Analytical Solutions of Non-Newtonian Fluid through a Reiner-Rivlin Liquid Sphere with Cell Surface

R. Selvi, P. Shukla

Abstract—**The present article is a theoretical attempt to perceive an analytical solution of an incompressible micropolar fluid through a non-Newtonian liquid sphere by adopting Mehta-Morse boundary condition. The framework of the flow is divided into two regions in which the non-Newtonian characteristic of Reiner-Rivlin liquid regulates the inner flow. However, the micropolar fluid keeping the microlevel properties of the fluid, regulates the outer flow surrounding the Reiner-Rivlin liquid sphere. An asymptotic series expansion involving the stream functions in terms of non-dimensional parameter S has been employed to derive the expression of the flow field for the Reiner-Rivlin liquid; however, an analytical expression has been derived for outer flow involving modified Bessel functions and Gagenbauer's polynomials. The graphical analysis demonstrating the superior outcomes of numerous parameters like cross-viscosity, solid volume fraction, micropolar parameter, and the coupling number on the drag coefficient are conducted, and outcomes are discussed comprehensively. The notable detection of the present work is that the drag is more resistance for an impermeable sphere as compared to a Reiner-Rivlin liquid sphere. With the rising value of permeability parameter, a continuous reduction in the drag force experienced by the sphere is observed however, an increasing value of the coupling number contributes to increase in the drag force experienced by sphere. The findings of the present work may leave valuable outcomes in analyzing the considerable industrial and clinical applications such as petroleum reservoir rocks, filtration process for wastewater treatment, and the flow of blood through the lungs, and the design of the digestive system. However, experimental verification is required for the proposed work.**

Index Terms—**Reiner-Rivlin liquid, Brinkman equation, Micropolar parameter, Axisymmetric flow, Coupling number, Drag force.**

I. Introduction

FLUID mechanics has recently undergone new developments, mostly focusing on the structures within a fluid. Newtonian law does not hold suitable for natural calamities, for example, volcanic lava, and for industrial applications such as include fluids, polymer fluids, drilling mud, cosmetics, food products, and exotic lubricants. Hence researchers across the globe have started to work on diverse flow problems, which are associated with several non-Newtonian fluids.

Manuscript received Jan 24, 2022; revised July 25, 2022. . This work was supported by The Institute of Mathematical Sciences, Chennai, Tamil Nadu 600113, INDIA.

R.Selvi is a post doctoral fellow in the Department of Mathematics, The Institute of Mathematical Sciences, Chennai, Tamil Nadu 600113, INDIA. (First and Corresponding author e-mail: selvir@imsc.res.in).

Pankaj Shukla is an Assistant Professor in the Department of Mathematics, Vellore Institute of Technology, Chennai, Tamil Nadu 600127, INDIA. (e-mail: pankaj.shukla@vit.ac.in).

The simplest theory considered for structural fluid is the micropolar theory attributed to Eringen [4] who developed the fluid mechanics in an article entitled “Simple Microfluids”. Micropolar fluids sustain couple stresses and body couples and show microrotational effects and inertia of microrotation. As the consequence of simple microfluids, he [5] developed a sub-class of fluids which is known as micropolar fluids that ignore microelement deformation but still allow micro-motion of the particle. The problem related to micropolar fluid with low Reynolds number flow through the solid sphere was solved by Rao and Rao [12] in which they found that the drag is higher in the case of micropolar fluid than that on a Newtonian fluid sphere. The drag of an axially symmetric body in viscous flow whose general expression was studied by Ramkissonon and Majumadar [9]. Neifer and Kaloni [8] explored the problems in two sections, in one section, the motion of the clear fluid through the micropolar fluid, and in the other section, the motion of the micropolar fluid through the clear fluid. Selvi [14] studied the analytical solution of micropolar fluid through a non-Newtonian fluid enclosed by porous medium. They concluded that the non-Newtonian liquid sphere is greater resistance to compare than classical fluid resistance. Selvi [15] researched the drag experienced on flow within the Reiner-Rivlin liquid sphere.

The cell model technique is a valuable tool to analyze the flow through a swarm of particles of even nano-size as it is complicated to apply the traditional methods of reflection, etcetera to study the flow through such cases. This method considers randomly oriented particles through which the flow is to be analyzed. Instead of reviewing every particle, an identical particle confined within a hypothetical cell is chosen. The importance of the cell surface is that the suitable boundary conditions on the cell surface are imposed, which considers the effect of other particles on the particle concerned. In this way, despite ignoring the other particles, we take into account the interaction of these particles on the particle-in-cell. Specific boundary conditions were recommended at the imaginary surface of the cell model. One of the real-time practical applications is to monitor blood flow through the arterial wall. Mehta and Morse [7] considered as the most important boundary condition as we are much fascinated by the flow behavior on a large scale. Non-Newtonian liquid sphere in a spherical container was obtained by Ramkissoon and Rahaman [11]. They found the drag exerted by the outer sphere. Saad [13] analysed the problem of a (micropolar fluid) non-Newtonian liquid through a clear fluid sphere with the help of cell models. Shukla [18] solved the problem of a non-Newtonian fluid past a coated film with no spin boundary condition. To the excellent and informative knowledge of authors and beliefs, the mathematical equations governing the flow of a Reiner-Rivlin liquid sphere enfolded in the micropolar fluid involving the Mehta-Morse boundary condition have not been solved earlier.

