Independent Set in Bipolar Fuzzy Graph

Ke Luo, and Wei Gao, *Member, IAENG*

***Abstract*—As an important tool to represent structured and uncertain data, graph models are widely used in computer networks and chemical molecular structure representation. If there are uncertainties in the vertices or the binary relationship between vertices in the graph, the membership function is introduced to the vertex set and the edge set, and the fuzzy graph is used to describe such uncertain structural features. The independent set is an important object to measure the topological structure of the graph, as the basis of many graph parameters. In this paper, a novel concept of independent set is proposed in bipolar fuzzy graph setting. The fuzzy topological parameters to measure its degree of independence are defined, and the characteristics of bipolar independence degree are obtained by using graph theory and fuzzy set theory, especially for two types of product bipolar fuzzy graphs. Finally, the algorithm for calculating the maximum (resp. minimum) positive (resp. negative) fuzzy subgraph with given positive (resp. negative) degree of independence is designed.**

***Index Terms*—bipolar fuzzy graph, independent set, product fuzzy graph, fuzzy molecular graph.**

I. Introduction

GRAPHS are an important model to characterize irregular data or structured data in computer science. For example, in the subway network, each station is represented by a vertex. If two stations are consecutively adjacent to each other on a certain subway line, then an edge is connected between these two stations. Thus the rail transit network of the entire city can be modelled by a graph. Another instance, every molecular structure can be characterized by a graphical model. Each atom is represented by a vertex, if there is a chemical bond between two atoms, then an edge is connected between the corresponding vertices. The graph obtained by this trick is called a molecular graph. Various applications of graph models can be found in [1-10].

In a large number of graphical model applications, we find that the basic elements of the model are uncertain in specific application scenarios. For example, in the field of computer networks, many application backgrounds, sites and channels have some uncertain factors, so it is necessary to use a fuzzy graph as a model to describe the performance of the network under the fuzzy framework. In the fuzzy graph setting, the uncertainties of vertices and edges are characterized by their respective membership functions (see Islam and Pal [11] and Samanta et al. [12]).

A bipolar fuzzy set traditionally uses a negative membership function to describe negative uncertainty. The corresponding structured fuzzy graph is called a bipolar fuzzy graph. That is, there are positive and negative membership functions defined on vertex set and edge set, thus we characterize positive and negative uncertainties for vertices and edges. In recent years, research on bipolar fuzzy graphs has become a hot topic in the field of fuzzy graphs. Mathew et al. [13] proposed several connectivity concepts in bipolar fuzzy graphs. Singh and Kumar [14] represented the lattice by means of bipolar fuzzy graph. Alnaser et al. [15] defined bipolar intuitionistic fuzzy graphs and introduced its matrices. Gong and Hua [16] studied the topological indices of bipolar fuzzy incidence graphs.

As we know, independent sets and independent numbers are critical properties to characterize the sparsity of graphs, occupying a central position in graph theory research. Gupta et al. [17] raised a new concept called ipsd-graph, and some general structural characterizations of separable ipsd-graphs are determined. Tait and Timmons [18] computed the upper bound on the independence number of polarity graphs. Zhao [19] summarized the tricks on regular graphs that are extremal with respect to the number of independent sets. Dyer and Muller [20] stated that the number of independent sets in cocomparability graphs can be counted in linear time. Ortiz and Villanueva [21] investigated the number of maximal independent sets in caterpillar graphs. Haviland [22] determined novel upper bounds for the independent domination number of regular graphs. Ortiz and Villanueva [23] studied the maximal independent sets in grid graphs. More contexts on independent set in graph can be referred to Coja-Oghlan and Efthymiou [24], Ge and Stefankovic [25], Gaspers et al. [26], and Oh and Lee [27].

Although independent sets have been studied a lot in various graph settings, the related research of independent sets is still open in the fuzzy graph situation, especially in the bipolar fuzzy graph setting (see Muhiuddin et al. [28]). Due to the importance of independent sets in mathematics, computer science, molecular science and other fields, this paper studies the independent sets in the bipolar fuzzy graph setting. We define the independent sets of bipolar fuzzy graphs from a new perspective, and gives its features on special graph classes.

The rest of this paper is organized as follows: first, the definition of bipolar fuzzy graphs and some preliminary knowledge about bipolar fuzzy sets are given; secondly, the main new concepts of this paper are presented and certain characteristics are described from a theoretical point of view; then, we discuss the degree of independence for two kinds of product bipolar fuzzy graphs; finally, an algorithm to find the subgraph with the extreme value of fixed independence number is proposed.

II. Preliminary knowledge

In this section, we give some basic definitions and terms in preparation for the introduction of the main concepts in the coming section.

Manuscript received April 23, 2022; revised June 30, 2022. This work was supported in part by “Research on the Application of Artificial Intelligence Technology in Accurate Book Procurement in Universities under Smart Library” (no. 21YBA179).

Ke Luo is a researcher of Department of Library, Shaoyang University, Shaoyang, 422001 China (e-mail: luoke00@qq.com).

W. Gao is a professor of School of Information and Technology, Yunnan Normal University, Kunming, 650500 China (e-mail: gaowei@ynnu.edu.cn).

A. Bipolar fuzzy graph

Let V be a universal set and it can be regarded as vertex set in bipolar fuzzy graph setting. The set $A = \{(v, \iota_A^P(v), \iota_A^N(v)) : v \in V\}$ is the bipolar fuzzy set on V where $\iota_A^P : V \to [0,1]$ and $\iota_A^N : V \to [-1,0]$ are the positive membership function and negative membership function on V respectively, and set $B = \{((v,v'), \iota_B^P(v,v'), \iota_B^N(v,v')) : v, v' \in V \times V\}$ is the bipolar fuzzy set on $V \times V$ where $\iota_B^P : V \times V \to [0,1]$ and $\iota_b^N : V \times V \to [-1,0]$ are the positive membership function and negative membership function on V^2 respectively. $G = (V, A, B)$ is a bipolar fuzzy graph if

$$\iota_B^P(v,v') \le \min\{\iota_A^P(v), \iota_A^P(v')\}$$

and

$$\iota_B^N(v,v') \ge \max\{\iota_A^N(v), \iota_A^N(v')\}$$

hold for any vertex pair $(v,v') \in V^2$, and

$$\iota_B^P(v,v') = \iota_B^N(v,v') = 0$$

if vv' is not an edge in bipolar fuzzy graph G. An edge vv' is trivial if $\iota_B^P(v,v') = 0$ or $\iota_B^N(v,v') = 0$. We say a bipolar fuzzy graph $G = (V, A, B)$ is complete if

$$\iota_B^P(v,v') = \min\{\iota_A^P(v), \iota_A^P(v')\}$$

and

$$\iota_B^N(v,v') = \max\{\iota_A^N(v), \iota_A^N(v')\}$$

hold for any vertex pair $(v,v') \in V^2$. Throughout this article, $\wedge$ and $\vee$ are expressed as minimum and maximum operations, respectively.

For two bipolar fuzzy sets $A_1 = \{(v, \iota_{A_1}^P(v), \iota_{A_1}^N(v)) : v \in V\}$ and $A_2 = \{(v, \iota_{A_2}^P(v), \iota_{A_2}^N(v)) : v \in V\}$. Set

$$A_3 = A_1 \wedge A_2 = \{(v, \iota_{A_3}^P(v), \iota_{A_3}^N(v)) : v \in V\}$$

and

$$A_4 = A_1 \vee A_2 = \{(v, \iota_{A_4}^P(v), \iota_{A_4}^N(v)) : v \in V\}$$

where

$$\iota_{A_3}^P(v) = \min\{\iota_{A_1}^P(v), \iota_{A_2}^P(v)\},$$
$$\iota_{A_3}^N(v) = \max\{\iota_{A_1}^N(v), \iota_{A_2}^N(v)\},$$
$$\iota_{A_4}^P(v) = \max\{\iota_{A_1}^P(v), \iota_{A_2}^P(v)\},$$

and

$$\iota_{A_4}^N(v) = \min\{\iota_{A_1}^N(v), \iota_{A_2}^N(v)\}.$$

We say $A_1 \le A_2$ if $\iota_{A_1}^P(v) \le \iota_{A_2}^P(v)$ and $\iota_{A_1}^N(v) \ge \iota_{A_2}^N(v)$ for any $v \in V$.

