Palindromic Labeling on Some Graphs Related to $H-$graph

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***Abstract*—This paper deals with the palindromic labeling on some graphs related to $H-$graph. A bijection $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ is called palindromic labeling on graph G, if for every edge $uv \in E(G)$, there exists an edge $xy \in E(G)$ such that $f(x) = |V(G)| + 1 - f(u)$ and $f(y) = |V(G)| + 1 - f(v)$. If the graph G admits palindromic labeling, then the graph G is called a palindromic graph. In this paper, discussion on palindromic labeling for $H-$graph, the graph obtained by duplication of every vertex by an edge in $H-$graph, the graph obtained by duplication of every edge by a vertex in $H-$graph, the graph obtained by attaching each vertex of $H-$graph by a path P_m, $H \odot G$ (corona product of H and G) and the graph $H(2n, G)$, where G is any finite graph has been done.**

***Index Terms*—Palindromic labeling, Automorphism of graph G, $H-$graph, Duplication of a vertex by an edge in graph G, Duplication of an edge by a vertex in graph G, Corona product $G_1 \odot G_2$ and the graph $H(2n, G)$.**

I. INTRODUCTION

THE palindromic labeling is introduced by Beeler and has a great impact in the field of graph theory, where the symmetry of the graph can be studied and the complexity of the problem can be explored. Palindromic graphs are like palindromic numbers with their reflectional symmetry. A graph labeling can be expressed as the allocation of integers to the vertices or edges, or both, where some conditions are followed [1].

In view of labeling, Beeler provided the notion of palindromic labeling firstly to explore the structure of different graphs. Palindromic graph is related to automorophisms of the graph of order two which fixed at most one vertex of the graph [2]. Further, in the continuation of palindromic graph, a theory has been reported in the literature [3] which says - "A graph G is palindromic if and only if there exists an automorphism ϕ on the vertices of G such that ϕ is an involution having at most one fixed vertex".

Further, the path graph P_n, cycle graph C_n, complete graph K_n and the $n-$dimensional hypercube graph Q_n are palindromic, for all n, where it is evident that the Petersen graph is not palindromic [3]. There are some important results related to palindromic graph mentioned below [3]:

1) The complete bipartite graph $K_{m,n}$ is palindromic if and only if $m = n$ or at least one of m or n is even.
2) The double star $S_{m,n}$ is palindromic if and only if $m = n$.
3) The wheel graph W_n is palindromic if and only if n is even.

The graphs $Pal_1(G, S, p)$ and $Pal_2(G, S, p)$ have been introduced and the conditions to be palindromic are discussed. The join of two graphs and their cartesian product along with necessary and sufficient conditions for palindromic labeling are also reported in the literature [3]. Further, in view of odd and even palindromic graphs, some researchers have introduced the concept of even and odd palindromic graphs which emphasizes the involution with no fixed vertices as well as involution with exactly one fixed vertex [4] [5].

In the present study palindromic labeling for some $H-$graphs have been investigated. The different graph operations on $H-$ graph reveal the palindromic labeling and the results have been illustrated with the help of examples.

A. Definition:

An automorphism of graph G is a permutation of the vertices of graph G that maps edges to edges of graph G and nonedges to nonedges of graph G [6].

B. Definition:

The $H-$graph of path P_n is the graph obtained from two copies of P_n with the vertices $u_1, u_2, u_3, ..., u_n$ and $v_1, v_2, v_3, ..., v_n$ by joining the vertices $u_{\frac{n+1}{2}}$ to $v_{\frac{n+1}{2}}$ if n is odd and $u_{\frac{n}{2}+1}$ to $v_{\frac{n}{2}}$ if n is even [7].

C. Definition:

Duplication of a vertex v_k by a new edge $e = v_k^{'} v_k^{''}$ in a graph G produces a new graph $G^{'}$ such that $N(v_k^{'}) = \{v_k, v_k^{''}\}$ and $N(v_k^{''}) = \{v_k, v_k^{'}\}$ [8].

D. Definition:

Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{u, v\}$ [8].

E. Definition:

The corona $G_1 \odot G_2$ of two graphs G_1 (with n_1 vertices and m_1 edges) and G_2 (with n_2 vertices and m_2 edges) is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. It follows from the definition of the corona $|V(G_1 \odot G_2)| = n_1 + n_1 n_2$ and $|E(G_1 \odot G_2)| = m_1 + n_1 m_2 + n_1 n_2$ [9].