In the proposed work, the analytical expressions have been executed to understand the flow of micropolar fluid through a non-Newtonian nature of Reiner-Rivlin liquid by taking the Mehta-Morse/Cunningham condition. The asymptotic series expansions in terms of stream functions have been utilized to solve the governing equation regulating the inner flow of Reiner-Rivlin fluid. Due to the presence of microlevel properties in the micropolar fluid, the coupled equations in terms of velocity and angular velocity of the micropolar fluid have been analytically derived for suitable boundary conditions involving the modified Bessel functions and Gagenbauers polynomials. The expressions for drag force and drag coefficient have been established. The drag force and drag coefficient dependence on the numerous parameters such as micropolar parameter, coupling number, dimensionless parameter S, and the volume fraction are presented pictorially. A comparative analysis has been done replicating in the bounded and unbounded medium of micropolar fluid through a Reiner-Rivlin liquid sphere. These results are compared with the corresponding earlier results.

The proposed study is separated into VII sections in which Section- II includes the model description and problem formulation describing the statement of the problem, governing equations, non-dimensionalization, and their solutions. Section- III demonstrates the appropriate boundary conditions delineating the conditions on the inner surface and the hypothetical cell surface. Section- IV reveals the calculations of the hydrodynamical quantities like drag force and drag coefficient. The limiting cases of the present study are discussed in Section- V. Based on the mathematical expressions of the flow quantities and numerical values of the parameters, Section- VI deals with the graphical analysis of the remarkable outputs. Section- VII delineates the summary and primary determinations of the present study.

II. PROBLEM FORMULATION AND MODEL DESCRIPTION

A. Statement of the Problem and Model Description

The flow of non-Newtonian characteristic of Reiner-Rivlin fluid sphere of radius $'a'$ embedded in the micropolar fluid of radius $'b'$ is shown in Fig. 1. The flow pattern is taken as steady, asymmetric, and incompressible by neglecting the body force and couple. The particle geometry is considered as a sphere, and hence the spherical polar coordinate systems (r, θ, ϕ) have been utilized to formulate the governing equations for micropolar and Reiner-Rivlin fluids. The characteristic (uniform) velocity U acting along the $Z-$direction of the fluid flow while taking O as the origin of the sphere is demonstrated in Fig. 1. The flow regime is separated into two regions: Region-I keeping $(a \leq r \leq b)$ reveals the flow of non-Newtonian Reiner-Rivlin fluid inside the micropolar fluid, and Region-II keeping $(r \leq a)$ replicates the flow of micropolar fluid outside the Reiner-Rivlin liquid sphere. The radius $'b'$ of the hypothetical cell is chosen as the ratio of particle to cell volume and is equal to the particle volume fraction γ.

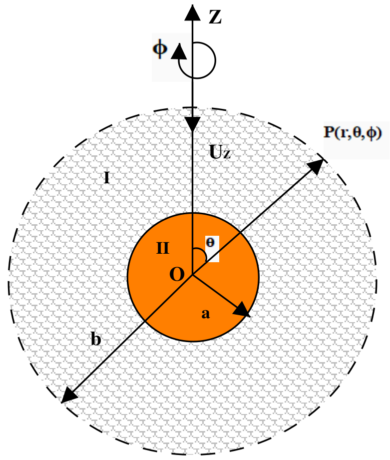

Fig. 1. Schematic representation motion of Reiner-Rivlin liquid cell of radius a covered by a micropolar fluid cell of radius b.

B. Governing Equations

The equations governing the flow of micropolar fluid for outer region are described below

$$\nabla \times \vec{q}^{(1)} = 0, \tag{1}$$

$$\kappa\nabla \times \vec{\omega}^{(1)} - (\mu_1 + \kappa)\nabla \times \nabla \times \vec{q}^{(1)} = \nabla p^{(1)}, \tag{2}$$

$$(\alpha+\beta+\gamma)\nabla(\nabla.\vec{\omega}^{(1)})+\kappa\nabla\times\vec{q}^{(1)}-\gamma\nabla\times\nabla\times\vec{\omega}^{(1)} = 2\kappa\vec{\omega}^{(1)}, \tag{3}$$

where p and $\vec{q}$ are flow parameters of the micropolar fluid describing the pressure, and velocity vector, respectively, κ denotes vertex viscosity and $\vec{\omega^{(1)}}$ referred as microrotation vector. The following inequalities are satisfied which has already been reported by Srinivascharya and Rajyalakshmi [17],

$$\kappa \geq 0, \quad 3\alpha+\beta+\gamma \geq 0, \quad 2\mu_1+\kappa \geq 0, \quad \gamma \geq |\beta \tag{4}$$

