Some New Conclusions for K-g-fusion Frames in Hilbert Spaces

Qianping Guo, Jinsong Leng, and Houbiao Li

Abstract—**In this paper, we present some new equalities and inequalities for K-g-fusion frames in Hilbert spaces with the help of operator theory. Our results generalize and improve the remarkable results which have been obtained by Ahmadi et al.**

Index Terms—**K-g-fusion frame, Parseval K-g-fusion frame, K-g-fusion dual, Positive operator.**

I. Introduction

FRAMES were first proposed by Duffin and Schaeffer [1] in 1952 to address some problems in nonharmonic Fourier series. Frames, which generalize the concept of bases, can provide non-unique representations for a given vector [2]. Now, frame theory has been applied in signal processing [3],[4], computer science[5,6], among others. For more information on frame theory and its applications, we refer the readers to $[2, 7, 8]$.

In the study of longstanding conjecture of signal processing community:a signal can be reconstructed without the information about the phase. Based on this fact, Balan et al. [9] discovered a surprising Parseval frame identities and the authors of $[10, 11]$ extended these identities to alternate dual frames. Later on, many authors improved and developed some results see $[12-15]$.

Recently, g-fusion frames were proposed by the combination of g-frames and fusion frames. K-frames were introduced by Găvruta [16] for studying the nature of atomic systems with a bounded linear operator $K \in B(\mathcal{H})$. As is well known, K-frames are more general than the classical frames. Many properties of frames may not hold for K-frames and g-fusion frames [15,17]. K-g-fusion frames were proposed by Ahmadi, Rahimlon, Sadri [15] et al., and they discussed the duality and stability of K-g-fusion frames. Then, which properties of the classical frames may be extended to the K-g-fusion frames? In this paper, we mainly study the equalities and inequalities for K-g-fusion frames from the point of view of operator theory.

We need to recall some notations and basic definitions.

Throughout this work, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $B(\mathcal{H},\mathcal{K})$ is the collection of all bounded linear operators of $\mathcal{H}$ into $\mathcal{K}$. If $\mathcal{K}=\mathcal{H}$, we write $B(\mathcal{H},\mathcal{H})$ as $B(\mathcal{H})$, and $\{\mathcal{H}_j\}_{j\in J}$ is a sequence of Hilbert spaces, where J is a subset of integers $\mathbf{Z}$. Also, π_V is the orthogonal projection from $\mathcal{H}$ onto a closed subspace $V \subset \mathcal{H}$. $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

Definition 1.1 [15] Let $W=\{W_j\}_{j\in J}$ be a collection of closed subspaces of $\mathcal{H}$, $\{v_j\}_{j\in J}$ be a family of positive weights, and let $\Lambda_j \in B(\mathcal{H},\mathcal{H}_j), j\in J$ and $K\in B(\mathcal{H})$. We say $\Lambda := (W_j, \Lambda_j, v_j)$ is a K-g-fusion frame for $\mathcal{H}$ if there exist $0 < A \le B < \infty$ such that for all $f \in \mathcal{H}$

$$A\|K^*f\|^2 \le \sum_{j\in J} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \le B\|f\|^2. \qquad (1)$$

The constants A and B are called the lower and upper K-g-fusion frame bounds, respectively. If $\sum_{j\in J} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 = \|K^*f\|^2$, we call Λ a Parseval K-g-fusion frame for $\mathcal{H}$. If $K=I_{\mathcal{H}}$, we call Λ a g-fusion frame (see [15]). If $\Lambda_j=\pi_{W_j}$ for each $j\in J$, we call Λ a K-fusion frame, and if $K=I_{\mathcal{H}}$ and $\Lambda_j=\pi_{W_j}$ for each $j\in J$, we call Λ a fusion frame. So a K-g-fusion frame is a generalization of a fusion frame, g-fusion frame and K-fusion frame [18].