F. Definition:

The cycle union of graph G is acquired from a cycle C_n by attaching each vertex of the cycle by a graph G and is represented by $C(n, G)$ [10].

In a similar way to $C(n, G)$, we have defined $H(2n, G)$ which is obtained from $H-$graph by attaching each vertex of the $H-$graph by a graph G.

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II. Main Results.

Theorem 2.1: The $H-$graph admits palindromic labeling.

Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \leq i \leq n\}$. So, $|V(H)| = 2n$ and $|E(H)| = 2n - 1$.

Case-1: If n is odd,
$E(H) = \left\{ u_i u_{i+1}, v_i v_{i+1}, u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} \middle| 1 \leq i \leq n-1 \right\}$.
Define a function $f : V(H) \to \{1, 2, ..., 2n\}$, for all $1 \leq i \leq n$ as $f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2n+1-i & \text{if } x = v_i \end{cases}$.
Clearly, f is one-one and onto.
As $f(v_i) = |V(H)|+1-f(u_i)$, for all $1 \leq i \leq n$, $H-$graph admits palindromic labeling when n is odd.

Case-2: If n is even,
$E(H) = \left\{ u_i u_{i+1}, v_i v_{i+1}, u_{\frac{n}{2}+1} v_{\frac{n}{2}} \middle| 1 \leq i \leq n-1 \right\}$.
Define a function $f : V(H) \to \{1, 2, ..., 2n\}$, for all $1 \leq i \leq n$ as $f(x) = \begin{cases} i & \text{if } x = u_i, \\ n+i & \text{if } x = v_i \end{cases}$.
Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(H)| + 1 - f(u_i)$, for all $1 \leq i \leq n$, $H-$graph admits palindromic labeling when n is even. Thus, case-1 and case-2 show that $H-$graph admits palindromic labeling.

Illustration 2.1: Figure 1 demonstrates the palindromic labeling for $H-$graph when n is odd. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for $H-$graph when $n = 3$. Further, Figure 2 demonstrates the palindromic labeling for $H-$graph when n is even. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for $H-$graph when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

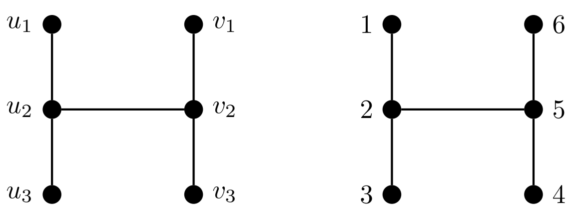

Fig. 1: The $H-$graph and it's corresponding palindromic labeling when $n = 3$.

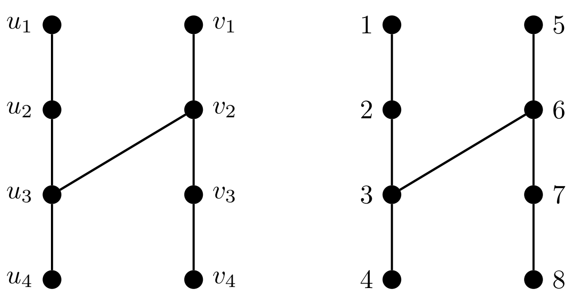

Fig. 2: The $H-$graph and it's corresponding Palindromic labeling when $n = 4$.

Theorem 2.2: The graph $G^{'}$ obtained by duplication of every vertex by an edge in $H-$graph admits palindromic labeling.

Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \leq i \leq n\}$. Let $G^{'}$ be the graph obtained by duplication of every vertex u_i by an edge $u_i^1 u_i^2$ and v_i by an edge $v_i^1 v_i^2$, for all $i = 1, 2, ..., n$. So, $|V(G^{'})| = 6n$ and $|E(G^{'})| = 8n - 1$.

Case-1: If n is odd,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{c} u_i^1 u_i^2, v_i^1 v_i^2, \\ u_i u_i^j, v_i v_i^j \end{array} \middle| \begin{array}{l} 1 \leq i \leq n, \\ j = 1, 2 \end{array} \right\}.$$

Define a function $f : V(G^{'}) \to \{1, 2, ..., 6n\}$, for all $1 \leq i \leq n$ and $j = 1, 2$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 6n+1-i & \text{if } x = v_i, \\ 2(i-1)+n+j & \text{if } x = u_i^j, \\ 6n+1-[2(i-1)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_i) = |V(G^{'})| + 1 - f(u_i)$ and $f(v_i^j) = |V(G^{'})| + 1 - f(u_i^j)$, for all $1 \leq i \leq n, j = 1, 2$, $G^{'}$ admits palindromic labeling when n is odd.