For region II, the state equations for the isotropic non-Newtonian liquid is described as below

$$\tau_{ij} = -p^{(2)}\delta_{ij} + 2\mu_2 d_{ij} + \mu_c d_{ik} d_{kj}, \tag{5}$$

where μ_2 is the apparent viscosity coefficient, and $d_{ij} = \frac{1}{2}\left(u_{ij}^{(2)} + u_{j,i}^{(2)}\right)$. τ_{ij} denotes the stress tensor, μ_c represents a cross viscosity of Reiner-Rivlin liquid. However, $p^{(2)}$ is pressure, and δ_{ij} denotes the Kronecker delta, respectively.

In order to find the solution of equations regulating the flow Reiner-Rivlin liquid embedded in the micropolar fluid, the following dimensionless variables are introduced. The dimensional and non-dimensional quantities/parameters are expressed with and without the tilde symbols, respectively. The non-dimensional variables are expressed in the following form:

$$R = ar, \quad q_\theta = U_{q_\theta}, \quad q_R = U_{q_r}, \quad d_{ij} = \frac{U}{a}d_{ij},$$
$$\tau_{ij} = \mu_i\frac{U}{a}\tau_{ij}, \quad \psi = Ua^2\psi, \quad p = \mu_i\frac{U}{a}p, \tag{6}$$

where U is the current velocity flow mentioned above.

The stream function operates independently of the azimutal angle ϕ in all regions: $\psi = \psi(r,\theta)$ as the flow is axially symmetric.

$$q^{(i)} = q_r^{(i)}(r,\theta)e_r + q_\theta^{(i)}(r,\theta)e_\theta, \tag{7}$$

The equations regulating the flow are coupled and complex to solve directly. In order to ensure the simplicity of the governing equations, the stream function and their derivatives are introduced in the following forms:

$$q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \quad \text{and} \quad q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \tag{8}$$

The pressure has been excluded from Eq. (2), for this reason the equation for the stream function in Region- I is derived as below

$$E^4(E^2 - m^2)\psi^{(1)} = 0, \tag{9}$$

where $E^2 = \frac{\partial}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$ is the Stokes operator, $m^2 = \frac{a^2\kappa(2\mu_1+\kappa)}{\gamma(\mu_1+\kappa)}$ is the micropolar parameter describing the microrotation of the fluid particle and $N = \frac{\mu_1}{\mu_1+\kappa}, (0 \le N \le 1)$ is the coupling number.

The analytical solution for the external flow field is obtained as

$$\psi^{(1)}(r,\zeta) = \Big[A_1r^2 + \frac{B_1}{r} + C_1r^4 + D_1r + E_1y_{-2}(mr) + F_1y_2(mr)\Big]G_2(\zeta), \tag{10}$$

where A_1, B_1, C_1, D_1, E_1 and F_1 are arbitrary constants. The functions $y_{-2}(mr)$ and $y_2(mr)$ are the modified Bessel functions and $G_2(\zeta)$ is the Gagenbauer's polynomial.

Using Eq. (3), the microspin component is obtained below

$$v_\phi^{(1)} = \frac{1}{2r\sin\theta}\left[E^2\psi^{(1)} + \frac{\gamma(\mu_1+\kappa)}{\kappa^2}E^4\psi^{(1)}\right] \tag{11}$$

By entering the values of $\psi^{(1)}$ into Eq. (11) the microrotation component is obtained under.

$$v_\phi^{(1)} = \frac{1}{2}\Big[5C_1r - D_1r^{-2} + \frac{m^2}{N}\Big(E_1y_{-2}(mr) + F_1y_2(mr)\Big)\Big]\sin\theta. \tag{12}$$

The solutions for the pressure and the stream function within the liquid core (the Reiner-Rivlin droplet) are very difficult to find analytically. To determine the asymptotic solutions for the stream function and the pressure, the power series expansions in terms of the small parameter S are introduced in the following form:

$$\psi^{(2)} = \psi_0 + \psi_1S + \psi_2S^2 + \ldots, \quad p^{(2)} = p_0 + p_1S + p_2S^2 + \ldots. \tag{13}$$

Ramkissoon [10] has already derived the differential equations for the zeroth ψ_0, first ψ_1 and second ψ_2 order stream functions

$$E^4\psi_0 = 0, \quad E^4\psi_1 = 8r\sin^2\theta\cos\theta, \quad E^4\psi_2 = \frac{32}{3}r^2\sin^2\theta, \tag{14}$$

