A Special Fractional Choosability of 1-planar Graphs Without 4- and 5-cycles

Mengjiao Li, Lei Sun* and Wei Zheng

Abstract—For a graph $G$ given three integers $a \geq b \geq i \geq 0$ and a certain vertex subset $S$ of $V(G)$, we say $G$ is $i$-weak $(a:b)$-choosable about set $S$ if for any list assignment $L$ of $G$ with the size of each list is $a$ there is a function so that each vertex $v$ of $S$ is given $i$ colors from $L(v)$ while each vertex $u$ of $V(G) \setminus S$ is given $b$ colors from $L(v)$ satisfying that each pair adjacent vertices have disjoint colors. In this paper, for $S = \{v : v \in V(G), a(v) = 4\}$, we prove that every 1-planar graph without 4- and 5-cycles is $i$-weak $(4i + i : b)$-choosable about set $S$. And we prove that every 1-planar graph without 4-vertices, 4- and 5-cycles is $4$-choosable.

Index Terms—1-planar graph, graph, fractional choosability, discharging.

I. INTRODUCTION

A LI graphs are finite and simple in this paper. The undefined but used notations could be found in [1].

A set coloring is a function that assigns each vertex of $G$ a color set so that none of adjacent vertices have common colors. A function $L$ is called a list assignment for a graph $G = (V(G), E(G))$ if each vertex $v$ of $V(G)$ is assigned a color set $L(v)$. If there is a function which is a special set coloring that assigns a $b$-element subset of $L(v)$ to each vertex $v$ of $G$, then $G$ admits a ($L:b$)-coloring and we say $G$ is ($L:b$)-colorable.

If $G$ is ($L:b$)-colorable for any list assignment $L = \{L(v) : v \in V(G)\} = a, v \in V(G)$, then $G$ is ($a:b$)-choosable. $G$ is ($a:b$)-colorable if $G$ has a ($L:b$)-coloring provided that $L(v) = \{1,2,\ldots,a\}$ for each $v \in V(G)$.

It is easy to find that the ($a:b$)-choosability is more difficult to reach than the ($a:b$)-colorability. The minimum fraction $\frac{a}{b}$ such that $G$ is ($L:b$)-colorable is called the fractional chromatic number of $G$ denoted by $\chi_f(G)$. While $G$ is ($a:1$)-choosable or ($a:1$)-colorable, it means that $G$ is proper vertex $a$-choosable or $a$-colorable. Denote the minimum number $a$ that $G$ is proper vertex $a$-colorable by $\chi(G)$. Obviously, $\chi_f(G)$ is no more than $\chi(G)$ for any graph $G$. $G$ is ($c_1,c_2,\ldots,c_k$)-colorable while there is a function $V(G) \to [k]$ such that $\Delta(G[V_i]) \leq c_i$, where $G[V_i]$ denotes the monochromatic induced subgraph of color $i$.

A graph which can be drawn in the plane with no two edges intersect is a planar graph. We say a graph is a 1-planar graph if it has a planar drawing so that each of its edges is crossed at most once besides its ends. Obviously, 1-planar graph is a generalization of planar graph. Such a planar drawing is called a plane graph or 1-plane graph respectively.

Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable in 1976 (see [2]). Though this conjecture is proved false by Addad et al. [3], there are many coloring results about planar graphs motivated by it. Borodin et al. proved that every planar graph without 4-, 7-cycles is 3-colorable in [4]. But whether any planar graphs without 4-, 5- and 6- cycles are 3-colorable is still unknown. Borodin et al. showed that every planar graph without 5- and 7-cycles and without adjacent triangles is 3-colorable (see [5]). Hill et al. [6] showed that every planar graph without 4- and 5-cycles is ($3,0,0$)-colorable, while under the same condition it is also ($2,0,0$)-colorable in [7] and is ($1,1,0$)-colorable in [8] by Chen et al. and Wu et al. respectively.

The Steinberg’s conjecture also promoted the following results about fractional coloring. Wu et al. proved that every planar graph without 4- and 6-cycles is ($7 : 2$)-colorable in [9]. Dvořák and Hu [10] showed that every planar graph without 4- and 5-cycles is ($11 : 3$)-colorable in 2019.