The synthesis and the analysis operators in the K-g-fusion frames are defined by [15]

$$T_\Lambda : \mathcal{H}_2 \to \mathcal{H}, \qquad T_\Lambda(\{f_j\}_{j\in J}) = \sum_{j\in J} v_j \pi_{W_j}\Lambda_j^* f_j,$$
$$T_\Lambda^* : \mathcal{H} \to \mathcal{H}_2, \qquad T_\Lambda^*(f) = \{v_j\Lambda_j\pi_{W_j}f\}_{j\in J}.$$

The K-g-fusion frame operator $S_\Lambda : \mathcal{H}\to\mathcal{H}$ defined by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in J} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f, \quad \forall f\in\mathcal{H}$$

which is positive, bounded and self adjoint [15]. It can be easily verify that

$$\langle S_\Lambda f, f\rangle = \sum_{j\in J} v_j^2\|\Lambda_j\pi_{W_j}f\|^2. \qquad (2)$$

Furthermore, if Λ is a K-g-fusion frame with bounds A and B, then from Equation (1), we have

$$\langle AKK^*f, f\rangle \le \langle S_\Lambda f, f\rangle \le \langle Bf, f\rangle. \qquad (3)$$

Like K-frames and K-fusion frames, the frame operator of the K-g-fusion frame is not invertible. But if $K\in B(\mathcal{H})$ has closed range, then S_Λ from $R(K)$ onto $S_\Lambda(R(K))$ is invertible [15].

Let $\Lambda=\{W_j,\Lambda_j,v_j\}$ be a K-g-fusion frame for $\mathcal{H}$ and $\tilde{\Lambda}=\{\tilde{W}_j,\tilde{\Lambda}_j,v_j\}$ be a K-g-fusion dual of Λ. Suppose that $I\subset J$ and we have the definition [15]

$$S_I f = \sum_{j\in I} v_j^2 \pi_{W_j}\Lambda_j^*\tilde{\Lambda}_j\pi_{\tilde{W}_j} f, \quad \forall f\in\mathcal{H}. \qquad (4)$$

Obviously, $S_I \in B(\mathcal{H})$, positive and $S_I + S_{I^c} = K$.

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Definition 1.2 [15] Let Λ be a K-g-fusion frame for $\mathcal{H}$. A g-fusion Bessel sequence $\tilde{\Lambda} = (\tilde{W}_j, \tilde{\Lambda}_j, v_j)$ is called a K-g-fusion dual of Λ, for each $f \in \mathcal{H}$, we have

$$Kf = \sum_{j\in J} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f.$$

Recently, Ahmadi et al. [15] obtained the following conclusions for K-g-fusion frames in Hilbert spaces based on the work in $[10, 11]$.

Theorem 1.1 [15] Let $K \in B(\mathcal{H})$ and $\Lambda = \{W_j, \Lambda_j, v_j\}$ be a K-g-fusion frame for $\mathcal{H}$. Suppose that $\tilde{\Lambda} = \{\tilde{W}_j, \tilde{\Lambda}_j, v_j\}$ is a K-g-fusion dual of Λ. Then for any $I \subset J$ and any $f \in \mathcal{H}$, we have

$$\begin{aligned} &\sum_{j\in I} v_j^2 \langle \tilde{\Lambda}_j \pi_{\tilde{W}_j} f, \Lambda_j \pi_{W_j} Kf\rangle - \|S_I f\|^2 \\ &= \sum_{j\in I^c} v_j^2 \overline{\langle \tilde{\Lambda}_j \pi_{\tilde{W}_j} f, \Lambda_j \pi_{W_j} Kf\rangle} - \|S_{I^c} f\|^2. \end{aligned} \tag{5}$$