The similar operator can be defined on V^2. For any two bipolar fuzzy sets $B_1 = \{((v,v'), \iota_{B_1}^P(v,v'), \iota_{B_1}^N(v,v')) : v, v' \in V\}$ and $B_2 = \{((v,v'), \iota_{B_2}^P(v,v'), \iota_{B_2}^N(v,v')) : v, v' \in V\}$, set

$$B_3 = B_1 \wedge B_2 = \{((v,v'), \iota_{B_3}^P(v,v'), \iota_{B_3}^N(v,v')) : v, v' \in V\}$$

and

$$B_4 = B_1 \vee B_2 = \{((v,v'), \iota_{B_4}^P(v,v'), \iota_{B_4}^N(v,v')) : v, v' \in V\},$$

where

$$\iota_{B_3}^P(v,v') = \min\{\iota_{B_1}^P(v,v'), \iota_{B_2}^P(v,v')\},$$
$$\iota_{B_3}^N(v,v') = \max\{\iota_{B_1}^N(v,v'), \iota_{B_2}^N(v,v')\},$$
$$\iota_{B_4}^P(v,v') = \max\{\iota_{B_1}^P(v,v'), \iota_{B_2}^P(v,v')\},$$

and

$$\iota_{B_4}^N(v,v') = \min\{\iota_{B_1}^N(v,v'), \iota_{B_2}^N(v,v')\}.$$

We say $B_1 \le B_2$ if $\iota_{B_1}^P(v,v') \le \iota_{B_2}^P(v,v')$ and $\iota_{B_1}^N(v,v') \ge \iota_{B_2}^N(v,v')$ for any $(v,v') \in V \times V$.

Let $G = (V, A, B)$ be a bipolar fuzzy graph and $S \subseteq V$ be a subset vertex set of V. The order of G is denoted by

$$O(G) = (O^P(G), O^N(G)),$$

where

$$O^P(G) = \sum_{v \in V} \iota_A^P(v)$$

and

$$O^N(G) = \sum_{v \in V} \iota_A^N(v)$$

are positive order and negative order of G, respectively. The size of G is denoted by

$$S(G) = (S^P(G), S^N(G)),$$

where

$$S^P(G) = \sum_{v,v' \in V \times V} \iota_B^P(v,v')$$

and

$$S^N(G) = \sum_{v,v' \in V \times V} \iota_B^N(v,v')$$

are positive size and negative size of G, respectively. For a given $S \subseteq V$, its bipolar cardinality of S can be written as

$$|S| = (\sum_{v \in S} \iota_A^P(v), \sum_{v \in S})\iota_A^N(v)),$$

where $\sum_{v \in S} \iota_A^P(v)$ and $\sum_{v \in S} \iota_A^P(v)$ are called positive cardinality and negative cardinality, respectively. For any $v \in V$, the neighborhood of v in G is denoted by

$$N(v) = \{v' \in V : \iota_B^P(v,v') > 0 \quad \text{or} \quad \iota_B^N(v,v') < 0\}.$$

The bipolar degree of v in G is formulated by

$$d(v) = (d^P(v), d^N(v)),$$

where

$$d^P(v) = \sum_{(v,v') \in V \times V} \iota_B^P(v,v')$$

and

$$d^N(v) = \sum_{(v,v') \in V \times V} \iota_B^N(v,v')$$

are positive degree and negative degree of v in G.

B. Brief overview on independent set in fuzzy graphs

In this subsection, we briefly review the existing definitions of independent sets for bipolar fuzzy graphs.

For a bipolar fuzzy graph $G = (V, A, B)$ and vv' is an edge in G. There are several definitions on strong edges and independent sets, and we introduce some of the most important ones. vv' is a positive strong edge if

$$\min\{\iota_A^P(v), \iota_A^P(v')\} \le 2\iota_B^P(v,v'),$$

and vv' is a negative strong edge if

$$\max\{\iota_A^N(v), \iota_A^N(v')\} \ge 2\iota_B^N(v,v').$$

vv' is a strong edge if it is both positive strong and negative strong. The independent strength of edge vv' is denoted by

$$I_G(vv') = (I_G^P(vv'), I_G^N(vv')),$$

where

$$I_G^P(vv') = \frac{\iota_B^P(v,v')}{\min\{\iota_A^P(v), \iota_A^P(v')\}}$$

and

$$I_G^N(vv') = \frac{\iota_B^N(v,v')}{\max\{\iota_A^N(v), \iota_A^N(v')\}}$$

are positive independent strength and negative independent strength, respectively. We expand it to $V \times V$ by defining $I_G^P(vv') = I_G^N(vv') = 0$ if vv' is not an edge in G. Moreover, $I_G(vv')$, $I_G^P(vv')$ and $I_G^N(vv')$ can be simply written by $I(vv')$, $I^P(vv')$ and $I^N(vv')$.

Two vertices in a bipolar fuzzy graph $G = (V, A, B)$ are said to be positive independent if there is no positive strong edge between them.They are negative independent if there is no negative strong edge between them. They are bipolar independent if there is no positive and negative strong edge between them. $S \subseteq V$ is a positive independent set of G if any two vertices in S are positive independent, S is a negative independent set of G if any two vertices in S are negative independent, and S is a bipolar independent set of G if any two vertices in S are bipolar independent. A positive independent set S is a maximal positive independent set if it has maximal cardinality among all positive independent sets, and this maximum cardinality is called the positive independence number of G which is denoted by $\gamma^P(G)$. A negative independent set S is a minimal negative independent set if it has minimal cardinality among all negative independent sets, and this minimal cardinality is called the negative independence number of G which is denoted by $\gamma^N(G)$. Thus, the independence number of bipolar fuzzy graph G is denoted by

$$\gamma(G) = (\gamma^P(G), \gamma^N(G)).$$

III. Bipolar independence in bipolar fuzzy graphs

The aim of this section is to introduce our novel idea of independent set in bipolar fuzzy graph setting. In crisp graph setting, any two vertices in an independent set have no edge to connect them, while in our setting, any vertex set is allowed to be an independent set with certain degree of independence.

A. Novel approach on the bipolar independence of bipolar fuzzy graph and theoretical analysis

Let $S \subseteq V$ be a vertex subset with n vertices. We define the degree of independence of S by

$$DI_n(S) = (DI_n^P(S), DI_n^N(S)),$$

where

$$DI_n^P(S) = 1 - \frac{\sum_{v,v' \in S} I^P(vv')}{\binom{n}{2}}$$

and

$$DI_n^N(S) = -1 + \frac{\sum_{v,v' \in S} I^N(vv')}{\binom{n}{2}}$$

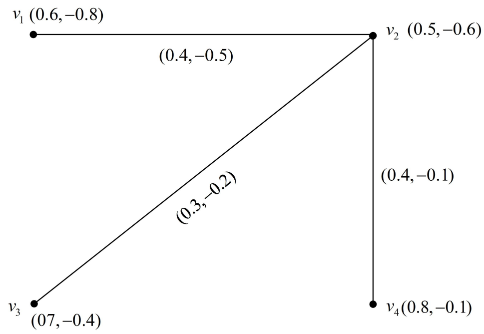

Fig. 1. The bipolar fuzzy graph G_1.

