A Special Fractional Choosability of 1-planar Graphs Without 4- and 5-cycles

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***Abstract*—For a graph G given three integers $a \geq b \geq i \geq 0$ and a certain vertex subset S of $V(G)$, we say G is i-weak $(a : b)$-choosable about set S if for any list assignment L of G with the size of each list is a there is a function so that each vertex v of S is given i colors from $L(v)$ while each vertex u of $V(G) \setminus S$ is given b colors from $L(v)$ satisfying that each pair adjacent vertices have disjoint colors. In this paper, for $S = \{v : v \in V(G), d(v) = 4\}$, we prove that every 1-planar graph without 4- and 5-cycles is i-weak $(4b + i : b)$-choosable about set S. And we prove that every 1-planar graph without 4-vertices, 4- and 5-cycles is 4-choosable.**

***Index Terms*—1-planar graph, cycle, fractional choosability, discharging.**

I. Introduction

All graphs are finite and simple in this paper. The undefined but used notations could be found in [1].

A set coloring is a function that assigns each vertex of G a color set so that none of adjacent vertices have common colors. A function L is called a list assignment for a graph $G = (V(G), E(G))$ if each vertex v of $V(G)$ is assigned a color set $L(v)$. If there is a function which is a special set coloring that assigns a b-element subset of $L(v)$ to each vertex v of G, then G admits a $(L : b)$-coloring and we say G is $(L : b)$-colorable.

If G is $(L : b)$-colorable for any list assignment $L = \{L(v) : |L(v)| = a, v \in V(G)\}$, then G is $(a : b)$-choosable. G is $(a : b)$-colorable if G has a $(L : b)$-coloring provided that $L(v) = \{1, 2, \ldots, a\}$ for each $v \in V(G)$. It is easy to find that the $(a : b)$-choosability is more difficult to reach than the $(a : b)$-colorability. The minimum fraction $\frac{a}{b}$ such that G is $(a : b)$-colorable is called the fractional chromatic number of G denoted by $\chi_f(G)$. While G is $(a : 1)$-choosable or $(a : 1)$-colorable, it means that G is proper vertex a-choosable or a-colorable. Denote the minimum number a that G is proper vertex a-colorable by $\chi(G)$. Obviously, $\chi_f(G)$ is no more than $\chi(G)$ for any graph G. G is $(c_1, c_2, \ldots, c_k)$-colorable while there is a function $V(G) \to [k]$ such that $\Delta(G[V_i]) \leq c_i$, where $G[V_i]$ denotes the monochromatic induced subgraph of color i.

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A graph which can be drawn in the plane with no two edges intersect is a planar graph. We say a graph is a 1-planar graph if it has a planar drawing so that each of its edges is crossed at most once besides its ends. Obviously, 1-planar graph is a generalization of planar graph. Such a planar drawing is called a plane graph or 1-plane graph respectively.

Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable in 1976 (see [2]). Though this conjecture is proved false by Addad et al. [3], there are many coloring results about planar graphs motivated by it. Borodin et al. proved that every planar graph without 4-,. . .,7-cycles is 3-colorable in [4]. But whether any planar graphs without 4-, 5- and 6- cycles are 3-colorable is still unknown. Borodin et al. showed that every planar graph without 5- and 7-cycles and without adjacent triangles is 3-colorable (see [5]). Hill et al. [6] showed that every planar graph without 4- and 5-cycles is (3,0,0)-colorable, while under the same condition it is also (2,0,0)-colorable in [7] and is (1,1,0)-colorable in [8] by Chen et al. and Wu et al. respectively.

The Steinberg's conjecture also promoted the following results about fractional coloring. Wu et al. proved that every planar graph without 4- and 6-cycles is $(7 : 2)$-colorable in [9]. Dvořák and Hu [10] showed that every planar graph without 4- and 5-cycles is $(11 : 3)$-colorable in 2019.