Case-2: If n is even,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{c} u_i^1 u_i^2, v_i^1 v_i^2, \\ u_i u_i^j, v_i v_i^j \end{array} \middle| \begin{array}{l} 1 \leq i \leq n, \\ j = 1, 2 \end{array} \right\}.$$

Define a function $f : V(G^{'}) \to \{1, 2, ..., 6n\}$, for all $1 \leq i \leq n$ and $j = 1, 2$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 5n+i & \text{if } x = v_i, \\ 2(i-1)+n+j & \text{if } x = u_i^j, \\ 6n+1-[2(n-i)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(G^{'})|+1-f(u_i)$ and $f(v_{n-i+1}^j) = |V(G^{'})|+1-f(u_i^j)$, for all $1 \leq i \leq n, j = 1, 2$, $G^{'}$ admits palindromic labeling when n is even. Thus, case-1 and case-2 show that $G^{'}$ admits palindromic labeling.

Illustration 2.2: Figure 3 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by duplication of every vertex by an edge in $H-$graph when n is odd. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 3$. Further, Figure 4 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by duplication of every vertex by an edge in $H-$graph when n is even. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

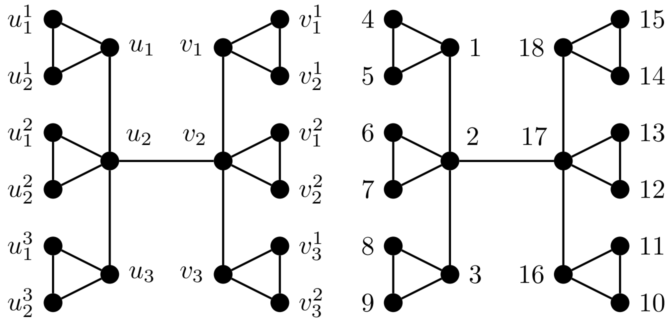

Fig. 3: The graph $G^{'}$ obtained by duplication of every vertex by an edge in $H-$graph and it's corresponding palindromic labeling when $n = 3$.

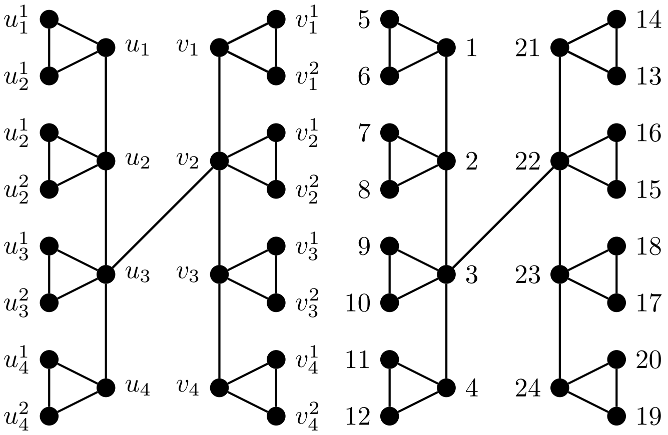

Fig. 4: The graph $G^{'}$ obtained by duplication of every vertex by an edge in $H-$graph and it's corresponding palindromic labeling when $n = 4$.

Theorem 2.3: The graph $G^{'}$ obtained by duplication of every edge by a vertex in $H-$graph admits palindromic labeling.
Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \leq i \leq n\}$. Let $G^{'}$ be the graph obtained by duplication of edge u_iu_{i+1} by a vertex $u_i^{'}$, duplication of edge v_iv_{i+1} by a vertex $v_i^{'}$ for all $i = 1, 2, ..., n-1$ and duplication of edge $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ by a vertex w when n is even and duplication of edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ by a vertex w when n is odd in $H-$graph. So, $|V(G^{'})| = 4n - 1$ and $|E(G^{'})| = 6n - 3$.
Case-1: If n is odd,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{l} u_{\frac{n+1}{2}}w, v_{\frac{n+1}{2}}w, u_iu_i^{'}, \\ u_{i+1}u_i^{'}, v_iv_i^{'}, v_{i+1}v_i^{'} \end{array} \middle| \; 1 \leq i \leq n \right\}.$$

