European Call Options Pricing Numerically using Finite Element Method

Erny Rahayu Wijayanti, Sumardi and Nanang Susyanto*

Abstract—**In this paper, we present the Black-Scholes model for European call option pricing that is transformed to the diffusion equation and added with noise (white noise) to describe the external stock factors that affect option prices. Therefore we propose stochastic partial differential equation (SPDE) which is stochastic heat equation by using finite element method to solve SPDE numerically. The discretization with respect to space is performed by piecewise linear finite elements and in time is performed by the Crank-Nicolson. The white noise is approximated by piecewise constant random processes. We estimate the European call option prices numerically using the finite element method (FEM).**

Index Terms—**option pricing, SPDEs, stochastic heat equations, white noise, finite element method**

I. Introduction

THE study of finance has drawn rising attention in mathematical research. The modeling approaches encompass methods and techniques from various different disciplines, stochastical and statistical methods, deterministic and stochastic partial differential equations (SPDE), and others. It is widely known that companies make investments to create and to take advantage of the opportunities to expand businesses and to make profits in competitive and uncertain markets. The opportunities appeared are later converted into investment projects which is very important for the role of financial securities. As a part of financial securities, options are mainly chosen to assure assets in order to cover generated risks of the stock prices changes. By this reason, mathematicians have become interested to study about option pricing extensively [2], [14], [15], [20].

Option is a financial contract that offer its owner the right to buy or sell a specified amount of a particular asset at a fixed price, which is called as exercise price (strike price) on or before a specified date called maturity date. Options that can be exercised at any time up to the maturity date are called American options, while options that can only be exercised on the maturity date are European options. The form of rights that occur based on point of view are divided by two types of options. Options that provide the right to buy the underlying asset are known as call options, whereas options conferring the right to sell the underlying asset are referred to put options.

Options traders need to consider additional variables that can affect an option price and the complexity of choosing the right strategy. Option pricing theory requires variables to theoretically value an option such as stock price, exercise price (strike price), volatility, interest rate, and time to expiration. Essentially, option provides an estimation of an option's fair value which traders incorporate into their strategies to maximize profits. Some commonly used models to value options are Black-Scholes, binomial option pricing and Monte-Carlo Simulation.

In the history of finance model, the Black-Scholes or Black-Scholes-Merton models [12] is one of the prominent option pricing model. The model represent that the mathematics plays an important role in the field of finance. The Black-Scholes model is applied to determine the theoretical price of European put and call options, neglecting any dividends paid during the option's lifetime. The Black-Scholes equation can be solved by both analytically and numerically.

Black and Scholes (1973) first found the solution based on previous research on option pricing that gave an idea of what the solution would look like. The analytic solution of Black-Scholes model can be obtained by transforming the equation into the heat equation as discussed in [22]. In [9] also discussed about analytic solution by Mellin transformation that was utilized to explain the Black-Scholes model, which did not require variable change or explaining dispersion condition. The Black-Scholes model which was modified with discrete dividend is solved by R. Company, A.I. Gonzalez and L. Jodar in [6]. They utilized a delta-characterizing grouping of generalized Dirac-Delta function and connected the Mellin transformation to acquire an integral formula.

Some numerical method to solve Black-Scholes model were studied by numerous researcher such as [3], [5], [2], [7], [11], [16]. In [3], M.N. Anwar and L.S. Andallah discussed about numerical solution of Black-Scholes for European call option by using explicit finite difference scheme and semi-implicit finite difference. Mardianto, et al. compared some numerical method such as binomial tree, explicit and implicit finite difference method on European put options. In [2] and [17], they proposed finite element method to solve the Black-Scholes for European options. Andalaft-Chachur in [2] also explain that the finite element method allows valuing American options as well as both call and put options.

However, the theory of Black-Scholes has wide margins of error due to deriving their values from other assets or other factors. In real life, many factors can affect the option price including the influence of external stock price factors. Therefore taking this into account, in this research, a noise will be added to represent some external factors which have a potential to affect options prices. Hence, we will work with the stochastic partial differential equation (SPDE).