Theorem 1.2 [15] Let $K \in B(\mathcal{H})$ and $\Lambda = \{W_j, \Lambda_j, v_j\}$ be a Parseval K-g-fusion frame for $\mathcal{H}$. If $I \subseteq J$ and $E \subseteq I^c$, then for every $f \in \mathcal{H}$, we get

$$\begin{aligned} \tfrac{3}{4}\|KK^* f\|^2 &\le Re(\sum_{j\in I} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle) \\ &\quad + \|\sum_{j\in I^c} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \\ &= Re(\sum_{j\in I^c} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle) \\ &\quad + \|\sum_{j\in I} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2. \end{aligned} \tag{6}$$

Motivated by the work of Balan et al. [10], in the Section 2, we continue this work about K-g-fusion frames and give some equalities and inequalities of these frames from the point of view of operator theory. Moreover, we also establish a new result for Parseval K-g-fusion frame associated with a scalar $\lambda \in [0, 1]$ and show that Theorem 1.2 is a particular case of our result when $\lambda = \frac{1}{2}$. Finally, we introduce some notations and get some results of the Parseval K-g-fusion frames.

II. Main results

To prove our main results, we shall briefly recall the following lemmas.

Lemma 2.1 [18] Suppose that $T \in B(\mathcal{H})$ has a closed range, then there exists a pseudo-inverse $T^\dagger \in B(\mathcal{H})$ of T such that

$$TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (T^\dagger)^* = (T^*)^\dagger.$$

Lemma 2.2 [12] Suppose that $U, V, T \in B(\mathcal{H})$, $U + V = T$, and the range of T is closed. Then we have

$$T^* T^\dagger U + V^* T^\dagger V = V^* T^\dagger T + U^* T^\dagger U.$$

Lemma 2.3 [13] If $U, V, M \in B(\mathcal{H})$ satisfy $U + V = M$, then

$$\begin{aligned} U^*U + \tfrac{1}{2}(V^*M + M^*V) &= V^*V + \tfrac{1}{2}(U^*M + M^*U) \\ &\ge \tfrac{3}{4} M^*M. \end{aligned}$$

Lemma 2.4 [13] If $U, V, M \in B(\mathcal{H})$ satisfy $U + V = MM^*$, then for any $\lambda \in [0, 1]$ we have

$$\begin{aligned} &U^*U + \lambda(V^*MM^* + MM^*V) \\ &= V^*V + (1-\lambda)(U^*MM^* + MM^*U) + (2\lambda - 1)(MM^*)^2 \\ &\ge (2\lambda - \lambda^2)(MM^*)^2. \end{aligned}$$

The following lemma is a known result for each Bessel sequence [2] and so is for K-g-fusion frames.

Lemma 2.5 Let K be a closed range operator and Λ be a K-g-fusion frame for $\mathcal{H}$. Then, for any $f \in \mathcal{H}$, we have

$$\|\sum_{j\in J} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \le \|S_\Lambda\| \sum_{j\in J} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2.$$

Firstly, we establish some inequalities for K-g-fusion frames.

Theorem 2.1 Let $K \in B(\mathcal{H})$ and $\Lambda = \{W_j, \Lambda_j, v_j\}$ be a K-g-fusion frame for $\mathcal{H}$. Suppose that $\tilde{\Lambda} = \{\tilde{W}_j, \tilde{\Lambda}_j, v_j\}$ is a K-g-fusion dual of Λ. Then for every $I \subset J$ and each $f \in \mathcal{H}$, we have

$$\begin{aligned} \tfrac{3}{4}\|Kf\|^2 &\le Re \sum_{j\in I} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle} \\ &\quad + \|\sum_{j\in I^c} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2 \\ &= Re \sum_{j\in I^c} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle} \\ &\quad + \|\sum_{j\in I} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2 \\ &\le \tfrac{3\|K\|^2 + \|S_I - S_{I^c}\|^2}{4} \|f\|^2, \end{aligned} \tag{7}$$

where the operator S_I is defined by Equation (4).