Different from the Four-Color Theorem, every 1-planar graph is 6-colorable (see [11]). In [12], [13], Song et al. respectively showed that every 1-planar graph is 5-colorable under the condition of without 4- and 5-cycles or without 4-cycles and adjacent 5-vertices. Chu et al. proved that every 1-planar graph with girth at least 7 is (1,1,1,0)-colorable in [14] and also (2,0,0,0)-colorable in [15]. The coloring problems of 1-planar graphs have been widely studied recently, such as equitable coloring in [16], [17] , k-(2,1)-total choosability in [18] and total coloring in [19], [20].

Let us consider a problem now. Assume we are conference sponsors who need to arrange the scheduled meetings. There are n listeners but some of them cannot attend the same meeting. Listeners who belongs to set M only need to join i meetings while the others want to take part in at least b meetings with $b \geq i$. How many meetings do we need to hold? To solve this problem, we construct a graph G with n vertices representing the listeners. Two vertices are adjacent if their corresponding listeners could not attend common meetings. Let S denote the set of vertices whose counterpart is in M. So the problem is turned to find the smallest integer a so that G is i-weak $(a : b)$-colorable about set S which is defined as following.

Based on such a problem, we defined a new coloring now. Given three integers $a \geq b \geq i \geq 0$ and a certain vertex subset S of $V(G)$, G is i-weak $(a : b)$-choosable about set S if for any list assignment L of G with the size of each list is a there is a function so that each vertex v of S is given i colors from $L(v)$ while each vertex u of $V(G)\backslash S$ is given b colors from $L(v)$ satisfying that each pair adjacent vertices have disjoint colors. For a graph G and a list assignment $L = \{L(v) = \{1, 2, \ldots, a\}, v \in V(G)\}$, if G admits a function which satisfies the condition defined above about set S, then we say this function is a i-weak $(a : b)$-coloring about set S of G and G is i-weak $(a : b)$-colorable about set S.

Based on the definitions above, we mainly prove the following result.

Theorem 1.1 For $S = \{v : v \in V(G), d(v) = 4\}$, every 1-planar graph without 4- and 5-cycles is i-weak $(4b+i : b)$-choosable about set S.

Corollary 1.2 and Corollary 1.3 are special cases of Theorem 1.1.

Corollary 1.2 For $S = \{v : v \in V(G), d(v) = 4\}$, every 1-planar graph without 4- and 5-cycles is 1-weak $(4b+1 : b)$-choosable about set S.

Corollary 1.3 For $S = \{v : v \in V(G), d(v) = 4\}$, every 1-planar graph without 4- and 5-cycles is 1-weak $(9 : 2)$-choosable about set S.

The following Theorem 1.4 is a result in the case that S is empty under the condition of Theorem 1.1.

Theorem 1.4 Every 1-planar graph without 4-vertices, 4- and 5-cycles is 4-choosable.

II. Preliminaries

We say vertex x is a k-vertex or k^+-vertex if its degree is k or at least k respectively. These definitions are also applied to faces. Making any crossing of the 1-plane graph G to a 4-vertex leads to the associated plane graph $G^\times$ of G. We define the vertices of $G^\times$ which do not appear in G as cross vertices, other vertices are called true vertices. The white dots and black dots shown in the following figures are respected the cross vertices and true vertices in $G^\times$ respectively.

A 4-face in $G^\times$ is ordinary or special if it is incident with exactly one or two cross vertices respectively. Let $m_3(v)$ (resp. $m_s(v)$) denote the number of 3-faces (resp. special 4-faces) of $G^\times$ incident with v for any $v \in G^\times$. The existence of 4- and 5-cycles is called the impossible event.

By the definition of 1-planar graph, we can directly find the following claim.

Claim 2.1 ([21]) There are not two adjacent cross vertices in $G^\times$.

Since the absence of 4- and 5-cycles in G, here comes the following claim.

Claim 2.2 The two subgraphs shown in Figure 1 could not appear in $G^\times$.

(1) (2)

Figure 1 Two subgraphs could not exist in $G^\times$

III. Proof of Theorem 1.1

We prove Theorem 1.1 by contradiction. Let G be a minimal counterexample of Theorem 1.1 which means that there is no other counterexample with vertices less than G. Then G has neither of 4-, 5-cycles and G is not i-weak $(4b+i : b)$-choosable about set S. The following discussions about G in this part and the next part are all about the one we just assumed.