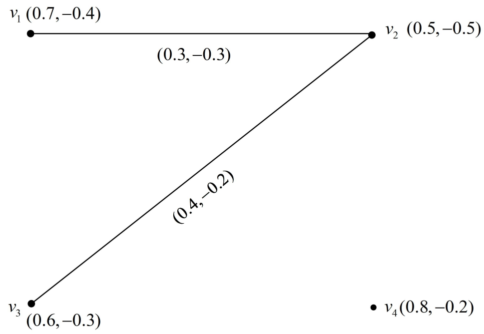

Fig. 2. The bipolar fuzzy graph G_2.

are the positive degree of independence of S and the negative degree of independence of S respectively, and $\binom{n}{2} = \frac{n(n-1)}{2}$.

Remark 1. If S is a set with no edges, $DI_n(S) = (1, -1)$ and such vertex subset become independent sets in the crisp graph. If bipolar fuzzy graph G is complete, and $|V| = n$ (G has n vertices). Then, $\sum_{v,v' \in V} I^P(vv') = \binom{n}{2}$ and $\sum_{v,v' \in V} I^N(vv') = \binom{n}{2}$, and thus $DI_n(V) = (0, 0)$. If $S = V$, then $DI_n(V)$, $DI_n^P(V)$ and $DI_n^N(V)$ can be written as $DI(G)$, $DI^P(G)$ and $DI^N(G)$ which denote the degree of independence, positive degree of independence and negative degree of independence of bipolar fuzzy graph G, respectively.

Example 1. Two bipolar fuzzy graphs G_1 and G_2 are described in Figure 1 and Figure 2, respectively.

In G_1, $V_1 = \{v_1, v_2, v_3, v_4\}$, $\iota_A^P(v_1) = 0.6$, $\iota_A^N(v_1) = -0.8$, $\iota_A^P(v_2) = 0.5$, $\iota_A^N(v_2) = -0.6$, $\iota_A^P(v_3) = 0.7$, $\iota_A^N(v_3) = -0.4$, $\iota_A^P(v_4) = 0.8$, $\iota_A^N(v_4) = -0.1$, $\iota_B^P(v_1, v_2) = 0.4$, $\iota_B^N(v_1, v_2) = -0.5$, $\iota_B^P(v_2, v_3) = 0.3$, $\iota_B^N(v_2, v_3) = -0.2$, $\iota_B^P(v_2, v_4) = 0.4$ and $\iota_B^N(v_2, v_4) = -0.1$. Thus,

$$\sum_{v,v' \in V_1} I^P(vv') = \frac{0.4}{0.5} + \frac{0.3}{0.5} + \frac{0.4}{0.5} = 2.2,$$

$$\sum_{v,v' \in V_1} I^N(vv') = \frac{-0.5}{-0.6} + \frac{-0.2}{-0.4} + \frac{-0.1}{-0.1} = \frac{7}{3},$$

$$DI^P(G_1) = 1 - \frac{2.2}{6} = \frac{19}{30}$$

and

$$DI^N(G_1) = -1 + \frac{7}{6 \times 3} = -\frac{11}{18}.$$

Hence, $DI(G_1) = (\frac{19}{30}, -\frac{11}{18})$.

In G_2, $V_2 = \{v_1, v_2, v_3, v_4\}$, $\iota_A^P(v_1) = 0.7$, $\iota_A^N(v_1) = -0.4$, $\iota_A^P(v_2) = 0.5$, $\iota_A^N(v_2) = -0.5$, $\iota_A^P(v_3) = 0.6$,

$\iota_A^N(v_3) = -0.3$, $\iota_A^P(v_4) = 0.8$, $\iota_A^N(v_4) = -0.2$, $\iota_B^P(v_1, v_2) = 0.3$, $\iota_B^N(v_1, v_2) = -0.3$, $\iota_B^P(v_2, v_3) = 0.4$, and $\iota_B^N(v_2, v_3) = -0.2$. Thus,

$$\sum_{v,v' \in V_2} I^P(vv') = \frac{0.3}{0.5} + \frac{0.4}{0.5} = \frac{7}{5},$$

$$\sum_{v,v' \in V_2} I^N(vv') = \frac{-0.3}{-0.4} + \frac{-0.2}{-0.3} = \frac{17}{12},$$

$$DI^P(G_2) = 1 - \frac{7}{5 \times 6} = \frac{23}{30}$$

and

$$DI^N(G_2) = -1 + \frac{17}{6 \times 12} = -\frac{55}{72}.$$

Hence, $DI(G_2) = (\frac{23}{30}, -\frac{55}{72})$.

In a bipolar fuzzy graph G, we say an edge vv' is a positive independent strong edge if $I^P(vv') \geq \frac{1}{2}$, i.e.,

$$\frac{\iota_B^P(v, v')}{\min\{\iota_A^P(v), \iota_A^P(v')\}} \geq \frac{1}{2}.$$

Say vv' is a negative independent strong edge if $I^N(vv') \geq \frac{1}{2}$, i.e.,

$$\frac{\iota_B^N(v, v')}{\max\{\iota_A^N(v), \iota_A^N(v')\}} \geq \frac{1}{2}.$$

Say vv' is an independent strong edge if it is both positive independent strong edge and negative independent strong edge.

In terms of the above definitions, we get the following characteristics.

Theorem 1. Let $G = (V, A, B)$ be a bipolar fuzzy graph. If there is an independent strong edge between every two vertices of G, then $DI^P(G) \leq \frac{1}{2}$ and $DI^N(G) \geq -\frac{1}{2}$.

Proof of Theorem 1. We only prove $DI^N(G) \geq -\frac{1}{2}$, and $DI^P(G) \leq \frac{1}{2}$ can be obtained using the similar trick.

Assume G has n vertices, and there is a negative independent strong edge between any two vertices in G, that is, $I^N(vv') \geq \frac{1}{2}$ for any $v, v' \in V$. Hence, we acquire

$$\frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n}{2}} \geq \frac{1}{2}$$

and

$$DI^N(G) = -1 + \frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n}{2}} \geq -1 + \frac{1}{2} = -\frac{1}{2}.$$

The desired result is proved. □

In terms of Theorem 1, we get the following corollary immediately.

Corollary 1. If a bipolar fuzzy graph G has no positive (resp. negative) independent strong edge, then $DI^P(G) > \frac{1}{2}$ (resp. $DI^N(G) < -\frac{1}{2}$).

Theorem 2. Let $H = (V', A, B)$ be a subgraph of bipolar fuzzy graph $G = (V, A, B)$ which is induced by V' with $|V| = n$ and $|V'| = n'$. Then, we have

$$\begin{aligned} DI^P(H) &= DI^P(G) - \frac{\binom{n}{2} - \binom{n'}{2}}{\binom{n}{2} \times \binom{n'}{2}} \sum_{v,v' \in V} I^P(vv') \quad (1) \\ &\quad + \frac{\sum_{v,v' \in V} I^P(vv') - \sum_{v,v' \in V'} I^P(vv')}{\binom{n'}{2}}. \end{aligned}$$

and

$$\begin{aligned} DI^N(H) &= DI^N(G) - \frac{\binom{n}{2} + \binom{n'}{2}}{\binom{n}{2} \times \binom{n'}{2}} \sum_{v,v' \in V} I^N(vv') \quad (2) \\ &\quad - \frac{\sum_{v,v' \in V} I^N(vv') - \sum_{v,v' \in V'} I^N(vv')}{\binom{n'}{2}}. \end{aligned}$$

Proof of Theorem 2. Here, we only give the detailed proof of (2), and the proof of (1) can be yielded in the same fashion.