We can obtain the exact solution of Black-Scholes equation by transforming it into a diffusion equation [22]. However, with the addition of noise in diffusion equation, the

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equation becomes a stochastic heat equation or SPDE and it will become too complex to get an exact solution, or explicitly said as no exact solution ([18], [1]), thus the equation can only be solved numerically. One of the numerical methods that can be applied to solve this SPDE is finite element method (FEM) ([1], [13]).

In this paper, we study the Black-Scholes equation which is transformed into a diffusion equation and added with white noise to model the European option, so that the equation becomes a stochastic partial differential equation (SPDE) that is a stochastic heat equation. Although the SPDE has been discussed quite a lot in [1], [4], [13], [18], [21], none of them apply it to real problems in daily life. In Section 2 we explain how to transform the Black-Scholes equation into the stochastic heat equation and solve that equation numerically using finite element method. Furthermore, we compute a stock option in various strike price, risk-free rate and expiration date in Section 3, combined with the expectation of the European call options prices within the 95% confidence interval and compare the result with exact solution of the corresponding deterministic equation. Finally, our conclusions are presented in Section 4.

II. Problem formulation and basic equations

A. Black-Scholes Model for European Call Option

Let us consider S as the price of the stock, which we consider as a random variable. In this paper, we will focus on application of Black-Scholes model for call option pricing. There are assumptions regarding the call option. The first one, the underlying stock pays no dividends during the option's life and the option can only be exercised at the expiration date, as it is a European option. The next one, constant composition returns are normally distributed and there are efficient markets. There is no taxes or transaction costs as well. The last one, volatility and risk-free rate are known and constant.

Suppose $V(S,t)$ is the value of an option as a function of time and stock price and E be the strike price. The risk-free interest rate denoted as r, σ be the volatility or the standard deviation of the stock return, and also the time in week denoted as t.

The important thing, we have to consider initial and boundary condition in the case of European call option. The value at the final time $t=T$ can be determined from the definition of call option. If the price at the final time stock is greater than strike price $(S>E)$ the call option will be worth $(S-E)$ since the buyer can buy the stock for E and sell it immediately for S and taking profit. But when $S<E$ then the buyer will not exercise the option and it will be worthless. Thus the option value known at $t=T$ is called the final condition and it is expressed as $V(S,T)=\max(S-E,0)$. For the boundary conditions, we consider the value of $V(0,t)=0$ and $V(S,t)\sim\infty$ as $S\to\infty$. Hence the Black-Scholes initial boundary value problem for European call option is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{1}$$

with

$$\begin{aligned}&V(S,T)=\max(S-E,0),\ \text{ for } 0\le S\\ &V(S,t)=0 \text{ for } S=0,\\ &V(S,t)\sim S \text{ as } S\to\infty,\ \text{ for } 0\le t\le T.\end{aligned}$$

According to [22] we can transform Equation (1) into a standard boundary value problem for diffusion equation. We set

$$S=Ee^x,\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^2},\quad V(S,t)=Ev(x,\tau). \tag{2}$$

Hence, the Equation (1) becomes

$$\frac{\partial v}{\partial\tau}=\frac{\partial^2v}{\partial x^2}+(c-1)\frac{\partial v}{\partial x}-kv, \tag{3}$$

where $c=\frac{r}{\frac{1}{2}\sigma^2}$ and we get initial conditions as $v(x,0)=\max(e^x-1,0)$. In order to eliminate the last two terms on the right hand side of Equation (3), we set

$$v(x,\tau)=e^{\alpha x+\beta\tau}, \tag{4}$$

for some constants α and β to be found, then differentiation gives

$$\beta u+\frac{\partial u}{\partial\tau}=\alpha^2u+2\alpha\frac{\partial u}{\partial x}+\frac{\partial^2u}{\partial x^2}+(c-1)\left(\alpha u+\frac{\partial u}{\partial x}\right)-cu. \tag{5}$$