Partial solutions of Eq.(14) may be represented as

$$\psi_0 = (r^4 - r^2)\sin^2\theta, \psi_1 = \frac{2}{21}r^5\sin^2\theta\cos\theta, \quad \psi_2 = \frac{2}{63}r^6\sin^2\theta. \tag{15}$$

Ramkissoon [10] demonstrated that the flow within the sphere containing a Reiner-Rivlin liquid. The stream function is given with the following expression:

$$\psi^{(2)}(r,\zeta) = \psi_0 + \psi_1S + \psi_2S^2 + \sum_{n=2}^{\infty}\left[a_nr^n + b_nr^{n+2}\right]G_n(\zeta). \tag{16}$$

After substituting of the expressions of stream functions ψ_0, ψ_1, ψ_2 into Eq. (16), the explicit expression of stream function ψ_2 is derived in terms of Gagenbauer's polynomial

$$\psi^{(2)}(r,\zeta) = \Big[(a_2-2)r^2 + (b_2+2)r^4 + \frac{4}{63}S^2r^6\Big]G_2(\zeta) + \Big[a_3r^3 + (b_3 + \frac{4}{21}S)r^5\Big]G_3(\zeta) + \sum_{n=4}^{\infty}\Big[a_nr^n + b_nr^{n+2}\Big]G_n(\zeta). \tag{17}$$

III. BOUNDARY CONDITIONS

In order to find out the closed form of the solutions by eliminating the arbitrary constants, the suitable boundary conditions are specified on the inner and hypothetical cell surfaces. The unknown constants involved in the solutions (Eqs. (10) and (17)) for the stream functions can be determined from the following boundary conditions.

A. *Conditions on the inner surface at $r = a$*

No penetration, the continuous tangential velocities and the continuity of tangential stresses are considered on the inner surface of the cell at $r = a$ as follows

$$q_r^{(1)} = 0 \qquad i.e., \qquad \psi_\theta^{(1)} = 0. \tag{18}$$

$$q_r^{(2)} = 0 \qquad i.e., \qquad \psi_\theta^{(2)} = 0. \tag{19}$$

$$q_\theta^{(1)} = q_\theta^{(2)} \qquad i.e., \qquad \psi_r^{(1)} = \psi_r^{(2)}. \tag{20}$$

and the continuity of the shear stresses $\tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)}$ which is equivalent to the mathematical expression given below

$$-\lambda\frac{(2-N)}{1-N}\frac{\partial\psi^{(1)}}{\partial r} + \frac{\lambda}{1-N}\frac{\partial^2\psi^{(1)}}{\partial r^2} = -2\frac{\partial\psi^{(2)}}{\partial r} + \frac{\partial^2\psi^{(2)}}{\partial r^2} \tag{21}$$

The no-spin condition on the micro-rotation velocity is considered, *i.e.*,

$$v_\phi^{(1)} = 0. \tag{22}$$

B. *Conditions on the hypothetical cell surface at $r = l(\frac{b}{a})$*

The conditions on the hypothetical cell surface are defined as

$$q_r^{(1)} = \cos\theta, \qquad i.e., \qquad \frac{\partial\psi^{(1)}}{\partial\theta} = r^2\sin\theta\cos\theta, \tag{23}$$

Mehta and Morse [7] assumed homogeneity on the cell surface,

$$q_\theta^{(1)} = -\sin\theta \quad i.e., \quad \frac{\partial\psi^{(1)}}{\partial r} = r\sin^2\theta. \tag{24}$$

Introducing all the boundary conditions from the Eqs. (18) to (24) into the Eqs. (10) and (17), the solutions of the equations governing the flow of non-Newtonian Reiner-Rivlin fluid embedded in micropolar fluid are reduced into the following algebraic linear equations:

$$A_1 + B_1 + C_1 + D_1 + E_1 y_{-2}(m) + F_1 y_2(m) = 0, \quad (25)$$

$$a_2 + b_2 = \frac{-4}{63} S^2, \quad (26)$$

$$2A_1 - B_1 + 4C_1 + D_1 + E_1 \left[m^2 y_{-1}(m) - y_{-2}(m)\right] + F_1 \left[m^2 y_1(m) - y_2(m)\right] - 2a_2 - 4b_2 = 4 + \frac{8}{21} S^2, \quad (27)$$

$$\frac{\lambda}{1-N}\Big[A_1(-2+2N) + B_1(4-N) + C_1(4+4N) + D_1(-2+N) + E_1\Big((4-N+m^2)y_{-2}(m) + y_{-1}(m)(-2m^2+m^2N)\Big) + F_1\Big((4-N+m^2)y_2(m) + y_1(m)(-2m^2+m^2N)\Big)\Big] + 2a_2 - 4b_2 = 12 + \frac{8}{7}S^2, \quad (28)$$