Proof For every $I \subset J$, applying Equation (4), we have $S_I + S_{I^c} = K$. It follows that

$$\begin{aligned} \langle K^* S_I f, f\rangle &= \langle S_I f, Kf\rangle \\ &= \langle \sum_{j\in I} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f, Kf\rangle \\ &= \sum_{j\in I} v_j^2 \langle \tilde{\Lambda}_j \pi_{\tilde{W}_j} f, \Lambda_j \pi_{W_j} Kf\rangle \\ &= \sum_{j\in I} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle}, \end{aligned}$$

that is, for each $f \in \mathcal{H}$, from Lemma 2.3, we have

$$\begin{aligned} &Re \sum_{j\in I} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle} + \|\sum_{j\in I^c} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2 \\ &= \tfrac{1}{2}(\langle S_I^* Kf, f\rangle + \langle K^* S_I f, f\rangle) + \|S_{I^c} f\|^2 \\ &= \tfrac{1}{2}(\langle S_{I^c}^* Kf, f\rangle + \langle K^* S_{I^c} f, f\rangle) + \|S_I f\|^2 \\ &= Re \sum_{j\in I^c} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle} + \|\sum_{j\in I} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2 \\ &\ge \tfrac{3}{4}\|Kf\|^2. \end{aligned}$$

Hence the left-hand inequality of the Equation (7) holds.

Next, we show that the right-hand inequality of the Equation (7). For any $f \in \mathcal{H}$, we obtain

$$\begin{aligned} &Re \sum_{j\in I^c} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} Kf, \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\rangle} + \|\sum_{j\in I} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2 \\ &= Re\langle S_{I^c}^* Kf, f\rangle + \langle S_I f, S_I f\rangle \\ &= Re\langle Kf, (K - S_I)f\rangle + \langle S_I f, S_I f\rangle \\ &= Re(\langle Kf, Kf\rangle - \langle Kf, S_I f\rangle) + \langle S_I f, S_I f\rangle \\ &= \langle Kf, Kf\rangle - Re\langle Kf, S_I f\rangle + \langle S_I f, S_I f\rangle \\ &= \langle Kf, Kf\rangle - Re\langle (K - S_I)f, S_I f\rangle \\ &= \langle Kf, Kf\rangle - Re\langle S_{I^c} f, S_I f\rangle \\ &= \langle Kf, Kf\rangle - \tfrac{1}{2}\langle S_I f, S_{I^c} f\rangle - \tfrac{1}{2}\langle S_{I^c} f, S_I f\rangle \\ &= \tfrac{3}{4}\|Kf\|^2 + \tfrac{1}{4}\langle S_I f + S_{I^c} f, S_I f + S_{I^c} f\rangle \\ &\quad -\tfrac{1}{2}\langle S_I f, S_{I^c} f\rangle - \tfrac{1}{2}\langle S_{I^c} f, S_I f\rangle \\ &= \tfrac{3}{4}\|Kf\|^2 + \tfrac{1}{4}\langle (S_I - S_{I^c})f, (S_I - S_{I^c})f\rangle \\ &\le \tfrac{3}{4}\|K\|^2\|f\|^2 + \tfrac{1}{4}\|S_I - S_{I^c}\|^2\|f\|^2 \\ &= \tfrac{3\|K\|^2 + \|S_I - S_{I^c}\|^2}{4}\|f\|^2. \end{aligned}$$

This completes the proof.

Remark 2.1 Note the equality of the Equation (7) involves the real parts of the complex numbers. Theorem 1.1 is a more general form which does not involve the real parts of the complex numbers. But the inequalities of the Equation (7) are new results.

In the sequel, we get a more general result. Suppose $\{t_j\}_{j\in J}$ is a bounded sequence of complex numbers. According to Lemma 2.3, we take

$$\begin{aligned} Uf &= \sum_{j\in J} t_j v_j^2 \pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f,\\ Vf &= \sum_{j\in J}(1-t_j) v_j^2 \pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f. \end{aligned}$$

Similarly to the proof of Theorem 2.1, we have the result as follows.