We can obtain an equation with no u term by choosing $\beta=\alpha^2+(c-1)\alpha-c$, while the choice $2\alpha+(c-1)=0$, eliminates the $\frac{\partial u}{\partial x}$ as well. The equations for α and β give

$$\alpha=-\frac{1}{2}(c-1),\quad \beta=-\frac{1}{4}(c+1)^2. \tag{6}$$

Therefore, we have

$$v=e^{-\frac{1}{2}(c-1)x-\frac{1}{4}(c+1)^2\tau}u(x,\tau), \tag{7}$$

and

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2u}{\partial x^2}\ \text{ for }-\infty<x<\infty, 0<\tau<\frac{1}{2}\sigma^2T, \tag{8}$$

where $u(x,0)=\max(e^{\frac{1}{2}(c+1)}-e^{\frac{1}{2}(c-1)},0)$, $u(0,\tau)=0$, and $\lim\limits_{x\to\infty}\frac{u(x,\tau)}{e^xe^{-\frac{1}{2}(c-1)x-\frac{1}{4}(c+1)^2\tau}}=1$.

In reality, there are some external factors of stock can be affect the option price. For example the condition of the company, the image of the CEO, the change of the chairman, the market confidence and so on. To represent this factor, in this simulation we add noise, which is white noise, into Equation (8), as follows

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2u}{\partial x^2}+gdW, \tag{9}$$

where g is a constant and W is Brownian motion or Wiener process.

Definition 2.1: A stochastic process $W=W(t), t\ge 0$ is called a Wiener process, if

1) W has independent increments, i.e. for any $0\le t_0<t_1<...<t_n$, $W(t_1)-W(t_0), W(t_2)-W(t_1),...,W(t_n)-W(t_{n-1})$ are independent.
2) W has Gaussian increments, i.e. $W(t)-W(s)$ is normally distributed with mean 0 and variance $t-s$, $W(t)-W(s)\sim N(0,t-s)$ for $t\ge s$.
3) $W(0)=0$.

4) W has continuous paths, i.e. the function $t \to W(t)$ is continuous in t.

Equation (9) is known as stochastic partial differential equation (SPDE) or more specifically stochastic heat equation. However it is too complicated to get the analytical solutions of the SPDE problems, stochastic heat equation in this case, or it does not exist. Therefore to investigate the prediction of SPDE models, it is necessary to approximate their solution numerically ([1],[13]). One of the numerical methods that may be used to approximate SPDEs problem is Finite Element Method (FEM) ([18], [19], [21], [4]).

B. Finite Element Method for Stochastic Parabolic Equations

Referring to [4] we can see Equation (9) as the heat equation on $[0,T] \times [0,\delta] = [0,T] \times D \in R^2, \delta > 0$, with a source term f as follows

$$\frac{du(t)}{dt} = \Delta u(t) + \frac{f(t)}{dt}, \tag{10}$$

with Dirichlet boundary condition and a smooth initial condition given by some function u_0.

Therefore, we have a parabolic Stochastic Partial Differential Equation (SPDE). We will transform the source term f into noise term later. According to the standard procedure of the Galerkin approximation method, Equation (10) will be multiplied with an element of the finite-dimensional space where the solution to that approximated problem lies. This element is called test function $\psi \in S_h$. Afterwards we integrated by parts and get the weak form as follows

$$\int_D \dot{u}(t,x)\psi(x)dx + \int_D u_x(t,x)\psi_x(x)dx = \int_D \dot{f}(t,x)\psi(x)dx$$

where $\dot{u}(t,x) = \frac{\partial u(t,x)}{\partial t}$, $u_x(t,x) = \frac{\partial u(t,x)}{\partial x}$, and $\dot{f}(t,x) = \frac{f(t,x)}{\partial t}$. The projection of X onto S_h denoted by u_h^e has the unique representation

$$u_h^e(t,x) = \sum_{j=1}^{n} \xi_j^e(t)\phi_j^e(x),$$

where $\xi_j^e(t)$ are solution values of $u(t,x)$ in S_h at time t and ϕ_j^e is a basis of S_h. The elements ϕ_j^e are called the trial functions. When we combine this representation with the Galerkin approximation above, we get the system