$$5C_1 - D_1 + \frac{m^2}{N}\left[E_1 y_{-2}(m) + F_1 y_2(m)\right] = 0, \quad (29)$$

$$A_1 l^2 + B_1 l^{-1} + C_1 l^4 + D_1 l + E_1 y_{-2}(ml) + F_1 y_2(ml) = l^2, \quad (30)$$

$$2A_1 l - B_1 l^{-2} + 4C_1 l^3 + D_1 + E_1\Big(m^2 y_{-1}(ml) - l^{-1} y_{-2}(ml)\Big) + F_1\left(m^2 y_1(ml) - l^{-1} y_2(ml)\right) = 2l, \quad (31)$$

$$5C_1 l - D_1 l^{-2} + \frac{m^2}{N}\left[E_1 y_{-2}(ml) + F_1 y_2(ml)\right] = 0. \quad (32)$$

The set of Eqs. (25) to (32) are solved using the Mathematica software. The analytical expressions are obtained for all unknown constants $A_1, B_1, C_1, D_1, E_1, F_1, a_2, b_2$. These expressions are not presented here because of their cumbersomeness. Thus the non-dimensional stream functions in regions I and II can be written respectively like Eq. (10)

$$\psi^{(1)}(r,\zeta) = \Big[A_1 r^2 + \frac{B_1}{r} + C_1 r^4 + D_1 r + E_1 y_{-2}(mr) + F_1 y_2(mr)\Big] G_2(\zeta), \quad 1 \le r \le \gamma^{-1} \quad (33)$$

and

$$\psi^{(2)}(r,\zeta) = \Big[(a_2-2)r^2 + (b_2+2)r^4 + \frac{4}{63}S^2 r^6\Big] G_2(\zeta) + \Big[a_3 r^3 + (b_3 + \frac{4}{21}S) r^5\Big] G_3(\zeta) + \sum_{n=4}^{\infty}\Big[a_n r^n + b_n r^{n+2}\Big] G_n(\zeta). \qquad r \le 1 \quad (34)$$

IV. CALCULATION OF THE DRAG FORCE

The drag force F exerted on a micropolar fluid sphere is estimated using the formula described by Srinivasacharya and Murthy [16]

$$F = 2\pi U(2\mu_1+\kappa)\alpha^2 \lim_{r\to\infty} r^3 \Big[\frac{\psi^{(1)} - \psi^{(1)}_{\infty}}{\tilde{\omega}^2}\Big], \quad (35)$$

where $\psi^{(1)}_{\infty} = \frac{1}{2} U r^2 \sin^2\theta$ and $\tilde{\omega} = r\sin\theta$.
Evaluating the above formula we found that

$$F = 2\pi U a(2\mu_1+\kappa)\alpha^2 D_1. \quad (36)$$

The drag coefficient D_N is determined by the following expression:

$$D_N = \frac{F}{-2\pi\mu_1 U a}, \quad (37)$$

$$D_N = \frac{2\pi U a(2\mu_1+\kappa)\alpha^2 D_1}{-2\pi\mu_1 U a}, \quad (38)$$

$$D_N = (2\mu_1+\kappa)\alpha^2 D_1. \quad (39)$$

V. LIMITING CASES:

Case- I: If $l \to 0, \quad m \to 0, \quad N \to 0$, then in the case of unbounded medium, the micropolar fluid sphere becomes a Reiner-Rivlin liquid sphere. The force applied to the droplet is obtained as,

$$F = \frac{-2a\pi U\mu_1}{3(1+\lambda)}\left[\frac{32}{63}S^2 + 6\lambda + 9\right], \quad \lambda = \mu_1/\mu_2, \quad (40)$$

where the expression of the force agreed with the result obtained by Ramkissoon [10].
Case- II: If $m \to \infty, \quad N \to 0$, then in case of spherical container, the micropolar fluid turns into a perfect Reiner-Rivlin spherical liquid. The force applied to the droplet is obtained as,

$$F = \frac{8a\pi U\mu_1 \begin{bmatrix} 16S^2\left(2 - 5\gamma + 3\gamma^{5/3}\right) \\ + 189\left(3 + 3\gamma^{5/3}(-1+\gamma) + 2\lambda\right) \end{bmatrix}}{189\left(-1+\gamma^{1/3}\right)^3 \begin{bmatrix} 4\gamma(-1+\lambda) + 4(1+\lambda) + \gamma^{2/3} \\ (-3+6\lambda) + \gamma^{1/3}(3+6\lambda) \end{bmatrix}}, \quad (41)$$