Theorem 2.2 Let $K\in B(\mathcal{H})$ and $\Lambda=\{W_j,\Lambda_j,v_j\}$ be a K-g-fusion frame for $\mathcal{H}$. Suppose that $\tilde{\Lambda}=\{\tilde{W_j},\tilde{\Lambda_j},v_j\}$ is a K-g-fusion dual of Λ. Then for all bounded sequence $\{t_j\}_{j\in J}$ and all $f\in\mathcal{H}$, we have

$$\begin{aligned} &Re\sum_{j\in J} t_j v_j^2 \overline{\langle \Lambda_j\pi_{W_j}Kf, \tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle}\\ &\quad +\|\sum_{j\in J}(1-t_j)v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\|^2\\ &= Re\sum_{j\in J}(1-t_j)v_j^2\overline{\langle \Lambda_j\pi_{W_j}Kf, \tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle}\\ &\quad +\|\sum_{j\in J} t_j v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\|^2\\ &\geq \tfrac{3}{4}\|Kf\|^2. \end{aligned}$$

Proof From the left-hand inequality of the Equation (7) if we take $I\subset J$,

$$t_j=\begin{cases} 1, & j\in I\\ 0, & j\in I^c \end{cases}$$

we conclude that the Theorem 2.2 holds.

This completes the proof.

Theorem 2.3 Suppose that $K\in B(\mathcal{H})$ is positive and it has closed range. Let $\Lambda=\{W_j,\Lambda_j,v_j\}$ be a K-g-fusion frame for $\mathcal{H}$ and $\tilde{\Lambda}=\{\tilde{W_j},\tilde{\Lambda_j},v_j\}$ be a K-g-fusion dual of Λ. Then for every $I\subset J$ and $f\in\mathcal{H}$, we obtain

$$\begin{aligned} &Re\sum_{j\in I} v_j^2\langle \tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \Lambda_j\pi_{W_j}K^\dagger Kf\rangle\\ &\quad +\langle \sum_{j\in I^c} v_j^2 K^\dagger\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &= Re\sum_{j\in I^c} v_j^2\langle \Lambda_j\pi_{W_j}K^\dagger Kf, \tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &\quad +\langle \sum_{j\in I} v_j^2 K^\dagger\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &\geq \tfrac{3}{4}\|K^{\frac{1}{2}}f\|^2, \end{aligned}$$

where $K^\dagger$ denotes the pseudo-inverse of K.

Proof Since $K\in B(\mathcal{H})$ is positive and has closed range, by using Lemma 2.1, we have $(K^\dagger)^*=(K^*)^\dagger=K^\dagger$. Obviously, for any $f\in\mathcal{H}$, $\langle K^\dagger S_I f, S_I f\rangle, \langle K^\dagger S_{I^c} f, S_{I^c} f\rangle\in\mathbf{R}$. According to Lemma 2.2, replace U and V by S_I and S_{I^c} yields that

$$\begin{aligned} &Re\sum_{j\in I} v_j^2\langle \tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \Lambda_j\pi_{W_j}K^\dagger Kf\rangle\\ &\quad +\langle \sum_{j\in I^c} v_j^2 K^\dagger\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &= Re\langle S_I f, K^\dagger Kf\rangle + \langle K^\dagger S_{I^c} f, S_{I^c} f\rangle\\ &= Re\langle K^*(K^\dagger)^* S_I f, f\rangle + \langle S_{I^c}^* K^\dagger S_{I^c} f, f\rangle\\ &= Re\langle (K^*K^\dagger S_I + S_{I^c}^* K^\dagger S_{I^c}) f, f\rangle\\ &= Re\langle (S_{I^c}^* K^\dagger K + S_I^* K^\dagger S_I) f, f\rangle\\ &= Re(\langle S_{I^c}^* K^\dagger Kf, f\rangle + \langle S_I^* K^\dagger S_I f, f\rangle)\\ &= Re(\langle K^\dagger Kf, S_{I^c} f\rangle + \langle K^\dagger S_I f, S_I f\rangle)\\ &= Re\overline{\langle S_{I^c} f, K^\dagger Kf\rangle} + \langle K^\dagger S_I f, S_I f\rangle\\ &= Re\sum_{j\in I^c} v_j^2\langle \Lambda_j\pi_{W_j}K^\dagger Kf, \tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &\quad +\langle \sum_{j\in I} v_j^2 K^\dagger\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle. \end{aligned}$$