$$\mathbf{R}\dot{\xi}(t) + \mathbf{Z}\xi(t) = \dot{\mathbf{b}}(t), \tag{11}$$

where $\mathbf{R} = (R_{ij}, i,j = 1,2,...,n), \mathbf{Z} = (Z_{ij}, i,j = 1,2,...,n)$, and $\dot{\mathbf{b}} = (\dot{b}_i, i = 1,2,...,n)$, with

$$R_{ij} = \int_D \phi_j(x)\phi_i(x)dx,\ Z_{ij} = \int_D (\phi_x)_j(x)(\phi_x)_i(x)dx,$$

$$\text{and } \dot{b}_i = \int_D \dot{f}(t,x)\phi_i(x)dx.$$

The hat function can be one of the possibilities functions to be chosen as a basis function ϕ. The dimension of the finite element space gives directly the number of approximation points, here $x_0, x_1, ..., x_n$, and as well as the number of finite elements ($[x_{i-1}, x_i], i = 1,2,...,n$). Thus we can get the global matrices $\mathbf{R}$ and $\mathbf{Z}$ by assembling sub-matrices. The sub-matrices of $\mathbf{R_k^e}$ and $\mathbf{Z_k^e}$, $k = 1,2,...,n$ for the hat function basis as follows

$$\begin{aligned} \mathbf{R_k^e} &= \left(\int_{x_e}^{x_{e+1}} \phi_j^e(x)\phi_i^e(x)dx, i,j = k-1,k \right), \\ \mathbf{Z_k^e} &= \left(\int_{x_e}^{x_{e+1}} (\phi_x)_j^e(x)(\phi_x)_i^e(x)dx, i,j = k-1,k \right). \end{aligned}$$

These elements are calculated for every $k = 1,2,...,n$ separately and then the global matrices are assembled in the end. The hat functions are all symmetric if the grid are in equidistant $(x_0, x_1, ..., x_n)$ with the same scaling, it means those sub-matrices are identical for all $k = 2,3,...,n-1$. Therefore, only the elements subject to the boundary conditions have to be updated in the very end. In this case of the model problem, a hat function basis given by

$$\phi_j(x_i) = \begin{cases} 1, & \text{for } j = i, \\ 0, & \text{for } j \neq i, \end{cases}$$

with

$$\sum_{j=1}^{n} \phi_j^e(x) = 1.$$

Then for each x

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & \text{for } x_{j-1} \le x \le x_j, \\ \frac{x_{j+1} - x}{h}, & \text{for } x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the calculation of $\mathbf{R}$ and $\mathbf{Z}$ does not depend on the discretization of the time, then the matrices can be computed in advance. For time discretization, the equidistant of time is $0 = t_0 < t_1 < t_2 < ... < t_N = T$ with step size $k = \frac{T}{N}$. The time derivatives are approximated by

$$\dot{\xi}(t) \approx \frac{\xi(t_i) - \xi(t_{i-1})}{k}, \quad t \in [t_i, t_{i+1}]$$

and

$$\dot{b}(t) \approx \frac{b(t_i) - b(t_{i-1})}{k}, \quad t \in [t_i, t_{i+1}].$$

Since $\mathbf{R}$ and $\mathbf{Z}$ are not time dependent, we obtained

$$\mathbf{R}\left(\xi(t_i) - \xi(t_{i-1})\right) + k\mathbf{Z}\xi(t) = b(t_i) - b(t_{i-1}), \quad t \in [t_i, t_{i+1}].$$

Next, we apply Crank-Nicolson approximation for $\xi(t)$ and we get

$$\xi(t) \approx \frac{\xi(t_i) + \xi(t_{i-1})}{2}, \quad t \in [t_i, t_{i+1}].$$

Therefore, we obtained the fully discrete equation as follows

$$\left(\mathbf{R} + \frac{k}{2}\mathbf{Z}\right)\xi(t_i) = \left(\mathbf{R} - \frac{k}{2}\mathbf{Z}\right)\xi(t_{i-1}) + \left(b(t_i) - b(t_{i-1})\right). \tag{12}$$

We can calculate the approximation for each time step by solving the system of linear equations, since each component of this equation is known.