Define a function $f : V(G^{'}) \rightarrow \{1, 2, ..., 4n-1\}$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, 1 \leq i \leq n, \\ 2n & \text{if } x = w, \\ 4n - i & \text{if } x = v_i, 1 \leq i \leq n, \\ n + i & \text{if } x = u_i^{'}, 1 \leq i \leq n-1, \\ 3n - i & \text{if } x = v_i^{'}, 1 \leq i \leq n-1 \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_i) = |V(G^{'})| + 1 - f(u_i)$, for all $1 \leq i \leq n$, $f(v_i^{'}) = |V(G^{'})| + 1 - f(u_i^{'})$, for all $1 \leq i \leq n-1$ and $f(w) = |V(G^{'})| + 1 - f(w)$, the graph $G^{'}$ admits palindromic labeling when n is odd.
Case-2: If n is even,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{l} u_{\frac{n}{2}+1}w, v_{\frac{n}{2}}w, u_iu_i^{'}, \\ u_{i+1}u_i^{'}, v_iv_i^{'}, v_{i+1}v_i^{'} \end{array} \middle| \; 1 \leq i \leq n \right\}.$$

Define a function $f : V(G^{'}) \rightarrow \{1, 2, ..., 4n-1\}$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, 1 \leq i \leq n, \\ 2n & \text{if } x = w, \\ 3n - 1 + i & \text{if } x = v_i, 1 \leq i \leq n, \\ n + i & \text{if } x = u_i^{'}, 1 \leq i \leq n-1, \\ 2n + i & \text{if } x = v_i^{'}, 1 \leq i \leq n-1 \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(G^{'})| + 1 - f(u_i)$, $f(v_{n-i+1}^{'}) = |V(G^{'})| + 1 - f(u_i^{'})$, for all $1 \leq i \leq n$ and $f(w) = |V(G^{'})| + 1 - f(w)$, the graph $G^{'}$ admits palindromic labeling when n is even. Thus, case-1 and case-2 show that $G^{'}$ admits palindromic labeling.

Illustration 2.3: Figure 5 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by duplication of every edge by a vertex in $H-$graph when n is odd. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 3$.

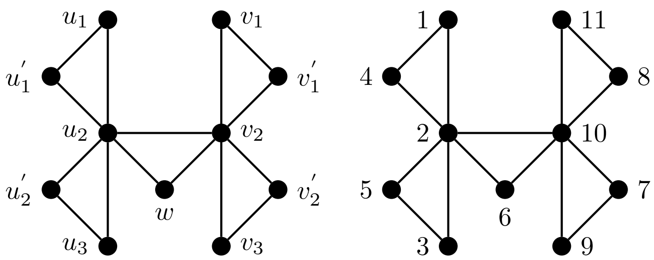

Fig. 5: The graph $G^{'}$ obtained by duplication of every edge by a vertex in $H-$graph and it's corresponding palindromic labeling when $n = 3$.

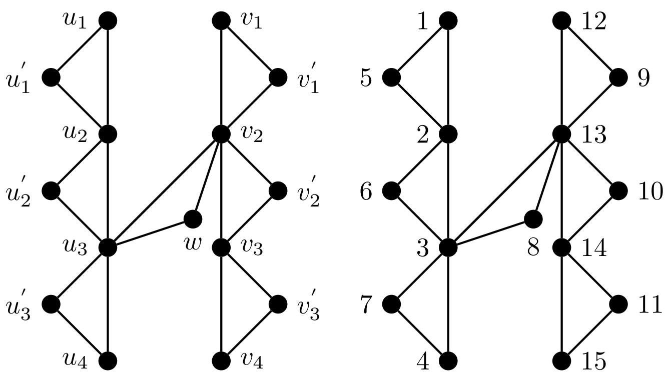

Fig. 6: The graph $G^{'}$ obtained by duplication of every edge by a vertex in $H-$graph and it's corresponding palindromic labeling when $n = 4$.

Further, Figure 6 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by duplication of every edge by a vertex in $H-$graph when n is even. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

Theorem 2.4: The graph $G^{'}$ obtained by attaching each vertex of $H-$graph by a path P_m admits palindromic labeling.

Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \leq i \leq n\}$. Let $G^{'}$ be a graph obtained by attaching each vertex of $H-$graph by a path P_m. Let $u_i^1, u_i^2, ..., u_i^{m-1}, u_i^m$ be the consecutive vertices of path P_m which is connected to vertex u_i of $H-$graph and $v_i^1, v_i^2, ..., v_i^{m-1}, v_i^m$ be the consecutive vertices of path P_m which is connected to vertex v_i in the graph $G^{'}$, for all $1 \leq i \leq n$. Without loss of generality, we may assume that $u_i^m = u_i$ and $v_i^m = v_i$, for all $1 \leq i \leq n$. So, $|V(G^{'})| = 2mn$ and $|E(G^{'})| = 2mn - 1$.