According to Lemma 2.1 and 2.2, we conclude that

$$\begin{aligned} &Re\sum_{j\in I} v_j^2\langle \tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \Lambda_j\pi_{W_j}K^\dagger Kf\rangle\\ &\quad +\langle \sum_{j\in I^c} v_j^2 K^\dagger\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f, \sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\tilde{\Lambda_j}\pi_{\tilde{W_j}} f\rangle\\ &= Re\langle (K^\dagger K S_I + S_{I^c}^* K^\dagger S_{I^c}) f, f\rangle\\ &= Re\langle (KK^\dagger(K - S_{I^c}) + S_{I^c}^* K^\dagger S_{I^c}) f, f\rangle\\ &= \langle Kf, f\rangle - Re\langle KK^\dagger S_{I^c}^* f, f\rangle + \langle S_{I^c}^* K^\dagger S_{I^c} f, f\rangle\\ &= \langle K^{\frac{1}{2}} f, K^{\frac{1}{2}} f\rangle - Re\langle K^{\frac{1}{2}}K^{\frac{1}{2}}K^\dagger S_{I^c} f, f\rangle\\ &\quad +\langle (K^{\frac{1}{2}}K^\dagger S_{I^c})^*(K^{\frac{1}{2}}K^\dagger S_{I^c}) f, f\rangle\\ &= \tfrac{3}{4}\|K^{\frac{1}{2}} f\|^2 + \langle \tfrac{1}{2}K^{\frac{1}{2}} f - K^{\frac{1}{2}}K^\dagger S_{I^c} f, \tfrac{1}{2}K^{\frac{1}{2}} f - K^{\frac{1}{2}}K^\dagger S_{I^c} f\rangle\\ &\geq \tfrac{3}{4}\|K^{\frac{1}{2}} f\|^2 \end{aligned}$$

for every $f\in\mathcal{H}$. This completes the proof.

In the following theorem, we establish a generalization of the result from Theorems 1.2 to Parseval K-g-fusion frames, where a scalar $\lambda\in[0,1]$ is involved.

Theorem 2.4 Let $K\in B(\mathcal{H})$ and $\Lambda=\{W_j,\Lambda_j,v_j\}$ is a Parseval K-g-fusion frame for $\mathcal{H}$. Then for any $\lambda\in[0,1]$, for all $I\subset J$ and $f\in\mathcal{H}$, we have

$$\begin{aligned} &2\lambda(Re\sum_{j\in I^c} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f, \Lambda_j\pi_{W_j} f\rangle})\\ &\quad +\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f\|^2\\ &= 2(1-\lambda)(Re\sum_{j\in I} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f, \Lambda_j\pi_{W_j} f\rangle})\\ &\quad +\|\sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f\|^2 + (2\lambda-1)\|KK^*f\|^2\\ &\geq (2\lambda-\lambda^2)\|KK^*f\|^2. \end{aligned}$$

Proof For $I\subset J$, we consider a new operator, let

$$S_{\Lambda I} f := \sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f. \tag{8}$$