According to the definition of negative degree of independence, we have

$$DI^N(G) = -1 + \frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n}{2}}$$

and

$$\begin{aligned} DI^N(H) &= -1 + \frac{\sum_{v,v' \in V'} I^N(vv')}{\binom{n'}{2}} \\ &= -1 + (\sum_{v,v' \in V} I^N(vv') - (\sum_{v,v' \in V} I^N(vv') \\ &\quad - \sum_{v,v' \in V'} I^N(vv'))) \setminus \binom{n'}{2}. \end{aligned}$$

Hence, we get

$$\begin{aligned} & DI^N(H) - DI^N(G) \\ = & -1 + (\sum_{v,v' \in V} I^N(vv') - (\sum_{v,v' \in V} I^N(vv') \\ & - \sum_{v,v' \in V'} I^N(vv'))) \setminus \binom{n'}{2} + 1 \\ & - \frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n}{2}} \\ = & -\frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n}{2}} + \frac{\sum_{v,v' \in V} I^N(vv')}{\binom{n'}{2}} \\ & - \frac{\sum_{v,v' \in V} I^N(vv') - \sum_{v,v' \in V'} I^N(vv')}{\binom{n'}{2}} \\ = & \frac{\binom{n}{2} - \binom{n'}{2}}{\binom{n}{2} \times \binom{n'}{2}} \sum_{v,v' \in V} I^N(vv') \\ & - \frac{\sum_{v,v' \in V} I^N(vv') - \sum_{v,v' \in V'} I^N(vv')}{\binom{n'}{2}}. \end{aligned}$$

The result for the negative part follows. □

In light of Theorem 2, we infer the following corollaries.

Corollary 2. Let $G = (V, A, B)$ be a bipolar fuzzy graph with n vertices and $x \in V$ be any vertex in G. Let H be a subgraph of G by deleting x from G. Then, we have

$$\begin{aligned} DI^P(H) &= DI^P(G) - \frac{4\sum_{v,v' \in V} I^P(vv')}{n(n-1)(n-2)} \\ &\quad + \frac{24\sum_{v,v' \in V} I^P(vv')}{(n-1)(n-2)}. \end{aligned} \quad (3)$$

and

$$\begin{aligned} DI^N(H) &= DI^N(G) + \frac{4\sum_{v,v' \in V} I^N(vv')}{n(n-1)(n-2)} \\ &\quad - \frac{24\sum_{v,v' \in V} I^N(vv')}{(n-1)(n-2)}. \end{aligned} \quad (4)$$

Proof of Corollary 2. Here, we only give the detailed proof of (4), and the proof of (3) can be yielded in the same fashion.

Using the definition of H, we have

$$\sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') = \sum_{x'\in V} \iota_B^N(x,x')$$

and $|V'| = n-1$.

In view of Theorem 2, we acquire

$$\begin{aligned} DI^N(H) &= DI^N(G) + \frac{\binom{n}{2}-\binom{n-1}{2}}{\binom{n}{2}\times\binom{n-1}{2}} \sum_{v,v'\in V} I^N(vv') \\ &\quad - \frac{\sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv')}{\binom{n-1}{2}} \\ &= DI^N(G) + \frac{4\sum_{v,v'\in V} I^N(vv')}{n(n-1)(n-2)} \\ &\quad - \frac{24\sum_{v,v'\in V} I^N(vv')}{(n-1)(n-2)}. \end{aligned}$$

Hence, the equation in corollary holds. □

Corollary 3. Let $G = (V, A, B)$ be a bipolar fuzzy graph and $H = (V', A, B)$ be a subgraph of G with $|V| = n$ and $|V'| = n'$. We have the following two statements.

(i) $DI^P(H) \le DI^P(G)$ if and only if

$$\begin{aligned} &\sum_{v,v'\in V} I^P(vv') - \sum_{v,v'\in V'} I^P(vv') \\ \le\ & \frac{\binom{n}{2}-\binom{n'}{2}}{\binom{n}{2}} \sum_{v,v'\in V} I^P(vv'). \end{aligned}$$

(ii) $DI^N(H) \le DI^N(G)$ if and only if

$$\begin{aligned} &\sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') \\ \ge\ & \frac{\binom{n}{2}-\binom{n'}{2}}{\binom{n}{2}} \sum_{v,v'\in V} I^N(vv'). \end{aligned}$$

Proof of Corollary 3. We only give the detailed proof of statement (ii).

Using Theorem 2 directly, we get

$$\begin{aligned} DI^N(H) &= DI^N(G) - \frac{\binom{n'}{2}-\binom{n}{2}}{\binom{n}{2}\times\binom{n'}{2}} \sum_{v,v'\in V} I^N(vv') \\ &\quad - \frac{\sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv')}{\binom{n'}{2}}. \end{aligned}$$

It follows that

$$\begin{aligned} & DI^N(H) \le DI^N(G) \\ \leftrightarrow\ & DI^N(G) - DI^N(H) \ge 0 \\ \leftrightarrow\ & \frac{\binom{n'}{2}-\binom{n}{2}}{\binom{n}{2}\times\binom{n'}{2}} \sum_{v,v'\in V} I^N(vv') \\ & + \frac{\sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv')}{\binom{n'}{2}} \ge 0 \\ \leftrightarrow\ & \sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') \\ & \ge \frac{\binom{n}{2}-\binom{n'}{2}}{\binom{n}{2}} \sum_{v,v'\in V} I^N(vv'). \end{aligned}$$

Hence, the statement (ii) is correct. □

Corollary 4. Let $G = (V, A, B)$ be a bipolar fuzzy graph and $H = (V', A, B)$ be a subgraph of G by deleting one vertex. Then

(1) $DI^P(H) = DI^P(G) \leftrightarrow \sum_{v,v'\in V} I^P(vv') - \sum_{v,v'\in V'} I^P(vv') = \frac{2}{n}\sum_{v,v'\in V} I^P(vv')$.

(2) $DI^P(H) < DI^P(G) \leftrightarrow \sum_{v,v'\in V} I^P(vv') - \sum_{v,v'\in V'} I^P(vv') < \frac{2}{n}\sum_{v,v'\in V} I^P(vv')$.

(3) $DI^P(H) > DI^P(G) \leftrightarrow \sum_{v,v'\in V} I^P(vv') - \sum_{v,v'\in V'} I^P(vv') > \frac{2}{n}\sum_{v,v'\in V} I^P(vv')$.

(4) $DI^N(H) = DI^N(G) \leftrightarrow \sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') = -\frac{2}{n}\sum_{v,v'\in V} I^N(vv')$.

(5) $DI^N(H) > DI^N(G) \leftrightarrow \sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') < -\frac{2}{n}\sum_{v,v'\in V} I^N(vv')$.

(6) $DI^N(H) < DI^N(G) \leftrightarrow \sum_{v,v'\in V} I^N(vv') - \sum_{v,v'\in V'} I^N(vv') > -\frac{2}{n}\sum_{v,v'\in V} I^N(vv')$.

Next, we introduce the classification of bipolar fuzzy graph in terms of positive degree of independence or negative degree of independence. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. The relation "$\sim^P$" and "$\sim^N$" between G_1 and G_2 are defined as follows:

- $G_1 \sim^P G_2 \leftrightarrow DI^P(G_1) = DI^P(G_2)$;
- $G_1 \sim^N G_2 \leftrightarrow DI^N(G_1) = DI^N(G_2)$.

After detailed analysis, we get the following result on $\sim^P$ and $\sim^N$ and we skip the detailed proof here.

Theorem 3. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. The following two statements are hold.

(i) The relation $\sim^P$ between G_1 and G_2 is an equivalence relation.

(ii) The relation $\sim^N$ between G_1 and G_2 is an equivalence relation.

The above two types of equivalence relations give two different classification methods of bipolar fuzzy graphs, which can be effectively divided according to actual application scenarios.

B. Bipolar independence in two classes of product bipolar fuzzy graph settings

In this subsection, we consider the bipolar independence for various types of products of bipolar fuzzy graphs.