Case-1: If n is odd,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{c} u_i^j u_i^{j+1}, \\ v_i^j v_i^{j+1} \end{array} \middle| \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq m-1 \end{array} \right\}.$$

Define a function $f : V(G^{'}) \rightarrow \{1, 2, ..., 2mn\}$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2mn + 1 - i & \text{if } x = v_i, \\ (m-1)(i-1) + n + j & \text{if } x = u_i^j, \\ 2mn + 1 - [(m-1)(i-1) + n + j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. Since, $f(v_i) = |V(G^{'})| + 1 - f(u_i)$ and $f(v_i^j) = |V(G^{'})| + 1 - f(u_i^j)$, for all $1 \leq i \leq n, 1 \leq j \leq m-1$. Hence, the graph $G^{'}$ admits palindromic labeling when n is odd.

Case-2: If n is even,

$$E(G^{'}) = E(H) \bigcup \left\{ \begin{array}{c} u_i^j u_i^{j+1}, \\ v_i^j v_i^{j+1} \end{array} \middle| \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq m-1 \end{array} \right\}.$$

Define a function $f : V(G^{'}) \rightarrow \{1, 2, ..., 2mn\}$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2mn - (n - i) & \text{if } x = v_i, \\ (m-1)(i-1) + n + j & \text{if } x = u_i^j, \\ 2mn + 1 - [(m-1)(n-i) + n + j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(G^{'})| + 1 - f(u_i)$ and $f(v_{n-i+1}^j) = |V(G^{'})| + 1 - f(u_i^j)$, for all $1 \leq i \leq n, 1 \leq j \leq m-1$, the graph $G^{'}$ admits palindromic labeling when n is even. Thus, case-1 and case-2 show that $G^{'}$ admits palindromic labeling.

Illustration 2.4: Figure 7 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by attaching each vertex of $H-$graph by a path P_2 when n is odd. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 3$. Further, Figure 8 demonstrates the palindromic labeling for the graph $G^{'}$ obtained by attaching each vertex of $H-$graph by a path P_2 when n is even. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $G^{'}$ when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

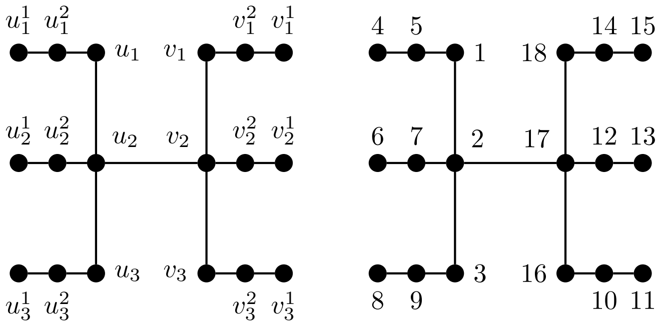

Fig. 7: The graph $G^{'}$ obtained by attaching each vertex of $H-$graph by a path P_2 and it's corresponding palindromic labeling when $n = 3$.

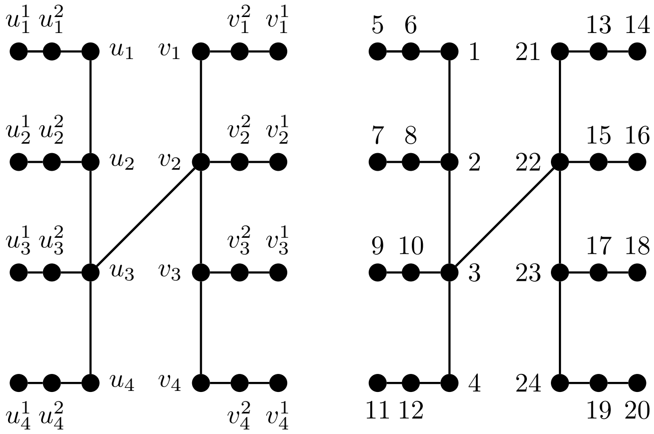

Fig. 8: The graph $G^{'}$ obtained by attaching each vertex of $H-$graph by a path P_2 and it's corresponding palindromic labeling when $n = 4$.

Theorem 2.5: The corona product $H \odot G$, where G is any finite graph admits palindromic labeling.

Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \leq i \leq n\}$. Let G be any graph with $|V(G)| = k$ and $|E(G)| = m$. Consider $2n$ copies of G as $G_1, G_2, ..., G_{2n}$. Let $u_i^1, u_i^2, ..., u_i^{k-1}, u_i^k$ be the consecutive vertices of graph G_i which are adjecent to u_i in the graph $H \odot G$, for all $1 \leq i \leq n$. Let $v_i^1, v_i^2, ..., v_i^{k-1}, v_i^k$ be the consecutive vertices of graph G_i which are adjecent to v_i in the graph $H \odot G$, for all $1 \leq i \leq n$. So, $|V(H \odot G)| = 2n(k+1)$ and $|E(H \odot G)| = 2n(m+k+1) - 1$.

Case-1: If n is odd,

$$E(H \odot G) = \bigcup_{i=1}^{2n} E(G_i) \bigcup E(H) \bigcup \left\{ u_i^j u_i, v_i^j v_i \middle| \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq k \end{array} \right\}.$$

Define a function $f : V(H \odot G) \rightarrow \{1, 2, ..., 2n(k+1)\}$, for all $1 \le i \le n$ and $1 \le j \le k$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2n(k+1)+1-i & \text{if } x = v_i, \\ k(i-1)+n+j & \text{if } x = u_i^j, \\ 2n(k+1)+1-[k(i-1)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_i) = |V(H \odot G)| + 1 - f(u_i)$ and $f(v_i^j) = |V(H \odot G)| + 1 - f(u_i^j)$, for all $1 \le i \le n, 1 \le j \le k$, the graph $H \odot G$ admits palindromic labeling when n is odd.

Case-2: If n is even,

$$E(H \odot G) = \bigcup_{i=1}^{2n} E(G_i) \bigcup E(H) \bigcup \left\{ u_i^j u_i, v_i^j v_i \,\middle|\, \begin{array}{l} 1 \le i \le n, \\ 1 \le j \le k \end{array} \right\}.$$

Define a function $f : V(H \odot G) \to \{1, 2, ..., 2n(k+1)\}$, for all $1 \le i \le n$ and $1 \le j \le k$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2n(k+1)-(n-i) & \text{if } x = v_i, \\ k(i-1)+n+j & \text{if } x = u_i^j, \\ 2n(k+1)+1-[k(n-i)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(H \odot G)| + 1 - f(u_i)$ and $f(v_{n-i+1}^j) = |V(H \odot G)| + 1 - f(u_i^j)$, for all $1 \le i \le n, 1 \le j \le k$, the graph $H \odot G$ admits palindromic labeling when n is even. Thus, case-1 and case-2 show that the graph $H \odot G$ admits palindromic labeling.

Illustration 2.5: Figure 9 demonstrates the palindromic labeling for the graph $H \odot 2K_1$ when n is odd. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $H \odot 2K_1$ when $n = 3$. Further, Figure 10 demonstrates the palindromic labeling for the graph $H \odot 2K_1$ when n is even. The left side figure shows the vertex labeling while the right side figure shows their corresponding palindromic labeling for the graph $H \odot 2K_1$ when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

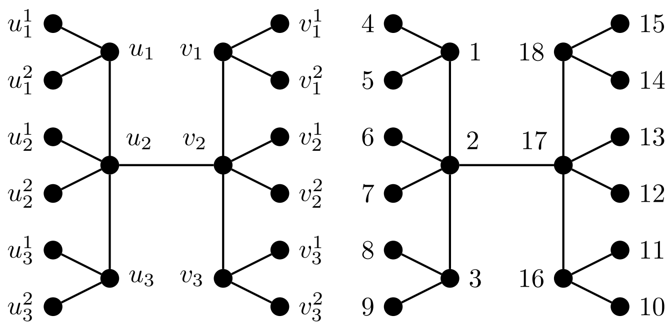

Fig. 9: The graph $H \odot 2K_1$ and it's corresponding palindromic labeling when $n = 3$.

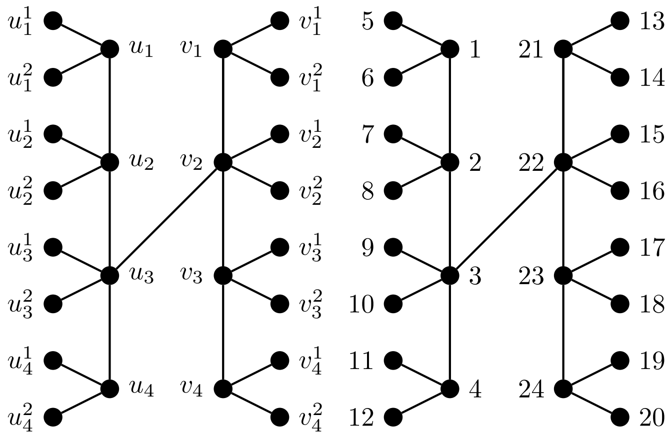

Fig. 10: The graph $H \odot 2K_1$ and it's corresponding palindromic labeling when $n = 4$.