where the equation (41) is previously derived by Ramkissoon and Rahaman [11].
Case(III): If $S \to 0$, then the Reiner-Rivlin liquid sphere behaves like a clear fluid sphere of radius $'a'$ in an infinite expanse, and hence the expression of drag force is obtained as

$$F = \frac{8\pi\mu_1 U a[3 + 2\lambda + 3\gamma^{5/3}(-1+\lambda)]}{(-1+\gamma^{1/3})^3 \begin{bmatrix} 4\gamma(-1+\lambda) + 4(1+\lambda) \\ + \gamma^{2/3}.(-3+6\lambda) + \gamma^{1/3}(3+6\lambda) \end{bmatrix}}, \quad (42)$$

where the equation (42) is obtained earlier by Jaiswal and Gupta [6].
Case- IV: If $\lambda \to 0$, then the Reiner-Rivlin spherical liquid sphere becomes a impermeable and solid sphere, Thus the expression of the drag force is obtained as

$$F = \frac{24\pi\mu_1 U a[1 + \gamma^{1/3} + \gamma^{2/3} + \gamma + \gamma^{4/3}]}{\left(-1+\gamma^{1/3}\right)^3\left(4 + 7\gamma^{1/3} + 4\gamma^{2/3}\right)}, \quad (43)$$

where the force agreed with the result obtained by Mehta and Morse [7].

Case- V: If $\gamma \to 0$, the drag on a solid sphere of radius $'a'$ in an infinite expanse of fluid is given as

$$F = 24\pi\mu_1 U a. \quad (44)$$

The earlier result is shown by Stokes [19].

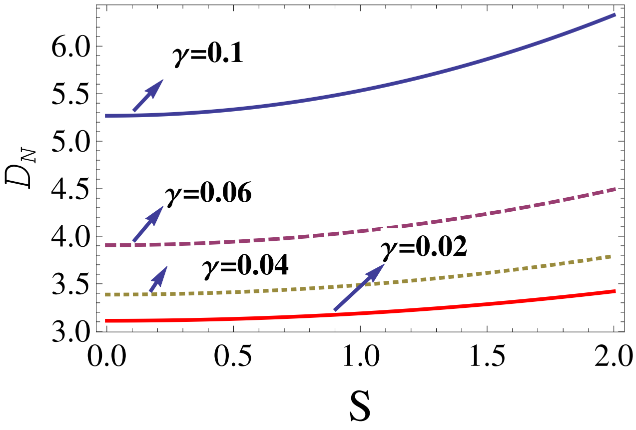

Fig. 2. Plot of non-dimensional drag D_N versus the cross-viscosity S for several aspect of the solid volume fraction γ

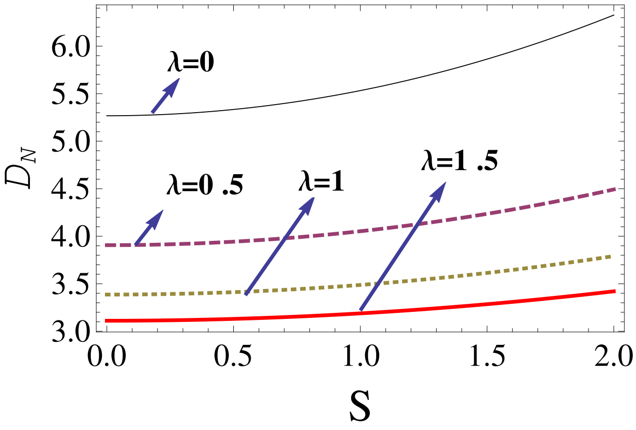

Fig. 3. The graph of dimensionless drag D_N with cross-viscosity S for different values of classical viscosity ratio λ.

VI. Results and Discussion

The analytical expressions of the drag force and drag coefficients are derived for the flow a Reiner-Rivlin liquid sphere embedded in a micropolar fluid sphere with zero-spin and Mehta-Morse boundary conditions. The impact of m(micropolar parameter), N(coupling number), cross-viscosity S and the viscosity coefficient λ on the drag coefficient for each of the problem are described and its variations are depicted in Figs. 2-5.

The impact of particle volume fraction γ on the drag coefficient D_N with the dimensionless cross-viscosity S is shown in Fig. 2 while keeping the fixed values of other parameters like m (micropolar parameter), viscosity coefficient λ, N (coupling number). It is perceived from the figure that an increase in the dimensionless cross-viscosity parameter S, the drag coefficient slightly increases for different values of the particle volume fraction γ. It is also noticed that a rising particle volume fraction enhances the drag coefficient D_N. This physically means that as the particle volume fraction increases, the fluid viscosity increases, and hence the fluid resistance increases due to the higher viscosity. It is important aspect to notice that for every corresponding small values of separation parameter, the drag effects on it is almost constant over the entire range of cross-viscosity. Here, the specific value of cross-viscosity $(S = 0)$ replicates that the case of Newtonian fluid.