Definition 1. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. The Cartesian product of G_1 and G_2 is denoted by $G_1 \times G_2 = (V_1 \times V_2, A_1 \times A_2, B_1 \times B_2)$ such that

(i) The vertex set of $G_1 \times G_2$ is denoted by $V_1 \times V_2 = \{(v_1, v_2)|v_1 \in V_1, v_2 \in V_2\}$.

(ii) Let E_1 and E_2 be the edge set of G_1 and G_2, respectively. The edge set of $G_1 \times G_2$ is denoted by $E = E' \cup E''$, where

$$E' = \{((v_1, v_2), (v_1, v_2^{'}))|v_1 \in V_1, v_2v_2^{'} \in E_2\}$$

and

$$E'' = \{((v_1, v_2), (v_1^{'}, v_2))|v_1v_1^{'} \in E_1, v_2 \in V_2\}.$$

(iii) For any $(v_1, v_2) \in V_1 \times V_2$,

$$\iota^P_{A_1\times A_2}(v_1, v_2) = \min\{\iota^P_{A_1}(v_1), \iota^P_{A_2}(v_2)\},$$

$$\iota^N_{A_1\times A_2}(v_1, v_2) = \max\{\iota^N_{A_1}(v_1), \iota^N_{A_2}(v_2)\}.$$

(iv) For any $((v_1,v_2),(v_1,v_2^{'}))\in E'$,

$$\iota^P_{B_1\times B_2}((v_1,v_2),(v_1,v_2^{'}))=\min\{\iota^P_{A_1}(v_1),\iota^P_{B_2}(v_2,v_2^{'})\},$$

$$\iota^N_{B_1\times B_2}((v_1,v_2),(v_1,v_2^{'}))=\max\{\iota^N_{A_1}(v_1),\iota^N_{B_2}(v_2,v_2^{'})\}.$$

For any $((v_1,v_2),(v_1^{'},v_2))\in E''$,

$$\iota^P_{B_1\times B_2}((v_1,v_2),(v_1^{'},v_2))=\min\{\iota^P_{B_1}(v_1,v_1^{'}),\iota^P_{A_2}(v_2)\},$$

$$\iota^N_{B_1\times B_2}((v_1,v_2),(v_1^{'},v_2))=\max\{\iota^N_{B_1}(v_1,v_1^{'}),\iota^N_{A_2}(v_2)\}.$$

The bound of degree of independence for the Cartesian product of bipolar fuzzy graphs can be characterized as follows (in Theorem 4, $DI^P_{n_1}(G_1)$, $DI^N_{n_1}(G_1)$, $DI^P_{n_2}(G_2)$, $DI^N_{n_2}(G_2)$, $DI^P_{n_1n_2}(G_1\times G_2)$ and $DI^N_{n_1n_2}(G_1\times G_2)$ can be simply written by $DI^P(G_1)$, $DI^N(G_1)$, $DI^P(G_2)$, $DI^N(G_2)$, $DI^P(G_1\times G_2)$ and $DI^N(G_1\times G_2)$, respectively).

Theorem 4. Let $G_1=(V_1,A_1,B_1)$ and $G_2=(V_2,A_2,B_2)$ be two bipolar fuzzy graphs with $|V_1|=n_1$ and $|V_2|=n_2$. Then, we have

$$DI^P(G_1\times G_2)\le 1-\frac{n_2\binom{n_1}{2}(1-DI^P(G_1))+n_1\binom{n_2}{2}(1-DI^P(G_2))}{\binom{n_1n_2}{2}},$$

$$DI^N(G_1\times G_2)\le -1+\frac{n_2\binom{n_1}{2}(-1+DI^N(G_1))+n_1\binom{n_2}{2}(-1+DI^N(G_2))}{\binom{n_1n_2}{2}}.$$

Proof of Theorem 4. We only present the detailed proof of the negative part.

By the definition of negative degree of independence, we have

$$DI^N(G_1)=-1+\frac{\sum_{v,v'\in V_1}I^N(vv')}{\binom{n_1}{2}}$$

and

$$DI^N(G_2)=-1+\frac{\sum_{v,v'\in V_2}I^N(vv')}{\binom{n_2}{2}}.$$

We discuss the two classes of edges E' and E'' in $G_1\times G_2$, respectively.

- For any $((v_1,v_2),(v_1,v_2^{'}))\in E'$, we have

$$\begin{aligned}
&I^N_{G_1\times G_2}((v_1,v_2),(v_1,v_2^{'}))\\
=&\frac{\iota^N_{B_1\times B_2}((v_1,v_2),(v_1,v_2^{'}))}{\max\{\iota^N_{A_1\times A_2}(v_1,v_2),\iota^N_{A_1\times A_2}(v_1,v_2^{'})\}}\\
=&\frac{\max\{\iota^N_{A_1}(v_1),\iota^N_{B_2}(v_2,v_2^{'})\}}{\max\{\iota^N_{A_1}(v_1),\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}}\\
=&\begin{cases}\frac{\max\{\iota^N_{A_1}(v_1),\iota^N_{B_2}(v_2,v_2^{'})\}}{\iota^N_{A_1}(v_1)}, & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}=\iota^N_{A_1}(v_1)\\ \frac{\iota^N_{B_2}(v_2,v_2^{'})}{\max\{\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}}, & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}\neq\iota^N_{A_1}(v_1)\end{cases}\\
=&\begin{cases}>I^N_{G_2}(v_2v_2^{'}), & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}=\iota^N_{A_1}(v_1)\\ =I^N_{G_2}(v_2v_2^{'}), & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_2}(v_2),\iota^N_{A_2}(v_2^{'})\}\neq\iota^N_{A_1}(v_1)\end{cases}\\
\ge& I^N_{G_2}(v_2v_2^{'}).
\end{aligned}$$

Hence,

$$\sum_{(v_1,v_2),(v_1,v_2^{'})\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1,v_2^{'}))\ge n_1I^N_{G_2}(v_2v_2^{'}).$$

- For any $((v_1,v_2),(v_1^{'},v_2))\in E''$, we get

$$\begin{aligned}
&I^N_{G_1\times G_2}((v_1,v_2),(v_1^{'},v_2))\\
=&\frac{\iota^N_{B_1\times B_2}((v_1,v_2),(v_1^{'},v_2))}{\max\{\iota^N_{A_1\times A_2}(v_1,v_2),\iota^N_{A_1\times A_2}(v_1^{'},v_2)\}}\\
=&\frac{\max\{\iota^N_{A_2}(v_2),\iota^N_{B_1}(v_1,v_1^{'})\}}{\max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'}),\iota^N_{A_2}(v_2)\}}\\
=&\begin{cases}\frac{\max\{\iota^N_{A_2}(v_2),\iota^N_{B_1}(v_1,v_1^{'})\}}{\iota^N_{A_2}(v_2)}, & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'}),\iota^N_{A_2}(v_2)\}=\iota^N_{A_2}(v_2)\\ \frac{\iota^N_{B_1}(v_1,v_1^{'})}{\max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'})\}}, & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'}),\iota^N_{A_2}(v_2)\}\neq\iota^N_{A_2}(v_2)\end{cases}\\
=&\begin{cases}>I^N_{G_1}(v_1v_1^{'}), & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'}),\iota^N_{A_2}(v_2)\}=\iota^N_{A_2}(v_2)\\ =I^N_{G_1}(v_1v_1^{'}), & \text{if}\quad \max\{\iota^N_{A_1}(v_1),\iota^N_{A_1}(v_1^{'}),\iota^N_{A_2}(v_2)\}\neq\iota^N_{A_2}(v_2)\end{cases}\\
\ge& I^N_{G_1}(v_1v_1^{'}).
\end{aligned}$$

Hence,

$$\sum_{(v_1,v_2),(v_1^{'},v_2)\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1^{'},v_2))\ge n_2I^N_{G_1}(v_1v_1^{'}).$$