Different from the Four-Color Theorem, every 1-planar graph is 6-colorable (see [11]). In [12], [13], Song et al. respectively showed that every 1-planar graph is 5-colorable under the condition of without 4- and 5-cycles or without 4-cycles and adjacent 5-vertices. Chu et al. proved that every 1-planar graph with girth at least 7 is ($1,1,1,0$)-colorable in [14] and also ($2,0,0,0$)-colorable in [15]. The coloring problems of 1-planar graphs have been widely studied recently, such as equitable coloring in [16], [17], $k$-($2,1$)-total choosability in [18] and total coloring in [19], [20].

Let us consider a problem now. Assume we are conference sponsors who need to arrange the scheduled meetings. There are $n$ listeners but some of them cannot attend the same meeting. Listeners who belongs to set $M$ only need to join $i$ meetings while the others want to take part in at least $b$ meetings with $b \geq i$. How many meetings do we need to hold? To solve this problem, we construct a graph $G$ with $n$ vertices representing the listeners. Two vertices are adjacent if their corresponding listeners could not attend common meetings. Let $S$ denote the set of vertices whose counterpart is in $M$. So the problem is turned to find the smallest integer $a$ so that $G$ is $i$-weak $(a:b)$-choosable about set $S$ which is defined as following.

Based on such a problem, we defined a new coloring now. Given three integers $a \geq b \geq i \geq 0$ and a certain vertex subset $S$ of $V(G)$, $G$ is $i$-weak $(a:b)$-choosable about set $S$ if for any list assignment $L$ of $G$ with the size of each list...
is a there is a function so that each vertex \( v \) of \( S \) is given \( i \) colors from \( L(v) \) while each vertex \( u \) of \( V(G) \setminus S \) is given \( b \) colors from \( L(v) \) satisfying that each pair adjacent vertices have disjoint colors. For a graph \( G \) and a list assignment \( L = \{ L(v) = \{1, 2, \ldots, a \}, v \in V(G) \} \), if \( G \) admits a function which satisfies the condition defined above about set \( S \), then we say this function is a \( i \)-weak \((a : b)\)-coloring about set \( S \) of \( G \) and \( G \) is \( i \)-weak \((a : b)\)-colorable about set \( S \).

Based on the definitions above, we mainly prove the following result.

**Theorem 1.1** For \( S = \{ v : v \in V(G), d(v) = 4 \} \), every 1-planar graph without 4- and 5-cycles is \( i \)-weak \((4b + i : b)\)-choosable about set \( S \).

**Corollary 1.2** and **Corollary 1.3** are special cases of **Theorem 1.1**.

**Corollary 1.2** For \( S = \{ v : v \in V(G), d(v) = 4 \} \), every 1-planar graph without 4- and 5-cycles is \( i \)-weak \((4b + i : b)\)-choosable about set \( S \).

**Corollary 1.3** For \( S = \{ v : v \in V(G), d(v) = 4 \} \), every 1-planar graph without 4- and 5-cycles is \( i \)-weak \((9 : 2)\)-choosable about set \( S \).

**Theorem 1.4** Every 1-planar graph without 4-vertices, 4- and 5-cycles is 4-choosable.

**II. Preliminaries**

We say vertex \( x \) is a \( k \)-vertex or \( k^+\)-vertex if its degree is \( k \) or at least \( k \) respectively. These definitions are also applied to faces. Making any crossing of the 1-plane graph \( G \) to a 4-vertex leads to the associated plane graph \( G^x \) of \( G \). We define the vertices of \( G^x \) which do not appear in \( G \) as cross vertices, other vertices are called true vertices. The white dots and black dots shown in the following figures are respected the cross vertices and true vertices in \( G^x \) respectively.

A 4-face in \( G^x \) is ordinary or special if it is incident with exactly one or two cross vertices respectively. Let \( m_3(v) \) (resp. \( m_4(v) \)) denote the number of 3-faces (resp. special 4-faces) of \( G^x \) incident with \( v \) for any \( v \in G^x \). The existence of 4- and 5-cycles is called the impossible event.

By the definition of 1-planar graph, we can directly find the following claim.