Fig. 3 demonstrates the effect of viscosity coefficient λ on the drag coefficient D_N with cross-viscosity parameter S. The drag increases slightly as dimensionless parameter increases which can explicitly seen from the figure. It is perceived from the figure that the drag increases gradually with increasing values of S between 0 and 1 and growth rate is slightly higher for smaller viscosity ratio $(\lambda = 0.0)$. It is also noticed that a rapid reduction is observed with rising viscosity ratio and the reduction rate become smaller for larger viscosity ratio λ. When $\lambda = 0$, it behaves as rigid sphere and the effective viscosity is the same as the clear viscosity when $\lambda = 1$. Physically it implies that the relative viscosity keeps on increasing the drag coefficient gets reduced. It's noteworthy to note that the drag experienced by a Reiner-Rivlin liquid sphere is less resistive than the drag experienced by a rigid spherical.

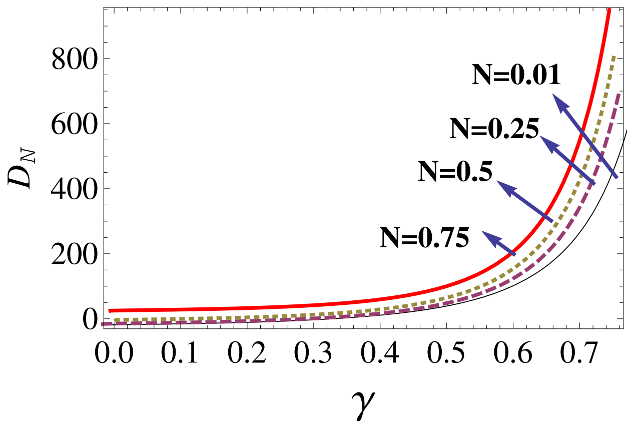

Fig. 4. Dependence of drag D_N versus the solid volume fraction γ for several aspect of coupling number N.

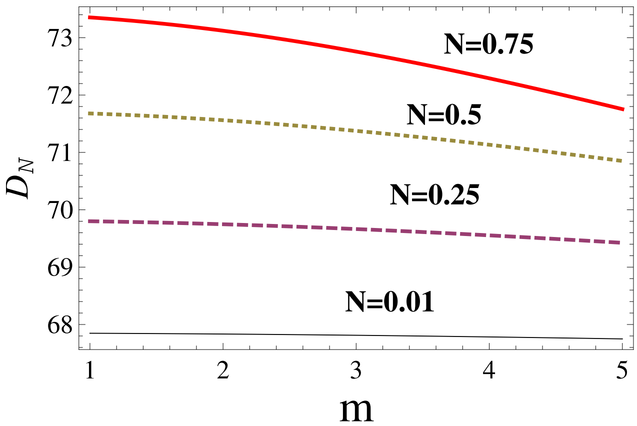

Fig. 5. Variation of drag D_N versus the micropolar parameter m for different values of the coupling number N.

The impact of coupling number N on the drag coefficient with particle volume fraction is illustrated in Fig. 4. It is observed that an increasing particle volume fraction contributes to enhance the drag coefficient gradually for the lower and moderate values of the particle volume fraction however a rapid growth in drag coefficient is observed for larger particle volume fraction $(0.65 \leq \gamma \leq 1.0)$. It is also noted that a rising coupling parameter leads to enhance the drag coefficient D_N for particle volume fraction $(0.65 \leq \gamma \leq 1.0)$. For smaller and moderate values of separation parameter γ, the drag increases gradually with equal amount for all values of coupling number N. It is a remarkable observation that the drag of Reiner-Rivlin liquid sphere is comparatively low when compared with the drag of micropolar fluid sphere.

The dimensionless drag D_N with micropolar parameter m is demonstrated in Fig. 5 for different values of coupling parameter N. It is observed that as the micropolar parameter increases, the drag decreases and the rate of decrement is higher for larger coupling number. Another observation is that an enhancement in the value of the drag coefficient is perceived with rising coupling number for the fixed value of the micropolar parameter which manifests that an effect of microlevel parameters leads to significant enhancement in microrotational viscosity. The non-dimensional drag coefficient, on the other hand, grows rapidly as the coupling number N increases, and the liquid sphere experiences more drag as vortex viscosity increases. It is also notice that the drag shows nonlinear behaviour with a coupling number for higher values of the coupling parameter N except $N = 0.01$.