Since $G_1\times G_2$ only has two kinds of edges E' and E'', we infer

$$\begin{aligned}
&\sum_{(v_1,v_2),(v_1^{'},v_2^{'})\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1^{'},v_2^{'}))\\
=&\sum_{(v_1,v_2),(v_1,v_2^{'})\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1,v_2^{'}))\\
&+\sum_{(v_1,v_2),(v_1^{'},v_2)\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1^{'},v_2))\\
\ge& n_1I^N_{G_2}(v_2v_2^{'})+n_2I^N_{G_1}(v_1v_1^{'})\\
=&n_2\binom{n_1}{2}(-1+DI^N(G_1))+n_1\binom{n_2}{2}(-1+DI^N(G_2)).
\end{aligned}$$

Theorem,

$$\begin{aligned}
&DI^N(G_1\times G_2)\\
\le&-1+\frac{\sum_{(v_1,v_2),(v_1^{'},v_2^{'})\in V_1\times V_2}I^N_{G_1\times G_2}((v_1,v_2),(v_1^{'},v_2^{'}))}{\binom{n_1n_2}{2}}\\
\le&-1+\frac{n_2\binom{n_1}{2}(-1+DI^N(G_1))+n_1\binom{n_2}{2}(-1+DI^N(G_2))}{\binom{n_1n_2}{2}}.
\end{aligned}$$

We finish the proof of Theorem 4. □

Now, we define the normal product of bipolar fuzzy graphs.

Definition 2. Let $G_1=(V_1,A_1,B_1)$ and $G_2=(V_2,A_2,B_2)$ be two bipolar fuzzy graphs. The normal product of G_1 and G_2 is denoted by $G_1\circ G_2=(V_1\times V_2,A,B)$ such that

(i) The vertex set of $G_1 \circ G_2$ is denoted by $V_1 \times V_2 = \{(v_1, v_2)|v_1 \in V_1, v_2 \in V_2\}$.
(ii) Let E_1 and E_2 be the edge set of G_1 and G_2, respectively. $((v_1, v_2), (v_1', v_2')) \in (V_1 \times V_2) \times (V_1 \times V_2)$ is an edge in $G_1 \circ G_2$ if one of the following conditions meets:

- $v_1 = v_1'$ and $v_2 v_2' \in E_2$;
- $v_2 = v_2'$ and $v_1 v_1' \in E_1$;
- $v_1 v_1' \in E_1$ and $v_2 v_2' \in E_2$.

(iii) For any $(v_1, v_2) \in V_1 \times V_2$,

$$\iota_A^P(v_1, v_2) = \min\{\iota_{A_1}^P(v_1), \iota_{A_2}^P(v_2)\},$$
$$\iota_A^N(v_1, v_2) = \max\{\iota_{A_1}^N(v_1), \iota_{A_2}^N(v_2)\}.$$

(iv) For any $((v_1, v_2), (v_1', v_2')) \in E(G_1 \circ G_2)$,

$$\begin{aligned}&\iota_B^P((v_1, v_2), (v_1', v_2'))\\ =& \begin{cases} \min\{\iota_{A_1}^P(v_1), \iota_{B_2}^P(v_2, v_2')\}, \\ \text{if} \quad v_1 = v_1' \quad \text{and} \quad v_2 v_2' \in E_2 \\ \min\{\iota_{A_2}^P(v_2), \iota_{B_1}^P(v_1, v_1')\}, \\ \text{if} \quad v_2 = v_2' \quad \text{and} \quad v_1 v_1' \in E_1 \\ \min\{\iota_{B_1}^P(v_1, v_1'), \iota_{B_2}^P(v_2, v_2')\}, \\ \text{if} \quad v_1 v_1' \in E_1 \quad \text{and} \quad v_2 v_2' \in E_2 \end{cases}\end{aligned}$$

$$\begin{aligned}&\iota_B^N((v_1, v_2), (v_1', v_2'))\\ =& \begin{cases} \max\{\iota_{A_1}^N(v_1), \iota_{B_2}^N(v_2, v_2')\}, \\ \text{if} \quad v_1 = v_1' \quad \text{and} \quad v_2 v_2' \in E_2 \\ \max\{\iota_{A_2}^N(v_2), \iota_{B_1}^N(v_1, v_1')\}, \\ \text{if} \quad v_2 = v_2' \quad \text{and} \quad v_1 v_1' \in E_1 \\ \max\{\iota_{B_1}^N(v_1, v_1'), \iota_{B_2}^N(v_2, v_2')\}, \\ \text{if} \quad v_1 v_1' \in E_1 \quad \text{and} \quad v_2 v_2' \in E_2 \end{cases}\end{aligned}$$

Theorem 5. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs with $|V_1| = n_1$, $|V_2| = n_2$, $|E_1| = m_1$ and $|E_2| = m_2$. Then, we have
(1) If $\iota_{B_1}^P(v_1, v_1') \leq \iota_{B_2}^P(v_2, v_2')$ and $\iota_{A_1}^P(v_1) \wedge \iota_{A_1}^P(v_1') \leq \iota_{A_2}^P(v_2) \wedge \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^P(G_1 \circ G_2) \\ \leq \;& 1 - \{(n_2 + 2m_2)\binom{n_1}{2}(1 - DI^P(G_1)) \\ & + n_1\binom{n_2}{2}(1 - DI^P(G_2))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(2) If $\iota_{B_1}^P(v_1, v_1') \geq \iota_{B_2}^P(v_2, v_2')$ and $\iota_{A_1}^P(v_1) \wedge \iota_{A_1}^P(v_1') \geq \iota_{A_2}^P(v_2) \wedge \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^P(G_1 \circ G_2) \\ \leq \;& 1 - \{(n_1 + 2m_1)\binom{n_2}{2}(1 - DI^P(G_2)) \\ & + n_1\binom{n_{\cdot}}{2}(1 - DI^P(G_1))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(3) If $\iota_{B_1}^P(v_1, v_1') \geq \iota_{B_2}^P(v_2, v_2')$ and $\iota_{A_1}^P(v_1) \wedge \iota_{A_1}^P(v_1') \leq \iota_{A_2}^P(v_2) \wedge \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^P(G_1 \circ G_2) \\ \leq \;& 1 - \{(n_1 + 2m_1)\binom{n_2}{2}(1 - DI^P(G_2)) \\ & + n_1\binom{n_{\cdot}}{2}(1 - DI^P(G_1))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(4) If $\iota_{B_1}^P(v_1, v_1') \leq \iota_{B_2}^P(v_2, v_2')$ and $\iota_{A_1}^P(v_1) \wedge \iota_{A_1}^P(v_1') \geq \iota_{A_2}^P(v_2) \wedge \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^P(G_1 \circ G_2) \\ \leq \;& 1 - \{(n_2 + 2m_2)\binom{n_1}{2}(1 - DI^P(G_1)) \\ & + n_1\binom{n_2}{2}(1 - DI^P(G_2))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(5) If $\iota_{B_1}^N(v_1, v_1') \geq \iota_{B_2}^N(v_2, v_2')$ and $\iota_{A_1}^N(v_1) \vee \iota_{A_1}^N(v_1') \geq \iota_{A_2}^N(v_2) \vee \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^N(G_1 \circ G_2) \\ \leq \;& -1 + \{(n_2 + 2m_2)\binom{n_1}{2}(1 - DI^N(G_1)) \\ & + n_1\binom{n_2}{2}(-1 + DI^N(G_2))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(6) If $\iota_{B_1}^N(v_1, v_1') \leq \iota_{B_2}^N(v_2, v_2')$ and $\iota_{A_1}^N(v_1) \vee \iota_{A_1}^N(v_1') \leq \iota_{A_2}^N(v_2) \vee \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^N(G_1 \circ G_2) \\ \leq \;& -1 + \{(n_1 + 2m_1)\binom{n_2}{2}(1 - DI^N(G_2)) \\ & + n_2\binom{n_1}{2}(-1 + DI^N(G_1))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(7) If $\iota_{B_1}^N(v_1, v_1') \geq \iota_{B_2}^N(v_2, v_2')$ and $\iota_{A_1}^N(v_1) \vee \iota_{A_1}^N(v_1') \leq \iota_{A_2}^N(v_2) \vee \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^N(G_1 \circ G_2) \\ \leq \;& -1 + \{(n_1 + 2m_1)\binom{n_2}{2}(1 - DI^N(G_2)) \\ & + n_2\binom{n_1}{2}(-1 + DI^N(G_1))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