To solve Equation (9), we discuss to approximate of the noise term. Consider to the problem we have $dW = \frac{\partial^2 W}{\partial t \partial x}(t,x)$ Gaussian White noise as the noise. We approximate the White noise by the formula $\frac{\partial^2 W}{\partial t \partial x} \approx \frac{\partial^2 \hat{W}}{\partial t \partial x}$. We approximate $\frac{\partial^2 \hat{W}}{\partial t \partial x}$ in each small rectangular block on a Brownian sheet by $\eta_{ij} \sim N(0,1)$. Then consider two dimensional Gaussian White noise $\frac{\partial^2 W}{\partial t \partial x}(t,x)$ where $W(t,x)$ is Brownian motion on a Brownian sheet.

We define the partition of $[0,T]\times[a,b]$ by rectangles $[t_i,t_{i+1}]\times[x_j,x_{j+1}]$ for $i=1,2,...,M$ and $j=1,2,...,N$ where $t_i=(i-1)\Delta t$, $x_j=(j-1)\Delta x$, $\Delta t=\frac{T}{M}$ and $\Delta x=\frac{1}{N}$. Then, a reasonable approximation on this partition is given by

$$\frac{\partial^2\hat{\mathrm{W}}}{\partial t\partial x}(t,x)=\frac{1}{\Delta t\Delta x}\sum_{i=1}^{M}\sum_{j=1}^{N}\eta_{ij}\sqrt{\Delta t\Delta x}\chi_i(t)\chi_j(x),$$

where

$$\chi_i(t)=\begin{cases}1, & \text{for} \quad t_i\leq t\leq t_{i+1}\\ 0, & \text{otherwise.}\end{cases},$$

and

$$\chi_j(x)=\begin{cases}1, & \text{for} \quad x_j\leq x\leq x_{j+1}\\ 0, & \text{otherwise.}\end{cases},$$

and also

$$\eta_{ij}=\frac{1}{\sqrt{\Delta t\Delta x}}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}dW(t,x),$$

i.e., $\eta_{ij}\in N(0,1)$ with $N(0,1)$ is a standard Gaussian random variable.

Then, if we set for the abstract source term f as the Gaussian White noise term $dW(t)$, we obtain for each entry $j=1,2,...,n$

$$\int_{t_{i-1}}^{t_i}\int_{x_{j-1}}^{x_j}\phi_j(x)d\hat{\mathrm{W}}(s,x)=\frac{1}{2}h\left(\sqrt{k}\eta_{ij}-\sqrt{k}\eta_{ij-1}\right),$$

where $\eta_{ij}\sim N(0,1)$ and $\eta_{ij-1}\sim N(0,1)$, which is we can write as

$$(b(t_i)-b(t_{i-1}))_j=\frac{1}{2}h\left(\Delta_i W(x_j)-\Delta_i W(x_{j-1})\right),$$

where $\Delta_i W(x)=W(t_i,x)-W(t_{i-1},x)$.

Furthermore, we want to investigate the convergence results of this method. We consider the Equation (9) where $g=1$ and the initial condition $u(x,0)=\sin(\pi x)$ for $[x\in[0,1]$ and Dirichlet boundary conditions $u(0,t)=u(1,t)=0$. The solution to the corresponding deterministic system

$$du(t)=\Delta u(t)dt$$

in this case is $u(x,t)=\exp(-\pi^2 t)\sin(\pi x)$. For the stochastic problem, the analytic solution is not available. According to [4], [1], [21], we approximate a finite element on a very refined grid as a substitute to an analytic solution. In this case, we select size of the grid $2^2\times 2^{14}$.