According to the K-g-fusion frame operator S_Λ, we get $S_{\Lambda I}$ is positive, bounded and self adjoint. Furthermore, by the definition of Parsevel K-g-fusion frame, we have

$$S_{\Lambda I} + S_{\Lambda I^c} = KK^*.$$

This, together with Lemma 2.4, replace U and V by $S_{\Lambda I}$

and $S_{\Lambda I^c}$, implies that

$$\begin{aligned}
&2\lambda(Re\sum_{j\in I^c} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f, \Lambda_j\pi_{W_j}f\rangle})\\
&\quad +\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2\\
&=\lambda(\langle S_{\Lambda I^c}KK^*f,f\rangle+\langle S_{\Lambda I^c}f,KK^*f\rangle)+\|S_{\Lambda I}f\|^2\\
&=\lambda(\langle S_{\Lambda I^c}KK^*f,f\rangle+\langle KK^*S_{\Lambda I^c}f,f\rangle)+\langle S_{\Lambda I}S_{\Lambda I}f,f\rangle\\
&=\langle S_{\Lambda I^c}S_{\Lambda I^c}f,f\rangle+(1-\lambda)(\langle KK^*S_{\Lambda I}f,f\rangle\\
&\quad +\langle S_{\Lambda I}KK^*f,f\rangle)+(2\lambda-1)\|KK^*f\|^2\\
&=2(1-\lambda)(Re\sum_{j\in I} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f, \Lambda_j\pi_{W_j}f\rangle})\\
&\quad +\|\sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2+(2\lambda-1)\|KK^*f\|^2\\
&\geq (2\lambda-\lambda^2)\|KK^*f\|^2
\end{aligned}$$

for any $\lambda\in[0,1]$ and every $f\in\mathcal{H}$ and the proof is finished.

Remark 2.2 Clearly, when $\lambda=\frac{1}{2}$ in Theorem 2.4, which was obtained the Theorem 1.2 (i.e., Theorem 3.6 in [15]) as a particular case from the above result. When $\lambda=0$ in Theorem 2.4, which was the operator in Equation (8).

Inequality (6) in Theorem 1.2 leads us to introduce the following concept, which is generalization of [11] for Parseval frames. Let Λ be a Parseval K-g-fusion frame, define

$$v_+(\Lambda,K,I)=\sup_{f\neq 0}\frac{Re(\sum_{j\in I^c} v_j^2\langle \Lambda_j\pi_{W_j}f,\Lambda_j\pi_{W_j}KK^*f\rangle)+\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2}{\|KK^*f\|^2},$$

and

$$v_-(\Lambda,K,I)=\inf_{f\neq 0}\frac{Re(\sum_{j\in I^c} v_j^2\langle \Lambda_j\pi_{W_j}f,\Lambda_j\pi_{W_j}KK^*f\rangle)+\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2}{\|KK^*f\|^2}.$$

Next, we will present some results of these notations.

Theorem 2.5 Let Λ is a Parseval K-g-fusion frame for $\mathcal{H}$. The following assertions hold:

(1) $\frac{3}{4}\leq v_-(\Lambda,K,I)\leq v_+(\Lambda,K,I)\leq \|K\|\|K^+\|(1+\|K\|)$.
(2) $v_+(\Lambda,K,I)=v_+(\Lambda,K,I^c)$,
$v_-(\Lambda,K,I)=v_-(\Lambda,K,I^c)$.
(3) $v_+(\Lambda,K,I)=v_-(\Lambda,K,I)=1$,
$v_+(\Lambda,K,\emptyset)=v_-(\Lambda,K,\emptyset)=1$.

Proof By the inequality (6), $\frac{3}{4}\leq v_-(\Lambda,K,I)$ holds trivially.