(8) If $\iota_{B_1}^N(v_1, v_1') \geq \iota_{B_2}^N(v_2, v_2')$ and $\iota_{A_1}^N(v_1) \vee \iota_{A_1}^N(v_1') \leq \iota_{A_2}^N(v_2) \vee \iota_{A_2}^P(v_2')$ for any $(v_1, v_1') \in E_1$ and $(v_2, v_2') \in E_2$, then

$$\begin{aligned} & DI^N(G_1 \circ G_2) \\ \leq \;& -1 + \{(n_2 + 2m_2)\binom{n_1}{2}(1 - DI^N(G_1)) \\ & + n_1\binom{n_2}{2}(-1 + DI^N(G_2))\} \setminus \binom{n_1 n_2}{2}. \end{aligned}$$

We skip the detailed proof of Theorem 5 which can be done using the same tricks and discussions as the proof of Theorem 4.

C. Algorithms to search the maximum positive independent graph and the minimum negative independent graph

For a given bipolar fuzzy graph $G = (V, A, B)$, we say a fuzzy subgraph $H = (V', A', B')$ of G is a maximal positive independent subgraph with a given positive degree of independence (denoted by DI_c^P) if there is no bipolar fuzzy subgraph $H' = (V'', A'', B'')$ with $V' \subseteq V''$, which makes the positive degree of independence of H' larger or equal to DI_c^P. Similarly, we have the following definition for the negative side. W say a fuzzy subgraph $H = (V', A', B')$ of G is a minimal negative independent subgraph with given negative degree of independence (denoted by DI_c^N) if there is no bipolar fuzzy subgraph $H' = (V'', A'', B'')$ with $V' \subseteq V''$, which makes the negative degree of independence of H' smaller or equal to DI_c^N.

We have the algorithms to search the maximum positive independent graph with detailed positive degree of independence and minimal negative independent graph with specific negative degree of independence, respectively. Here, we only present the second algorithm to search the minimal negative independent graph with given DI_c^N.

Algorithm A: Find the minimal negative independent graph
Input: A bipolar fuzzy graph $G = (V, A, B)$ with fixed negative degree of independence DI_c^N.
A1: Compute the $I_G^N(vv')$ for each edge vv' in bipolar fuzzy graph G.
A2: Consider Θ_v as the sum of negative independent strengths of edges incident to a vertex $v \in V$. In the initialization, $\Theta_v = 0$ for all vertices v in V. Set Υ as the sum of the negative independent strengths of all edges in bipolar fuzzy graph G. Calculate $\Upsilon = \sum_{v,v'} \iota^N(vv')$.
A3: Compute $DI_n^N = -1 + \frac{\Upsilon}{\binom{n}{2}}$.
A4: If $DI_n^N \geq DI_c^N$, then the given bipolar fuzzy graph is the desired minimal negative fuzzy independent subgraph. On the contrary, we go to the step A5.
A5: Find Θ_v for every vertex v in G:

For each vertex v in bipolar fuzzy graph G
For each vertex v' in bipolar fuzzy graph G
$\Theta_v = \Theta_v + I_G^N(vv')$
End For
End For

A6: Find

$$\Omega = \min_{v \in V}\{\Theta_v\}$$

and we mark

$$v_{min} = \text{argmin}_{v \in V}\{\Theta_v\}.$$

A7: Remove v_{min} from the vertex set and re-set $V = V - v_{min}$. Moreover, the Θ_v for every vertex v is updated in the following procedure.

For each vertex v in bipolar fuzzy graph G.
$\Theta_v = \Theta_v - \iota^N(v, v_{min})$
End For

A8: Update $\Upsilon = \Upsilon - \Omega$ and $n = n - 1$.
A9: Find $DI_n^N = -1 + \frac{\Upsilon}{\binom{n}{2}}$.
A10: Judge whether the minimal negative independent subgraph is achieved or not in terms of the following programme.

If $DI_n^N \geq DI_c^N$, then
The desired minimal negative independent subgraph is achieved which is induced by the current vertex set V.
Else
Go to step A6
End For

Output: The minimal negative independent graph with given DI_c^N.

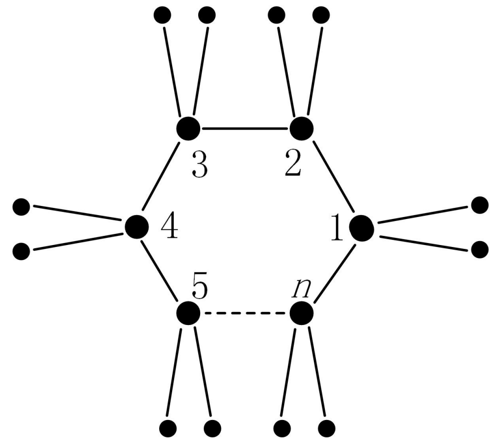

Fig. 3. The structure of cycloalkane C_n^2.

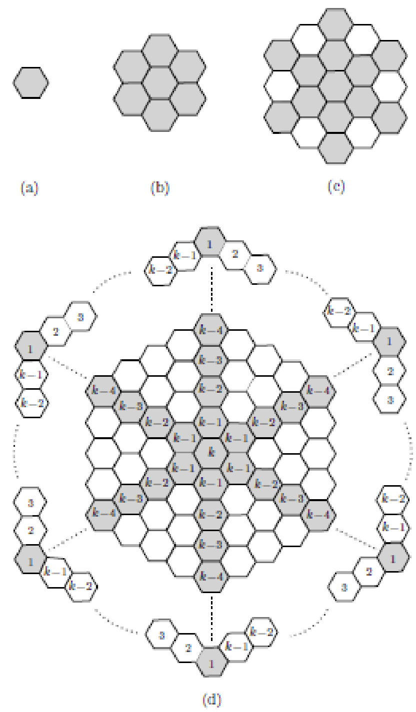

Fig. 4. The structure of Benzene H_k.

IV. Application in Chemical Compound

In this section, we apply the concept of bipolar independence to bipolar fuzzy graphs of chemical molecules. We consider the following chemical structures: cycloalkane C_n^2 (see Figure 3), Benzene H_k (see Figure 4) and One-sided polyamine chain $C(m,n)$ (see Figure 5). The bipolar membership function of the vertices is described according to the types and rotation characteristics of the atoms, and the bipolar membership functions of the edges in the chemical bipolar molecular graph are described according to the types of chemical bonds and the properties of the atoms at both ends of the chemical bonds.

Let $n = 20$ in C_n^2, $k = 6$ in H_k and $(m, n) = (5, 6)$ in $C(m, n)$. We determined that $DI^P(C_n^2) = 0.8$, $DI^N(C_n^2) = -0.4$, $DI^P(H_k) = 0.5$, $DI^N(H_k) = -0.6$, $DI^P(C(m, n)) = 0.2$ and $DI^N(C(m, n)) = -0.9$. It implies that $DI^P(C_n^2) > DI^P(H_k) > DI^P(H_k)$ and $DI^N(C_n^2) < DI^N(H_k) < DI^N(H_k)$.