The expectation of error was determined by taking the square root of the average over the realizations of the maximum over all the time steps of the normalized sum of the squared difference of the solution on the finest grid and the solution on the coarser grids, i.e. let $(\hat{U}_i, i=1,2,...,N)$ be the set of simulated paths on time grid $(t_j, j=0,1,...,n)$ and a space grid $(x_k, k=0,1,...,L)$, and the $(u_i, i=1,2,...m)$ the set of the simulated paths of the "substitution analytic" solution on the coarser grid in time and space. Therefore, we can calculate the error as follows

$$e_N=\sqrt{\frac{1}{N}\sum_{i=1}^{N}\max_{j=0,1,...,n}\frac{1}{m}\sum_{k=1}^{m}\left(\hat{U}_i(t_j,x_k)-u_i(t_j,x_k)\right)^2}. \tag{13}$$

The increments are added depending to the grid size for the coarser grid. Even though the number of realizations was not that big, it was $N=100$, but the convergence is already observable. The discretization was selected starting on a grid size 2×2^7 up to grid of $2^7\times 2^{13}$.

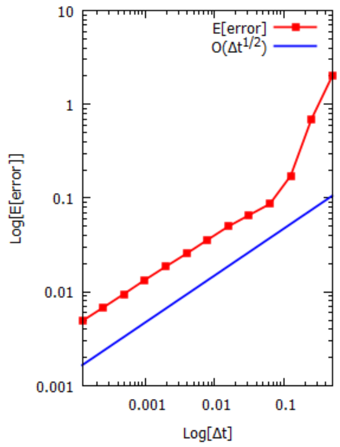

Fig. 1: Graph of error expectation with respect to the time discretization and a line as reference slope on a logarithmic scale.

Figure 1 displays the errors expectation with respect to the time discretization and a line as reference slope on a logarithmic scale. We can observe that the order convergence of expectations error is $\frac{1}{2}$.

III. Result and Discussion

In this section, we estimate the price of a European call option focused on Zoom Video Communications, Inc (ZM) stock. We set the strike price $Sk=\$240,\$260,\$280$ to be taken over 30, 90 and 210 days. The volatility of the underlying stock is computed using return data for the company taken from Yahoo Finance on November, 17^{th} 2019-November, 16^{th} 2021. The risk-free rate of return is taken from the U.S. Treasury website. We get 5.56% for the volatility (σ) and the risk-free rate (r) is 0.043%. With Equation (9) we will simulate the European call option pricing of Zoom stock. We set for $g=\frac{\sigma}{10}$.

Firstly, consider European call option of ZM with strike price $Sk=\$280$ and time period $T=90$ days. The volatility of the stock price is $\sigma=5.56\%$ and the risk-free rate $r=2.25\%$. The finite element solutions are shown in Fig.2, as follows.

Figure 2 presents the price of the option at different time during the period time. The value of the call option is plotted over a range of stock prices $\$160\leq S\leq\$340$ surrounding the strike price.

Furthermore, we simulated the European option model with various parameters. These parameters are used to purpose knowing the influence of strike price, the expiration date and risk-free rate of return on behavior of a call option value.

We presents comparison of stock price at the beginning of the option's time period and its 95% confidence interval of the expectation call option prices in Table I. For example, if the stock price at the beginning of the time period is \$250.94,

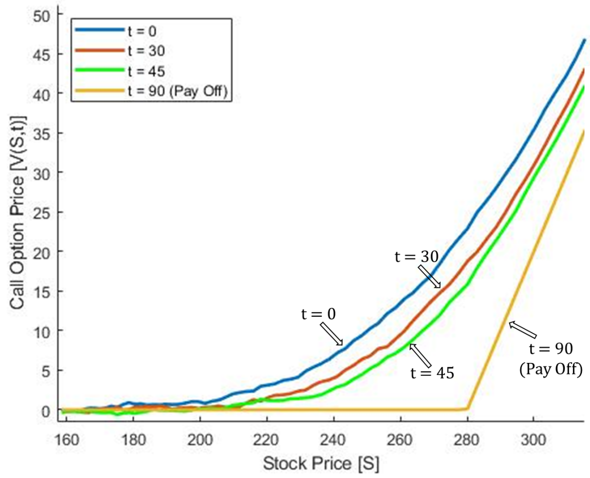

Fig. 2: FEM expectation for a European call option different time.