**Claim 2.1** ([21]) There are not two adjacent cross vertices in \( G^x \).

Since the absence of 4- and 5-cycles in \( G \), here comes the following claim.

**Claim 2.2** The two subgraphs shown in Figure 1 could not appear in \( G^x \).

**III. Proof of Theorem 1.1**

We prove Theorem 1.1 by contradiction. Let \( G \) be a minimal counterexample of Theorem 1.1 which means that there is no other counterexample with vertices less than \( G \). Then \( G \) has neither of 4-, 5-cycles and \( G \) is not \( i \)-weak \((4b + i : b)\)-choosable about set \( S \). The following discussions about \( G \) in this part and the next part are all about the one we just assumed.

**Lemma 3.1** The minimum degree of \( G \) is at least 5.

**Proof**: Suppose not, \( G \) contains a vertex \( v \) with degree no more than 4. By minimality, \( G' = G \setminus \{ v \} \) is \( i \)-weak \((4b + i : b)\)-choosable about set \( S \). For any \( L = \{ L(v) : |L(v)| = 4b + i, v \in V(G) \} \), there is an \( i \)-weak \((4b + i : b)\)-coloring \( \varphi_L \) about set \( S \) in \( G \). Let us discuss in two cases.

If \( v \) has less than 4 neighbors, then \( v \) still has \( |L(v) \setminus \bigcup_{u \in N_G(v)} \varphi_L(w)| \geq 4b+i-3b = b+i \geq b \) colors now. Otherwise \( v \) belongs to \( S \), \( v \) remains \( |L(u) \setminus \bigcup_{u \in N_G(v)} \varphi_L(w)| \geq 4b+i-4b = i \) which could be used. In either case, we can extend \( \varphi_L \) to \( G \), a contradiction.

We are going to use the discharging method on \( G^x \) to prove that no counterexample of Theorem 1.1 exists. Let us set the initial charges of \( G^x \) by \( \chi_0(v) = \frac{1}{2}d(v) - 3 \) for each vertex \( v \) of \( V(G^x) \), and \( \chi_0(f) = d(f) - 3 \) for any face \( f \) which belongs to \( F(G^x) \).

By Euler’s formula we have \( |V(G^x)| = |E(G^x)| + |F(G^x)| = 2 \), thus the following equation is established.

\[
\sum_{v \in V(G^x)} \frac{1}{2}d(v) - 3 + \sum_{f \in F(G^x)} (d(f) - 3) = -6
\]

The symbol \( \chi(x) \) denotes the final charge of any \( x \in V(G^x) \cup F(G^x) \) after the following discharging rules below. Note that the discharging rules could not change the sum of the charges of \( V(G^x) \cup F(G^x) \), thus the sum of the final charges should still be \(-6\).

\[
\sum_{x \in V(G^x) \cup F(G^x)} \chi_0(x) = \sum_{x \in V(G^x) \cup F(G^x)} \chi(x) = -6
\]

The discharging rules are given as follows.

**R1.** Each 4\(^+\)-face \( f \) of \( G^x \) sends to each incident cross vertex \( v \):
(a) \( 1/4 \) for \( m_3(v) = 0 \).
(b) \( 1/3 \) for \( m_3(v) = 1 \).
(c) \( 1/2 \) for \( m_3(v) = 2 \).

**R2.** Each 4\(^+\)-face \( f \) of \( G^x \) sends to each incident 5-vertex \( v \):
(a) \( 1/4 \) if \( f \) is an ordinary 4-face.
(b) \( 1/8 \) if \( f \) is a special 4-face.
(c) \( 3/8 \) if \( f \) is a 5-face.
(d) \( 1/2 \) if \( f \) is a 6\(^+\)-face.

After setting the discharging rules, we are going to check the final charges of \( V(G^x) \cup F(G^x) \). Check the final charges of vertices in \( G^x \) firstly.

It is obvious that the minimum degree of \( G^x \) is at least 4 by Lemma 3.1. The following three lemmas show that \( \chi(v) \geq 0 \) for any \( v \) of \( V(G^x) \). The upper bounds of the number of 3-faces and special 4-faces incident with \( v \) are based on Claim 2.1, Claim 2.2 and the absence of 4- and 5-cycles in \( G \).