VII. Summary and Concluding Remarks

The flow of non-Newtonian behavior of Reiner-Rivlin fluid sphere embedded in the micropolar fluid has been studied in the present work by considering the zero-spin and Mehta-Morse conditions on the surface of the liquid sphere.The mathematical equations governing the flow of micropolar fluid enfolded over the non-Newtonian Reiner-Rivlin liquid are solved using an asymptotic series expansion in terms of the non-dimensional parameter S and hence the expressions of drag force and drag coefficients are demonstrated in closed form of m (micropolar parameter), N(coupling number), separation parameter γ, cross-viscosity S and the viscosity coefficient λ. Graphical representation revealing the significant results of non-dimensional drag force and drag coefficient on the cell surface are shown and explained in detail. The characteristic of the flow quantities are analyzed by comparing the numerical values of the drag with the particular parameters.

The following remarkable observations are pointed out and mentioned below:

1) It is perceived that the drag coefficient D_N is higher for the micropolar fluid sphere in comparison to the Reiner-Rivlin liquid sphere.
2) Additionally, it has been found that a liquid Reiner-Rivlin sphere's drag coefficient D_N is significantly lower than that of a permeable sphere.
3) It is observed that the particle volume fraction increases, the fluid viscosity increases and hence the fluid resistance increases due to the higher viscosity.

These results drawn here could be of substantial influence to explore the critical industrial and clinical applications like petroleum reservoir rocks, filtration process for wastewater treatment, physiological fluid flow through lungs and the design of the digestive system. The present work is required to validate experimentally for future development of the clinical or filtration processes.

References

[1] Abramowitz M, and Stegun I.A, *Hand book of Mathematical Function*, New York:Dover Publications, 1970.

[2] Cunningham E, "On the steady state fall of spherical particles through fluid medium" , *Proc R Soc Lond Ser A*, vol. **83**, pp. 357-369, 1910.

[3] Dejam M, "Derivation os dispersion coefficient in an electro-osmotic flow of a viscoelastic fluid through a porous-walled microchannel", *Chem. Engg.Sci*, vol.**204**, pp. 298-309, 2019.

[4] Eringen A.C, "Simple Micropolar fluids", *Int. J. Engg. Sci*, vol. **2**, pp. 205-217, 1964.

[5] Eringen A.C, "Theory of Micropolar fluids", *J. Math. Mech*, vol. **16**, pp. 1-18, 1966.

[6] Jaiswal B.R, and Gupta B.R, "Cell models for viscous flow past a swarm of Reiner- Rivlin liquid spherical drops", *Meccanica*, vol. **52**, pp. 69-89, 2016.

[7] Mehta G, and Morse T.F, "Flow through charged membranes", *J Chem Phys*, vol. **63**, pp. 1878-1889, 1975.

[8] Niefer R, and Kalon P.N, "On the motion of a micropolar fluid drop in a viscous fluid", *Journal of Engineering Mathematics*, vol. **14**, pp. 107-116, 1980.

[9] Ramkissoon H, and Majumadar S.R, "Drag on the axially symmetric body in the stokes flow of micropolar fluid", *Physics of Fluids*, vol. **19**, pp. 16-21, 1976.

[10] Ramkissoon H, "Stokes flow past a Reiner-Rivlin fluid sphere", *J. Appl. Math. Mech*, vol. **69**, pp. 259-261, 1989.

[11] Ramkissoon H, and Rahaman K, "Non-Newtonian fluid sphere in a spherical container", *Acta Mechanica*, vol. **149**, pp.239-245, 2001.

[12] Rao S.K.L, and Rao P.B, "Slow stationary flow of a micropolar fluid past a sphere", *J. Eng. Math*, vol. **4**, pp. 209-217, 1970.

[13] Saad E.I, "Cell models for micropolar fluid past a viscous fluid sphere", *Meccanica*, vol. **47**, pp. 2055-2068, 2012.

[14] Ramasamy Selvi, Shukla P. and Filippov A.N, "Flow around a Liquid Sphere Filled with a Non-Newtonian Liquid and Placed into a Porous Medium", *Colloid J*, vol.**82**, pp. 152-160, 2020.

[15] Selvi R, Shukla P, and Singh A, "Drag on a Reiner-Rivlin liquid sphere embedded in a porous region placed in a micropolar fluid", *Journal of Porous Media*, vol **23**, no. 6, pp. 613-626. 2020.

[16] Srinivasacharya D, and Murthy J, "Flow past an axisymmetric body embedded in a saturated porous medium", *C. R. Mecanique*, vol. **330**, pp. 417-423, 2002.

[17] Srinivasacharya D, and Rajyalakshmi I, "Creeping flow of Micropolar fluid past a porous sphere", *Applied Mathematics and Computation*, vol. **153**, no. 4, pp. 843-854, 2004.

[18] Shukla P, "Micropolar fluid past a sphere coated with a thin fluid film", *Indian Jour. Sci. Tech*, vol. **8**, IPL0384, 2015.

[19] Stokes G.G, "On the effects of internal friction of fluids on pendulums", *J. Trans. Camb. philos. Soc*, vol. **9**, pp. 8-106, 1851.