TABLE I: Expectation of European call option price and the Zoom stock price at t = 0 using Finite Element Method

No	*Stock Price*	Interval of Option Price Sk = 240	Sk = 260	Sk = 280
1	229.34	[5.56, 6.05]	[1.46 ,1.76]	[0.11, 0.43]
2	231.65	[6.40, 7.16]	[1.77, 2.33]	[0.19, 0.74]
3	233.98	[7.34, 8.08]	[2.13, 2.59]	[0.69, 1.08]
4	236.33	[8.40, 9.10]	[2.57, 2.90]	[0.41, 0.65]
5	238.70	[9.41, 10.18]	[3.17, 3.53]	[0.68, 0.90]
6	241.10	[10.73, 11.48]	[3.86, 4.25]	[0.97, 1.24]
7	243.52	[12.36, 12.99]	[4.51, 4.90]	[1.19, 1.4
8	245.97	[13.89, 14.46]	[5.36, 5.79]	[1.61, 1.8
9	248.44	[15.30, 16.13]	[6.05, 6.87]	[1.79, 2.4
10	250.94	[17.09, 17.87]	[7.97, 9.59]	[2.33, 3.0
11	253.46	[18.86, 19.49]	[8.18, 8.96]	[2.79, 3.3
12	256.01	[20.87, 21.51]	[9.52, 10.38]	[3.46, 4.0
13	258.58	[22.76, 23.36]	[10.46, 11.37]	[3.99, 4.6
14	261.18	[24.80, 25.38]	[12.68, 15.07]	[4.65, 5.3
15	263.81	[26.83, 27.33]	[13.38, 14.24]	[5.29, 5.9

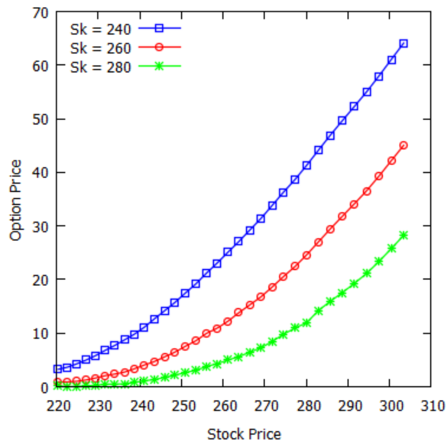

Fig. 3: Graphic of comparison toward a European Call option movement with different strike price (Sk) and risk-free rate $r = 0.0225$.

then with strike price $Sk = \$260$ for 30 days the option value would be in between \$7.97 and \$9.59.

From Fig. 3, it can be seen that the various value of strike price Sk influence to European call option price. There is an inverse relationship between the strike price and call option price. If there is an increase in strike price Sk, the call option price tends to decrease.

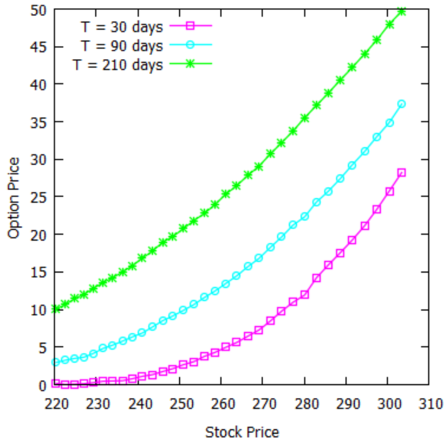

Fig. 4: Graphic of comparison toward a European Call option movement with different expiration date (T) and fix strike price $Sk = \$280$.

Next, we compare the option price due to the various time period. We set the fix strike price $Sk = \$280$ and the risk-free rate $r = 0.0225$. It can be seen in Fig. 4 that the time period influence to European call option's price. The call option prices increases as the longer time period of the option to be exercise.