**Lemma 3.2** If \( v \) is a 4-vertex of \( G^x \), then \( \chi(v) \geq 0 \).

**Proof**: Note that \( \chi_0(v) = -1 \) and \( m_3(v) \leq 2 \). Classify \( v \) by the number of 3-faces incident with it.

(1) If \( m_3(v) = 0 \), then \( \chi(v) = -1 + 4 \times 1/4 = 0 \) by R1(a).
Lemma 3.3 If \( v \) is a 5-vertex of \( G^x \), then \( ch(v) \geq 0 \).

Proof: Note that \( ch(v) = -1/2 \) and \( m_3(v) \leq 4 \).

(1) If \( m_3(v) = 0 \), then \( ch(v) \geq -1/2 + 5 \times 1/8 = 1/8 \geq 0 \) by \( R2 \).

(2) If \( m_3(v) = 1 \), then \( m_3(v) \leq 3 \) and \( ch(v) \geq -1/2 + 1/4 + 3 \times 1/8 = 1/8 \geq 0 \) by \( R2 \).

(3) If \( m_3(v) = 2 \), then \( m_3(v) \leq 2 \) and \( ch(v) \geq -1/2 + 1/4 + 2 \times 1/8 = 0 \) by \( R2 \).

(4) If \( m_3(v) = 3 \), then \( m_3(v) \leq 1 \).

Case 1: \( v \) is not incident with any special 4-faces.

Then we discuss the condition of the remaining unknown face \( f' \) incident with \( v \) and is neither a 3-face nor \( f \). Since \( m_3(v) = 3 \) and \( m_3(v) = 1 \), then \( f' \) is either a 5\(^{+}\)-face or an ordinary 4-face.

Case 2.1: \( f' \) is a 5\(^{+}\)-face, then \( ch(v) \geq -1/2 + 1/4 + 3 \times 1/8 = 1/8 \geq 0 \) by \( R2(a) \), \( R2(c) \) and \( R2(d) \).

Case 2.2: \( f' \) is an ordinary 4-face. Note that \( v \) is incident with \( f, f' \) and three 3-faces. Considering the two faces \( f_1, f_2 \) adjacent to \( f \) and incident with \( v, f_1 \) and \( f_2 \) are either two 3-faces or one is 3-face and another one is \( f' \). Suppose \( f_1, f \) and \( f_2 \) are counterclockwise in the plane (see Figure 2(1)).

Case 2.2.1: Both \( f_1 \) and \( f_2 \) are 3-faces. The Figure 2(2) shows that \( G \) has at least one 4-cycles.

Case 2.2.2: By symmetry, suppose \( f_1 \) is a 3-face, \( f_2 \) is \( f' \). As is shown in the Figure 2(3), one 5-face appears in \( G \).

Both the cases above contradict the fact that the impossible event could not occur. So Case 2.2 could not happen. Thus \( ch(v) \geq 0 \) for any 5-vertex with \( m_3(v) = 3 \).

Lemma 3.4 If \( v \) is a 6\(^{+}\)-vertex of \( G^x \), then \( ch(v) \geq 0 \).

Proof: There is no discharging rule about \( v \), so \( ch(v) = ch_0(v) = \frac{1}{2}d(v) - 3 = \frac{1}{2}(d(v) - 6) \geq 0 \).

The following three lemmas show that \( ch(f) \geq 0 \) for any \( f \in F(G^x) \).

Lemma 3.5 If \( f \) is a cross 4-face of \( G^x \), then \( ch(f) \geq 0 \).

Proof: Note that \( ch_0(f) = 1 \). According to Claim 2.1, \( f \) is incident with either one or two cross vertices which means that \( f \) is either ordinary or special.

(1) \( f \) is an ordinary 4-face, then the only cross vertex incident with \( f \) is not incident with any 3-faces. Otherwise, as illustrated by the following Figure 5(2-3), the impossible event happens, a contradiction.

Thus \( ch(f) \geq 1 - 1/4 - 3 \times 1/4 = 0 \) by \( R1(a) \) and \( R2(a) \).