Theorem 2.6: The graph $H(2n, G)$, where G is any finite graph admits palindromic labeling.

Proof: Let $H-$graph with $2n$ vertices and the vertex set $V(H) = \{u_i, v_i \mid 1 \le i \le n\}$. Let G be any finite graph with $|V(G)| = k$ and $|E(G)| = m$. Let $H(2n, G)$ be the graph obtained from $H-$graph by attaching each vertex of the $H-$graph by a graph G. Consider $2n$ copies of G as $G_1, G_2, ..., G_{2n}$. Let $u_i^1, u_i^2, ..., u_i^{k-1}, u_i^k$ be the consecutive vertices of graph G_i for first path in $H-$graph and $v_i^1, v_i^2, ..., v_i^{k-1}, v_i^k$ be the consecutive vertices of graph G_i for second path in $H-$graph, for all $1 \le i \le n$. Without loss of generality, we may assume that the union vertex of G_i with u_i is u_i^k and the union vertex of G_i with v_i is v_i^k. So, $|V(H(2n, G))| = 2nk$ and $|E(H(2n, G))| = 2n(m+1) - 1$.

Case-1: If n is odd,

$$E(H(2n, G)) = \bigcup_{i=1}^{2n} E(G_i) \bigcup E(H).$$

Define a function $f : V(H(2n, G)) \to \{1, 2, ..., 2nk\}$, for all $1 \le i \le n$ and $1 \le j \le k-1$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2nk+1-i & \text{if } x = v_i, \\ (k-1)(i-1)+n+j & \text{if } x = u_i^j, \\ 2nk+1-[(k-1)(i-1)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_i) = |V(H(2n, G))| + 1 - f(u_i)$ and $f(v_i^j) = |V(H(2n, G))| + 1 - f(u_i^j)$, for all $1 \le i \le n, 1 \le j \le k-1$, the graph $H(2n, G)$ admits palindromic labeling when n is odd.

Case-2: If n is even,

$$E(H(2n, G)) = \bigcup_{i=1}^{2n} E(G_i) \bigcup E(H).$$

Define a function $f : V(H(2n, G)) \to \{1, 2, ..., 2nk\}$, for all $1 \le i \le n$ and $1 \le j \le k-1$ as

$$f(x) = \begin{cases} i & \text{if } x = u_i, \\ 2nk-(n-i) & \text{if } x = v_i, \\ (k-1)(i-1)+n+j & \text{if } x = u_i^j, \\ 2nk+1-[(k-1)(n-i)+n+j] & \text{if } x = v_i^j \end{cases}.$$

Clearly, f is one-one and onto. As $f(v_{n-i+1}) = |V(H(2n,G))| + 1 - f(u_i)$ and $f(v_{n-i+1}^j) = |V(H(2n,G))| + 1 - f(u_i^j)$, for all $1 \leq i \leq n, 1 \leq j \leq k-1$, the graph $H(2n,G)$ admits palindromic labeling when n is even. Thus, case-1 and case-2 show that the graph $H(2n,G)$ admits palindromic labeling.

Illustration 2.6: Figure 11 demonstrates the palindromic labeling for the graph $H(2n,K_4)$ when n is odd. The upper part of Figure 11 shows the vertex labeling while the lower part of Figure 11 shows their corresponding palindromic labeling for the graph $H \odot 2K_1$ when $n = 3$. Further, Figure 12 demonstrates the palindromic labeling for the graph $H(2n,K_4)$ when n is even. The upper part of Figure 12 shows the vertex labeling while the lower part of Figure 12 shows their corresponding palindromic labeling for the graph $H \odot 2K_1$ when $n = 4$. Thus, both the cases are justified and admit palindromic labeling.