Lemma 3.1 The minimum degree of G is at least 5.

Proof: Suppose not, G contains a vertex v with degree no more than 4. By minimality, $G' = G - \{v\}$ is i-weak $(4b + i : b)$-choosable about set S. For any $L = \{L(v) : |L(v)| = 4b + i, v \in V(G)\}$, there is a i-weak $(4b + i : b)$-coloring φ_L about set S in G. Let us discuss in two cases.

If v has less than 4 neighbors, then v still has $|L(v) \setminus \bigcup_{w\in N_{G^\times}(v)} \varphi_L(w)| \geq 4b + i - 3b = b + i \geq b$ colors now. Otherwise v belongs to S, v remains $|L(u) \setminus \bigcup_{w\in N_{G^\times}(u)} \varphi_L(w)| \geq 4b+i-4b = i$ colors which could be used. In either case, we can extend φ_L to G, a contradiction. ■

We are going to use the discharging method on $G^\times$ to prove that no counterexample of Theorem 1.1 exists. Let us set the initial charges of $G^\times$ by $ch_0(v) = \frac{1}{2}d(v) - 3$ for each vertex v of $V(G^\times)$, and $ch_0(f) = d(f) - 3$ for any face f which belongs to $F(G^\times)$.

By Euler's formula we have $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$, thus the following equation is established.

$$\sum_{v\in V(G^\times)} (\frac{1}{2}d(v) - 3) + \sum_{f\in F(G^\times)} (d(f) - 3) = -6$$

The symbol $ch(x)$ denotes the final charge of any $x \in V(G^\times) \cup F(G^\times)$ after the following discharging rules below. Note that the discharging rules could not change the sum of the charges of $V(G^\times) \cup F(G^\times)$, thus the sum of the final charges should still be –6.

$$\sum_{x\in V(G^\times)\cup F(G^\times)} ch_0(x) = \sum_{x\in V(G^\times)\cup F(G^\times)} ch(x) = -6$$

The discharging rules are given as follows.

$R1$. Each 4^+-face f of $G^\times$ sends to each incident cross vertex v:

(a) 1/4 for $m_3(v)$ = 0.
(b) 1/3 for $m_3(v)$ = 1.
(c) 1/2 for $m_3(v)$ = 2.

$R2$. Each 4^+-face f of $G^\times$ sends to each incident 5-vertex v:

(a) 1/4 if f is an ordinary 4-face.
(b) 1/8 if f is a special 4-face.
(c) 3/8 if f is a 5-face.
(d) 1/2 if f is a 6^+-face.

After setting the discharging rules, we are going to check the final charges of $V(G^\times) \cup F(G^\times)$. Check the final charges of vertices in $G^\times$ firstly.

It is obvious that the minimum degree of $G^\times$ is at least 4 by Lemma 3.1. The following three lemmas show that $ch(v) \geq 0$ for any v of $V(G^\times)$. The upper bounds of the number of 3-faces and special 4-faces incident with v are based on Claim 2.1, Claim 2.2 and the absence of 4- and 5-cycles in G.

Lemma 3.2 If v is a 4-vertex of $G^\times$, then $ch(v) \geq 0$.

Proof: Note that $ch_0(v) = -1$ and $m_3(v) \leq 2$. Classify v by the number of 3-faces incident with it.

(1) If $m_3(v) = 0$, then $ch(v) = -1 + 4 \times 1/4 = 0$ by $R1(a)$.

(2) If $m_3(v) = 1$, then $ch(v) = -1 + 3 \times 1/3 = 0$ by $R1(b)$.

(3) If $m_3(v) = 2$, then $ch(v) = -1 + 2 \times 1/2 = 0$ by $R1(c)$. ■

Lemma 3.3 If v is a 5-vertex of $G^{\times}$, then $ch(v) \geq 0$.

Proof: Note that $ch_0(v) = -1/2$ and $m_3(v) \leq 4$.

(1) If $m_3(v) = 0$, then $ch(v) \geq -1/2 + 5 \times 1/8 = 1/8 > 0$ by $R2$.