(2) \( f \) is a special 4-face. Suppose \( v \) and \( u \) are the two cross vertices incident with \( f \). Note that each of \( v \) and \( u \) is at most incident with two 3-faces. So we discuss the number of 3-faces incident with the cross vertices \( v \) and \( u \).

Case 1: Both of \( v \) and \( u \) are incident with at most one 3-face. Then \( ch(f) \geq 1 - 2 \times 1/3 - 2 \times 1/8 = 1/12 \geq 0 \) by \( R1(a) \), \( R1(b) \) and \( R2(b) \).

Case 2: One of \( v \) and \( u \) is incident with two 3-faces. In general, suppose \( v \) is such a vertex. Then \( u \) could not incident with a 3-face as the Figure 6(2-4) show below. Otherwise no matter how the 3-face incident with \( u \) appears, the impossible event happens, a contradiction.

Then \( ch(f) \geq 1 - 1/4 - 1/2 - 2 \times 1/8 = 0 \) by \( R1(a), R1(c) \) and \( R2(b) \).
Figure 6 $d(f) = 4$, $m_3(v) = 2$

**Lemma 3.6** If $f$ is a 5-face of $G^*$, then $ch(f) \geq 0$.

**Proof:** Note that $ch_0(f) = 2$. According to Claim 2.1, there are less than three cross vertices incident with $f$.

1. There is exactly one cross vertex $v$ incident with $f$ shown in Figure 7(1). In this case, $m_3(v) \leq 1$. If not, Figure 7(2) shows that a 5-cycle will occur in $G$. $ch(f) \geq 2 - 1/3 - 4 \times 3/8 = 1/6 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$.

2. There are exactly two cross vertices $v$ and $u$ incident with $f$.

Case 1: $m_3(v) \leq 1$ and $m_3(u) \leq 1$. $ch(f) \geq 2 - 2 \times 1/3 - 3 \times 3/8 = 5/24 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$.

Case 2: Generally suppose $v$ is incident with two 3-faces, then $m_3(u) \leq 1$. Otherwise, $G$ will contain a 5-cycle, as shown in Figures 8(2-3) and 8(5-6). Thus $ch(f) \geq 2 - 1/3 - 1/2 - 3 \times 3/8 = 1/24 > 0$ by $R1(a)$, $R1(b)$ and $R2(c)$.

**Figure 7** $d(f) = 5$, $f$ has exactly one cross vertex $v$

**Figure 8** $d(f) = 5$, $m_3(v) = 2$

**Lemma 3.7** If $f$ is a $6^+$-face of $G^*$, then $ch(f) \geq 0$.

**Proof:** By the discharging rules, $f$ sends at most $1/2$ charges to each of its incident vertices. Thus $ch(f) \geq ch_0(f) - \frac{1}{2}d(f) = d(f) - 3 - \frac{1}{2}d(f) = \frac{1}{2}(d(f) - 6) \geq 0$.

According to Lemma 3.2–Lemma 3.7, every $x$ of $V(G^*) \cup F(G^*)$ satisfies $ch(x) \geq 0$. Thus the final sum of the charges of $V(G^*) \cup F(G^*)$ is non-negative which contradicts the fact that it should be $-6$. Thus there is no counterexample $G$ existing which makes the Theorem 1.1 valid.

**IV. PROOF OF THEOREM 1.4**

We prove Theorem 1.4 by contradiction. Suppose $G$ is a counterexample of Theorem 1.4 with vertices as few as possible, then $G$ has no 4-vertices. To prove Theorem 1.4, we need to prove Lemma 4.1 firstly.

**Lemma 4.1** The minimum degree of $G$ is at least 5.

**Proof:** Suppose not, $G$ contains a vertex $v$ with degree less than 4. By minimality, $G' = G - v$ is 4-choosable. For any list assignment $L = \{L(v) : |L(v)| = 4, v \in V(G)\}$, there is a function $\varphi_L$ which is a 4-coloring of $G'$. Then $v$ still have $|L(v) \setminus \bigcup_{u \in N_G(v)} \varphi_L(u)| \geq 4 - 3 = 1$ colors which could be used. So we can extend $\varphi_L$ to $G$, a contradiction.

The proof of the rest of Theorem 1.4 is done in the same way as in Theorem 1.1, and we’re going to omit it here.

**REFERENCES**