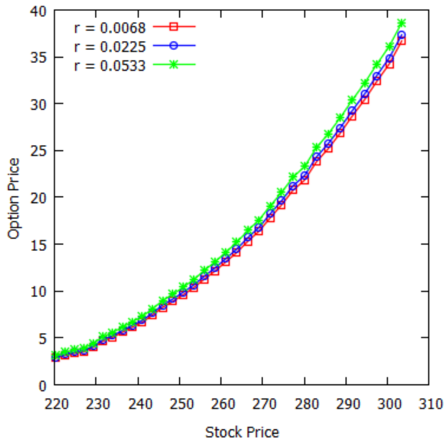

Fig. 5: Graphic of comparison toward a European Call option movement with different risk-free rate (r) and fix strike price $Sk = \$280$.

Next, for various parameters r, the risk-free rate of return. The simulation results show that if the value r is high, the price of European call option will be more expensive as well.

Furthermore, we compared the expectation of call option price which obtained from the finite element approximation with the exact solution of the deterministic Black-Scholes that correspond to the Eq. (9) for the various risk-free rate.

The result are presented in Table II and Table III for fix strike price $Sk = \$280$, time period $T = 90$ days and risk-free rate (r) 0.68% and 2.25%. The finite element approximation expectation of call option price is presented in the table by its 95% confidence interval.

From Table II and Table III, it can be seen that several values of exact solutions from the deterministic equation

TABLE II: Comparison of European call option pricing by FEM and the exact solution of the deterministic heat equation that correspond to the stochastic heat equation with $r = 0.68\%$, $Sk = \$280$ and $T = 90$ days.

No	Stock Price	Numerical Result	Exact Deterministic Solution
1	269.14	[15.63, 17.17]	17.27
2	271.84	[17.09, 18.57]	18.55
3	274.57	[18.47, 19.93]	19.91
4	277.33	[20.06, 21.49]	21.33
5	280.12	[21.16, 22.55]	22.82
6	282.94	[23.12, 24.48]	24.38
7	285.78	[24.52, 25.88]	26.01
8	288.65	[26.21, 27.49]	27.71
9	291.55	[28.06, 29.28]	29.49
10	294.48	[29.86, 30.96]	31.34

TABLE III: Comparison of European call option pricing by FEM and the exact solution of the deterministic heat equation that correspond to the stochastic heat equation with $r = 2.25\%$, $Sk = \$280$ and $T = 90$ days.

No	Stock Price	Numerical Result	Exact Deterministic Solution
1	269.14	[16.03, 17.58]	17.69
2	271.84	[17.51, 19.01]	18.99
3	274.57	[18.91, 20.39]	20.37
4	277.33	[20.52, 21.97]	21.81
5	280.12	[21.64, 23.05]	23.32
6	282.94	[23.62, 25.00]	24.90
7	285.78	[25.04, 26.43]	26.55
8	288.65	[26.76, 28.05]	28.28
9	291.55	[28.63, 29.87]	30.07
10	294.48	[30.44, 31.57]	31.94

which correspond to stochastic heat equation that represent the option pricing model are in the interval of FEM expectation solution of the problem. It means that the stochastic term in the model is sufficient to represent the external factors of stock prices that can affect the determination of option prices.

IV. Conclusion

In this paper, we discuss the European call option pricing modeled by the Black-Scholes and transformed into the diffusion equation. Since in real life the option price is also affected by the other things from external stock factors, thus, we add noise called Gaussian white noise. Therefore we have a stochastic partial differential equation (SPDE), or specifically called as stochastic heat equation. We solve the SPDE problem numerically by the finite element method (FEM). We regularize the noise, which is white noise by using the piece-wise constant random function. For the partial differential equations, we discretized the time by using Crank-Nicolson and the space by piece-wise linear finite element. We investigate the order convergence of the FEM to solve stochastic heat equation. The result shows that the order convergence of finite element method is $\frac{1}{2}$ in time.

The numerical solution of the European call option prices is presented over a range of stock prices surrounding the strike price. The 95% confidence interval of the expectation of the option prices is also presented. The various parameters is given in order to investigate the influence of strike price, risk-free rate of return and the time period on behavior of a call option price.

The analytic solution of the deterministic equation that correspond to stochastic is given and compared with FEM expectation solution. Several values of the exact solution are in the interval of the numeric solution. It clearly shows us that the stochastic term is sufficient enough to represent the external stock factors that can affect to the determination of option price.

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