(2) If $m_3(v) = 1$, then $m_s(v) \leq 3$ and $ch(v) \geq -1/2 + 1/4 + 3 \times 1/8 = 1/8 > 0$ by $R2$.

(3) If $m_3(v) = 2$, then $m_s(v) \leq 2$ and $ch(v) \geq -1/2 + 1/4 + 2 \times 1/8 = 0$ by $R2$.

(4) If $m_3(v) = 3$, then $m_s(v) \leq 1$.

Case 1: v is not incident with any special 4-faces.

$ch(v) \geq -1/2 + 2 \times 1/4 = 0$ by $R2$.

Case 2: v is incident with exactly one special 4-face f. Then we discuss the condition of the remaining unknown face f' incident with v and is neither a 3-face nor f. Since $m_3(v) = 3$ and $m_s(v) = 1$, then f' is either a 5^+-face or an ordinary 4-face.

Case 2.1: f' is a 5^+-face, then $ch(v) \geq -1/2 + 1/8 + 3/8 = 0$ by $R2(a)$, $R2(c)$ and $R2(d)$.

Case 2.2: f' is an ordinary 4-face. Note that v is incident with f, f' and three 3-faces. Considering the two faces f_1, f_2 adjacent to f and incident with v, f_1 and f_2 are either two 3-faces or one is 3-face and another one is f'. Suppose f_1, f and f_2 are counterclockwise in the plane (see Figure 2(1)).

Case 2.2.1: Both f_1 and f_2 are 3-faces. The Figure 2(2) shows that G has at least one 4-cycles.

Case 2.2.2: By symmetry, suppose f_1 is a 3-face, f_2 is f'. As is shown in the Figure 2(3), one 5-face appears in G.

Both the cases above contradict the fact that the impossible event could not occur. So Case 2.2 could not happen. Thus $ch(v) \geq 0$ for any 5-vertex with $m_3(v) = 3$.

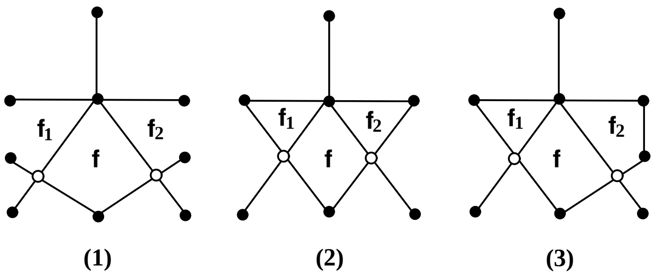

Figure 2 $d(v) = 5$, $m_3(v) = 3$, $m_s(v) = 1$, f is an ordinary 4-face

In fact, if $m_3(v) = 3$ and $m_s(v) = 1$, the Figure 3(1) shows the only possibility. If f' is a 3- or 4-face shown in the Figure 3(2-3), here comes a 4-face in G. Thus f' must be a 5^+-face.

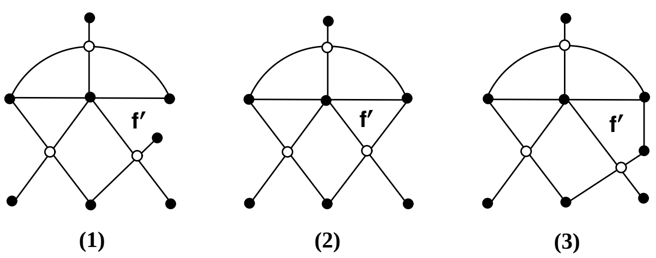

Figure 3 $d(v) = 5$, $m_3(v) = 3$, $m_s(v) = 1$

(5) If $m_3(v) = 4$, the only case is shown in the following Figure 4(1). Consider the condition of the face f incident with v but is not a 3-face. Since the absence of 4-cycles in G, Figure 4(2-3) could not happen, then f must be a 6^+-face. Thus $ch(v) = -1/2 + 1/2 = 0$ by R2(d). ■