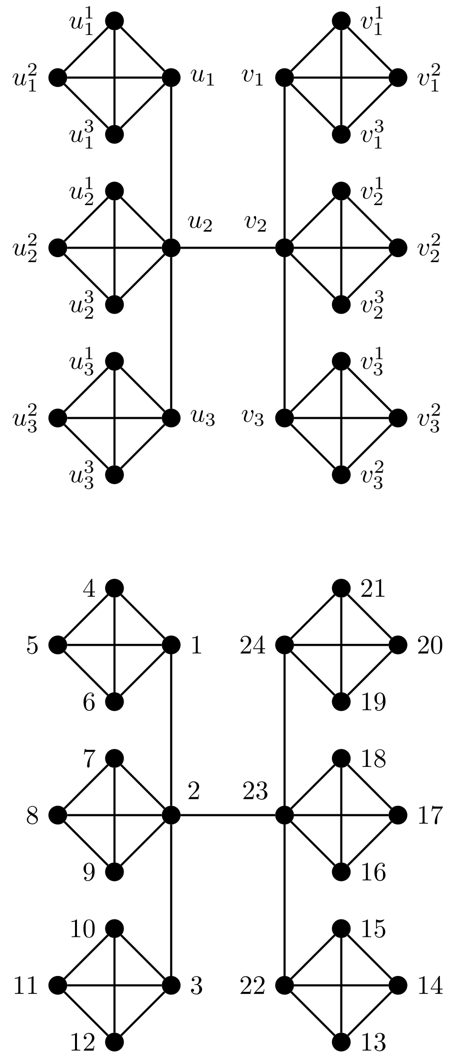

Fig. 11: The graph $H(2n,K_4)$ and it's corresponding palindromic labeling when $n = 3$.

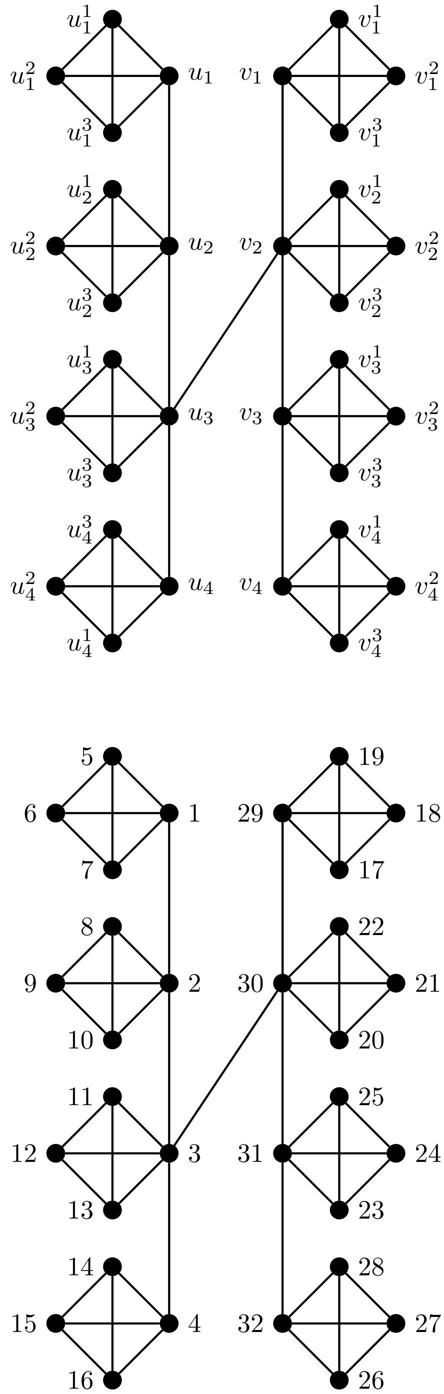

Fig. 12: The graph $H(2n,K_4)$ and it's corresponding palindromic labeling when $n = 4$.

III. Conclusion

In this paper, the palindromic labeling on some graphs related to $H-$graphs is investigated. The different operations like corona product of $H-$graph with finite graphs, duplication of every vertex by an edge and duplication of every edge by a vertex in $H-$graph for palindromic labeling was incorporated. The family of different $H-$graphs with the above mentioned operations were taken to anticipate the palindromic labeling and the obtained results justify the condition of palindromic labeling successfully. The results show that the family of different $H-$graphs are palindromic graphs. The obtained results are interpreted graphically with the help of examples where n is taken as odd and even in $H-$graphs. The graphical representations show the vertex labeling and their corresponding palindromic labeling. Further, in view of palindromic labeling and with the help of our results, analogous results for the different families of graphs like cycle graph, wheel graph etc. can be investigated. Thus, from figures 1-12, it is evident that the family of different $H-$graphs are palindromic graphs.

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