4082 (0.945)	0.4093 (0.955)	0.4063 (0.943)	0.4023 (0.949)
		50	I	0.3957 (0.944)	0.407 (0.94)	0.3949 (0.937)	0.3893 (0.946)
			II	0.394 (0.957)	0.4051 (0.948)	0.3931 (0.944)	0.3875 (0.953)
			III	0.3931 (0.95)	0.3989 (0.944)	0.3917 (0.947)	0.3861 (0.947)
5	30	20	I	0.5716 (0.969)	0.5348 (0.984)	0.5418 (0.951)	0.5569 (0.971)
			II	0.5651 (0.959)	0.5279 (0.98)	0.5357 (0.95)	0.5496 (0.963)
			III	0.5261 (0.959)	0.5093 (0.976)	0.5157 (0.957)	0.5137 (0.956)
	40	30	I	0.4899 (0.955)	0.4722 (0.961)	0.4718 (0.959)	0.4788 (0.949)
			II	0.4843 (0.958)	0.4673 (0.963)	0.4662 (0.962)	0.4735 (0.958)
			III	0.4526 (0.966)	0.4491 (0.978)	0.4453 (0.962)	0.4439 (0.964)
	50	30	I	0.4571 (0.943)	0.4519 (0.962)	0.4503 (0.952)	0.4485 (0.941)
			II	0.4546 (0.951)	0.4473 (0.972)	0.447 (0.96)	0.4451 (0.952)
			III	0.4074 (0.949)	0.4112 (0.964)	0.4075 (0.952)	0.4001 (0.951)
		40	I	0.4294 (0.969)	0.4235 (0.966)	0.4189 (0.968)	0.4213 (0.967)
			II	0.4268 (0.96)	0.4196 (0.952)	0.4155 (0.965)	0.4184 (0.956)
			III	0.4057 (0.955)	0.4055 (0.956)	0.4006 (0.953)	0.398 (0.952)
	60	40	I	0.405 (0.946)	0.4065 (0.955)	0.955 (0.96)	0.3983 (0.944)
			II	0.4037 (0.954)	0.4037 (0.957)	0.399 (0.955)	0.3966 (0.949)
			III	0.3724 (0.951)	0.3768 (0.952)	0.3721 (0.954)	0.3655 (0.947)
		50	I	0.3962 (0.954)	0.4078 (0.947)	0.3957 (0.949)	0.3903 (0.949)
			II	0.3843 (0.965)	0.3828 (0.963)	0.3782 (0.96)	0.3779 (0.961)
			III	0.3723 (0.949)	0.3734 (0.953)	0.3688 (0.956)	0.3659 (0.946)