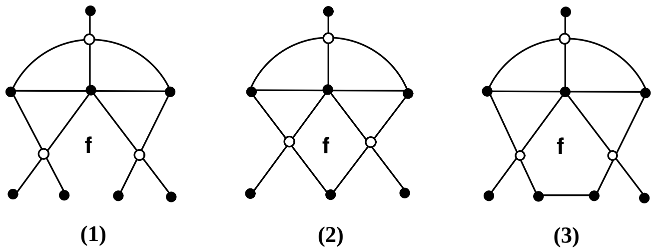

Figure 4 $d(v) = 5$, $m_3(v) = 4$

Lemma 3.4 If v is a 6^+-vertex of $G^{\times}$, then $ch(v) \geq 0$.

Proof: There is no discharging rule about v, so $ch(v) = ch_0(v) = \frac{1}{2}d(v) - 3 = \frac{1}{2}(d(v) - 6) \geq 0$. ■

The following three lemmas show that $ch(f) \geq 0$ for any $f \in F(G^{\times})$.

Lemma 3.5 If f is a cross 4-face of $G^{\times}$, then $ch(f) \geq 0$.

Proof: Note that $ch_0(f) = 1$. According to Claim 2.1, f is incident with either one or two cross vertices which means that f is either ordinary or special.

(1) f is an ordinary 4-face, then the only cross vertex incident with f is not incident with any 3-faces. Otherwise, as illustrated by the following Figure 5(2-3), the impossible event happens, a contradiction.

Thus $ch(f) \geq 1 - 1/4 - 3 \times 1/4 = 0$ by $R1(a)$ and $R2(a)$.

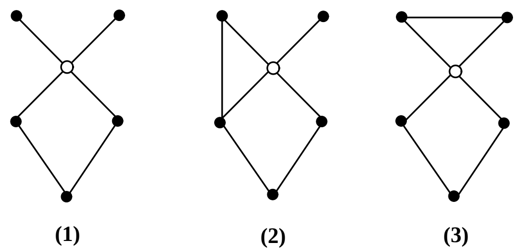

Figure 5 $d(f) = 4$, f has only one cross vertex

(2) f is a special 4-face. Suppose v and u are the two cross vertices incident with f. Note that each of v and u is at most incident with two 3-faces. So we discuss the number of 3-faces incident with the cross vertices v and u.

Case 1: Both of v and u are incident with at most one 3-face. Then $ch(f) \geq 1 - 2 \times 1/3 - 2 \times 1/8 = 1/12 > 0$ by $R1(a)$, $R1(b)$ and $R2(b)$.

Case 2: One of v and u is incident with two 3-faces. In general, suppose v is such a vertex. Then u could not incident with a 3-face as the Figure 6(2-4) show below. Otherwise no matter how the 3-face incident with u appears, the impossible event happens, a contradiction.

Then $ch(f) \geq 1 - 1/4 - 1/2 - 2 \times 1/8 = 0$ by $R1(a)$, $R1(c)$ and $R2(b)$. ■

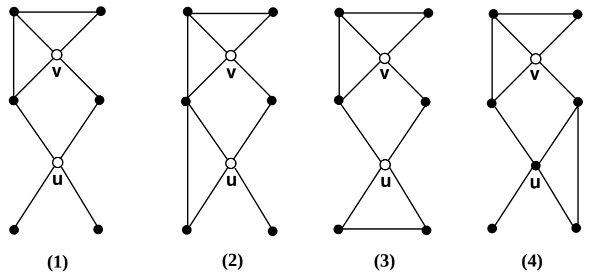

Figure 6 $d(f) = 4$, $m_3(v) = 2$

Lemma 3.6 If f is a 5-face of $G^{\times}$, then $ch(f) \geq 0$.

Proof: Note that $ch_0(f) = 2$. According to Claim 2.1, there are less than three cross vertices incident with f.

(1) There is exactly one cross vertex v incident with f shown in Figure 7(1). In this case, $m_3(v) \leq 1$. If not, Figure 7(2) shows that a 5-cycle will occur in G. $ch(f) \geq 2 - 1/3 - 4 \times 3/8 = 1/6 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$.