Since Λ is a Bessel sequence, by Lemma 2.5 we get

$$\begin{aligned}
\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2 &\leq \|S_\Lambda\|\sum_{j\in I} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\\
&\leq \|S_\Lambda\|\sum_{j\in J} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\\
&\leq \|K\|^2\|K^*f\|^2\\
&= \|K\|^2\|K^\dagger KK^*f\|^2\\
&\leq \|K\|^2\|K^\dagger\|^2\|KK^*f\|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&Re(\sum_{j\in I^c} v_j^2\langle \Lambda_j\pi_{W_j}f,\Lambda_j\pi_{W_j}KK^*f\rangle)\\
&\leq (\sum_{j\in I^c} v_j^2\|\Lambda_j\pi_{W_j}f\|^2)^{\frac{1}{2}}(\sum_{j\in I^c} v_j^2\|\Lambda_j\pi_{W_j}KK^*f\|^2)^{\frac{1}{2}}\\
&=\|K^*f\|\|K^*KK^*f\|\\
&=\|K^\dagger KK^*f\|\|K^*KK^*f\|\\
&\leq \|K^\dagger\|\|KK^*f\|\|K^*\|\|KK^*f\|\\
&=\|K^\dagger\|\|K\|\|KK^*f\|^2.
\end{aligned}$$

Hence,

$$v_-(\Lambda,K,I)\leq v_+(\Lambda,K,I)\leq \|K\|\|K^\dagger\|(1+\|K\|\|K^\dagger\|).$$

According to the proof of Theorem 3.5 in [16], for any $f\in\mathcal{H}$ we observed that

$$\langle S_I^2f,f\rangle+\langle S_{I^c}KK^*f,f\rangle=\langle KK^*S_If,f\rangle+\langle S_{I^c}^2f,f\rangle.$$

Thus,

$$\begin{aligned}
&\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2+\sum_{j\in I^c} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}\\
&=\|\sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2+\sum_{j\in I} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}.
\end{aligned}$$

Obviously, (2) holds.

Finally, (3) is easy to check.

In fact, according to the result above-mentioned, we can present some equivalent results for Parseval K-g-fusion frames as follows.

Corollary 2.1 Let Λ be a Parseval K-g-fusion frame for $\mathcal{H}$. Then for any $I\subset J$ and $f\in\mathcal{H}$, the following statements are equivalent.

(1) $v_+(\Lambda,K,I)=v_-(\Lambda,K,I)=1$.
(2) $\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2=Re\sum_{j\in I}\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}v_j^2$.
(3) $\|\sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2=Re\sum_{j\in I^c}\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}v_j^2$.

Proof $(2)\Leftrightarrow(3)$ is clearly.

Also, $(1)\Rightarrow(2)$ holds by a direct computation. Now, let (2) hold, then

$$\begin{aligned}
&\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2+\sum_{j\in I^c} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}\\
&=\sum_{j\in J} v_j^2\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}\\
&=\langle KK^*f,S_\Lambda f\rangle=\|KK^*f\|^2,
\end{aligned}$$

i.e., (1) holds.

Hence $(1)\Leftrightarrow(3)$ and similarly $(1)\Leftrightarrow(2)$.

Corollary 2.2 Let Λ be a Parseval K-g-fusion frame for $\mathcal{H}$. Then for any $I\subset J$ and $f\in\mathcal{H}$, the following statements are equivalent.

(1) $\|\sum_{j\in I} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2=\sum_{j\in I}\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}v_j^2$.
(2) $\|\sum_{j\in I^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\|^2=\sum_{j\in I^c}\overline{\langle \Lambda_j\pi_{W_j}KK^*f,\Lambda_j\pi_{W_j}f\rangle}v_j^2$.
(3) $S_If\perp S_{I^c}f$.
(4) $f\perp S_{I^c}S_If$.

Proof By Equation (8), $(1)\Leftrightarrow(2)$ holds trivially.

Since S_I and S_{I^c} are positive, for each $f\in\mathcal{H}$, we have

$$\langle S_{I^c}f,S_If\rangle=\langle f,S_{I^c}S_If\rangle=\langle (KK^*S_I-S_I^2)f,f\rangle.$$

This implies that, $(3)\Leftrightarrow(4)$ and $(1)\Leftrightarrow(4)$.

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