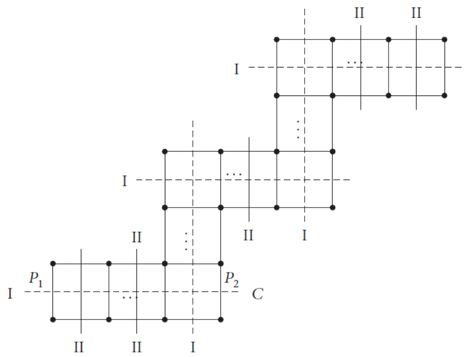

Fig. 5. The structure of one-sided polyamine chain $C(m,n)$.

V. Conclusion

In this paper, we introduce the degree of independence in bipolar fuzzy graph setting, and several characteristics are determined from a theoretical point of view, including the two classes of products of bipolar fuzzy graphs. The results obtained in this paper have guiding significance for the analysis of network structure and chemical molecular structure with uncertainty. The following content can be used as the subject of the ongoing research.

• The independent set in the network reflects the sparseness and connectivity of the network, especially the device-based network. Therefore, the user will be disconnected at special moments (for example, shutting down at night, no signal in the basement or virgin forest), and the connection status of each node in the network is uncertain. Under this condition, the fuzzy graph can be used to model the device-based network, analyze its independence, and obtain the relevant information (up and down link capacity, throughput, bandwidth, frequency, etc.) of the corresponding network.

• There are certain uncertainties in the structure of chemical molecules, such as the spin angle of atoms, the characteristics of chemical bond splitting and recombination, etc. We can define the membership function of atoms and chemical bonds, so as to convert molecular graphs into fuzzy molecular graphs, and analyze it from the perspective of fuzzy mathematics.

References

[1] M. H. Lan, J. Xu, and W. Gao, "Ontology feature extraction via vector learning algorithm and applied to similarity measuring and ontology mapping," *IAENG International Journal of Computer Science*, vol. 43, no. 1, pp. 10-19, 2016.

[2] L. L. Zhu, Y. Pan, M. K. Jamil, et al., "Boosting based ontology sparse vector computation approach," *Engineering Letters*, vol. 25, no. 4, pp. 406-415, 2017.

[3] L. L. Zhu, W. G. Tao, X. Z. Min, et al., "Theoretical characteristics of ontology learning algorithm in multi-dividing setting," *IAENG International Journal of Computer Science*, vol. 43, no. 2, pp. 184-191, 2016.

[4] S. Z. Zhou, "A result on fractional (a,b,k)-critical covered graphs," *Acta Mathematicae Applicatae Sinica-English Series*, vol. 37, no. 4, 657–664, 2021.

[5] W. Gao, L. Yan, Y. F. Li, et al., "Network performance analysis from binding number prospect," *Journal of Ambient Intelligence and Humanized Computing*, vol. 13, no. 3, 1259–1267, 2022.

[6] W. Gao, Y. J. Chen, and Y. Q. Wang, "Network vulnerability parameter and results on two surfaces," *International Journal of Intelligent Systems*, vol. 36, no. 8, 4392–4414, 2021.

[7] W. Gao, W. F. Wang, and Y. J. Chen, "Tight bounds for the existence of path factors in network vulnerability parameter settings," *International Journal of Intelligent Systems*, vol. 36, no. 3, 1134-1158, 2021.

[8] W. Gao, J. L. G. Guirao, and H. L. Wu, "Nordhaus-Gaddum type inequalities for some distance-based indices of bipartite molecular graphs," *Journal of Mathematical Chemistry*, vo. 58, no. 7, 1345–1352, 2020.

[9] S. Gong, X. X. Huang, C. H. Qiu, et al., "Locally ontology relaxed stability analysis in various ontology learning settings," *IAENG International Journal of Applied Mathematics*, vol. 52, no.2, 432–440, 2022.

[10] J. Z. Wu, J. B. Yuan, and W. Gao, "Toughness and fractional covered graph," *IAENG International Journal of Applied Mathematics*, vol. 52, no.2, 380–385, 2022.

[11] S. R. Islam and M. Pal, "First Zagreb index on a fuzzy graph and its application," *Journal of Intelligent & Fuzzy Systems*, vol. 40, no. 6, 10575–10587, 2021.

[12] S. Samanta, M. Pal, R. Mahapatra, et al., "A study on semi-directed graphs for social media networks," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, 1034–1041, 2021.

[13] S. Mathew, M. S. Sunitha, and N. Anjali, "Some connectivity concepts in bipolar fuzzy graphs," *Annals of Pure and Applied Mathematics,* vol. 7, no. 2, 98–100, 2014.

[14] P. K. Singh and Ch. A. Kumar, "Bipolar fuzzy graph representation of concept lattice," *Information Sciences*, vol. 288, 437–448, 2014.

[15] A. M. A. Alnaser, W. A. AlZoubi, and M. O. Massadeh, "Bipolar intuitionistic fuzzy graphs and its matrices," *Applied Mathematics & Information Sciences*, vol. 14, no. 2, 205–214, 2020.

[16] S. Gong and G. Hua, "Topological indices of bipolar fuzzy incidence graph," *Open Chemistry*, vol. 19, 894–903, 2021.

[17] P. Gupta, A. Goyal, and R. Jain, "Independent point-set dominating sets in graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 17, no. 1, 229–241, 2020.

[18] M. Tait and C. Timmons, "Independent sets in polarity graphs," *SIAM Journal on Discrete Mathematics*, vol. 30, no. 4, 2115–2129, 2016.

[19] Y. Z. Zhao, "Extremal regular graphs: independent sets and graph homomorphisms," *American Mathematical Monthly*, vol. 124, no. 9, 827–843, 2017.

[20] M. Dyer and H. Muller, "Counting independent sets in cocomparability graphs," *Information Processing Letters*, vol. 144, 31–36, 2019.

[21] C. Ortiz and M. Villanueva, "Maximal independent sets in caterpillar graphs," *Discrete Applied Mathematics*, vol. 160, no. 3, 259–266, 2012.

[22] J. Haviland, "Independent dominating sets in regular graphs," *Journal of Combinaitoral Optimization*, vol. 26, no. 1, 120–126, 2013.

[23] C. Ortiz and M. Villanueva, "Maximal independent sets in grid graphs," *International Transactions in Operational Research*, vol. 24, no. 1-2, 369–385, 2016.

[24] A. Coja-Oghlan and C. Efthymiou, "On independent sets in random graphs," *Random Structures & Algorithms*, vol. 47, no. 3, 436–486, 2015.

[25] Q. Ge and D. Stefankovic, "A graph polynomial for independent sets of bipartite graphs," *Combinatorics Probability & Computing*, vol. 21, no. 5, 695–714, 2012.

[26] S. Gaspers, D. Kratsch, M. Liedloff, "On independent sets and bicliques in graphs," *Algorithmica*, vol. 62, no. 3-4, 637–658, 2012.

[27] S. Oh and S. Lee, "Enumerating independent vertex sets in grid graphs," *Linear Algebra and Its Applications*, vol. 510, 192-204, 2016.

[28] G. Muhiuddin, T. Pramanik, A. M. Alanazi, et al., "Independent fuzzy graph: a new approach," *Proceedings of the National Academy of Sciences, India, Section A: Physical Sciences*, https://doi.org/10.1007/s40010-022-00769-w, 2022.

Ke Luo received the masters degree in computer science and theory from Yunnan Normal University, in 2010, He is currently an Librariar with the Shaoyang University Library, His research interests include image information encryption, artificial intelligence technology, and library modern technology.

Wei Gao was born in the city of Shaoxing, Zhejiang Province, China on Feb. 13, 1981. He acts as professor in the School of Information Science and Technology, Yunnan Normal University. As a researcher in computer science and mathematics, his interests are covering two disciplines: Graph theory, Statistical learning theory, Information retrieval, and Artificial Intelligence.