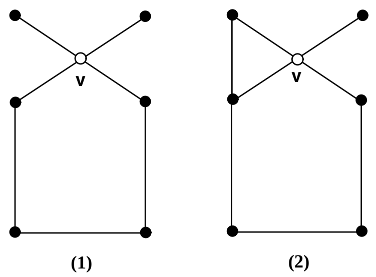

Figure 7 $d(f) = 5$, f has exactly one cross vertex v

(2) There are exactly two cross vertices v and u incident with f.

Case 1: $m_3(v) \leq 1$ and $m_3(u) \leq 1$. $ch(f) \geq 2 - 2 \times 1/3 - 3 \times 3/8 = 5/24 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$.

Case 2: Generally suppose v is incident with two 3-faces, then $m_3(u) \leq 1$. Otherwise, G will contain a 5-cycle, as shown in Figures 8(2-3) and 8(5-6). Thus $ch(f) \geq 2 - 1/3 - 1/2 - 3 \times 3/8 = 1/24 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$. ■

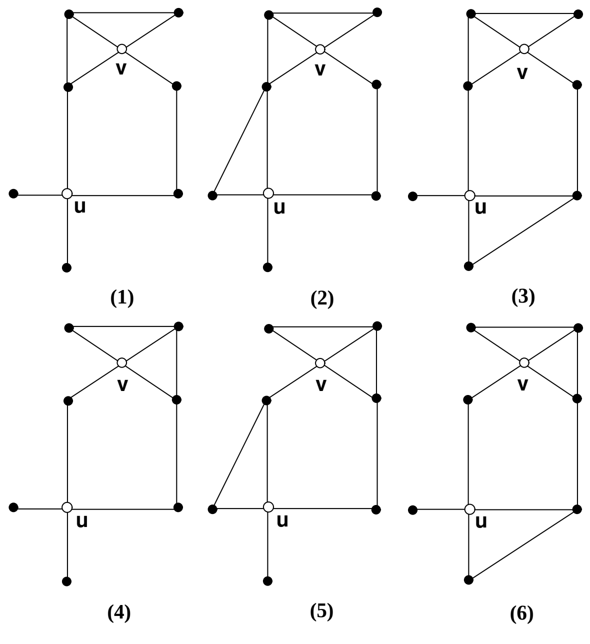

Figure 8 $d(f) = 5$, $m_3(v) = 2$

Lemma 3.7 If f is a 6^{+}-face of $G^{\times}$, then $ch(f) \geq 0$.

Proof: By the discharging rules, f sends at most 1/2 charges to each of its incident vertices. Thus $ch(f) \geq ch_0(f) - \frac{1}{2}d(f) = d(f) - 3 - \frac{1}{2}d(f) = \frac{1}{2}(d(f) - 6) \geq 0$. ■

According to Lemma 3.2–Lemma 3.7, every x of $V(G^{\times}) \cup F(G^{\times})$ satisfies $ch(x) \geq 0$. Thus the final sum of the charges of $V(G^{\times}) \cup F(G^{\times})$ is non-negative which contradicts the fact that it should be -6. Thus there is no counterexample G existing which makes the Theorem 1.1 valid.

IV. Proof of Theorem 1.4

We prove Theorem 1.4 by contradiction. Suppose G is a counterexample of Theorem 1.4 with vertices as few as possible, then G has no 4-vertices. To prove Theorem 1.4, we need to prove Lemma 4.1 firstly.

Lemma 4.1 The minimum degree of G is at least 5.

Proof: Suppose not, G contains a vertex v with degree less than 4. By minimality, $G' = G - v$ is 4-choosable. For any list assignment $L = \{L(v) : |L(v)| = 4, v \in V(G)\}$, there is a function φ_L which is a 4-coloring of G'. Then v still have $|L(v) \setminus \bigcup_{u \in N_{G^{\times}}(v)} \varphi_L(u)| \geq 4 - 3 = 1$ colors which could be used. So we can extend φ_L to G, a contradiction. ■

The proof of the rest of Theorem 1.4 is done in the same way as in Theorem 1.1, and we're going to omit it here.

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