Finite Horizon LQ Optimal Control of a Two-Time-Scale Discrete System

Tahia Zerizer [1], Lina Soundous Hacini [2] *

Abstract—We consider a finite-horizon LQ optimal control problem governed by a time-varying discrete system with two time-scales. Both fixed and free final state approaches for the asymptotic analysis and solution of this problem are proposed. In the first approach, an iterative algorithm converging to the solution of the corresponding two-point boundary value problem, which is not exactly solvable, is constructed, thereby finding an approximate optimal control value. We propose a simplified and lower-cost feedback controller in order to find the approximate optimal index in the second approach. Two examples are given to illustrate the proposed method.

Index Terms—discrete optimization, time-scale, perturbation, finite horizon, open-loop, closed-loop, Riccati difference equation

1 Introduction

Algorithms solving optimization problems are of great interest to engineers and are very well documented in recent control literature (see e.g. [1, 2, 3, 4, 7, 16, 28]). The control of discrete systems at two time-scales has received much attention over the past decades; these high-dimensional multi-scale systems are numerically challenging and often computationally unaffordable. The goal of their study is to simplify their structure using lower-order models and to ensure that the controls obtained from the lower-order model can be used to control the original system. In this article, we focus on the time-varying discrete-time system:

$$\begin{aligned} x(t+1) =& A_\varepsilon(t)x(t) + B_\varepsilon(t)u(t),\ t \in [i, f-1], \\ x(i) =& \bar{x} \ \text{ is given}; \end{aligned} \tag{1}$$

where $[i, f-1]$ denotes the set $\{i, i+1, i+2, \cdots, f-1\}$,

$$\begin{aligned} x(t) &= \begin{bmatrix} y(t) \\ z(t) \end{bmatrix},\ A_\varepsilon(t) = \begin{bmatrix} A_{11}(t) & \varepsilon A_{12}(t) \\ A_{21}(t) & \varepsilon A_{22}(t) \end{bmatrix}, \\ B_\varepsilon(t) &= \begin{bmatrix} \varepsilon B_1(t) \\ B_2(t) \end{bmatrix}, y(t) \in \mathrm{R}^{n_1},\ z(t) \in \mathrm{R}^{n_2}, \end{aligned} \tag{2}$$

and $u(t) \in \mathbf{R}^r$ is the control. This system is known as a ***C-model*** [6, 8, 9, 10, 12, 13], i.e., the small parameter ε is located on a column of the state matrix. The purpose of the control inputs $\{u(t)\}_i^f$ is to minimize the following index J:

$$\frac{1}{2}x^T(f)Q(f)x(f) + \frac{1}{2}\sum_i^{f-1} x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \tag{3}$$

where the $n \times n$ state weighting matrices $Q(t)$ and the $r \times r$ input weighting matrices $R(t)$ are assumed to be symmetric and positive definite; $n := n_1 + n_2$ is the order of the whole system. Our goal is to study the possibility of approaching the original problem (P) by the reduced problem where the small parameter ε is removed and to implement the corresponding less expensive controls. We checked that the small number ε can be insignificant but can also become very critical according to the calculation. In this article, we specify an upper bound ε_0 within a tolerable range for ε, so that the approximation and the algorithm can be used with confidence. We show that the positive value ε_0 can be determined given the system coefficients and the step size N. Note that similar problems have been studied since the early 1980s, see [13, 8] and the references therein; different kinds of models have been approached by reduced-order systems and approximate controllers have been designed. For instance, in [14] and [11], open-loop and closed-loop control strategies were investigated for a time-invariant case of the problem (P). The authors used a heuristic singular perturbation method. Their methodology cannot be extended to the corresponding general time-varying case. The purpose of our note is to present a new method for dealing with the linear-quadratic problem (P) to ensure that the controls obtained from the reduced-model can be used to control the original system. This paper

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is organized as follows. Section 2 is devoted to the open-loop control problem, i.e. when the final state is fixed, which is equivalent to solving a two-point boundary value problem (TPBVP). In general, it is impossible to give exact methods acceptable for most discrete TPBVP classes; numerical methods are required to construct approximate solutions, and there is generally no guarantee that the optimal solution will be found. For ε small enough, we prove the existence of a solution to the corresponding TPBVP and we implement an asymptotic algorithm which converges to the exact solution. The results presented in this article are mathematically justified. We apply the perturbation technique that we used in [15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In section 3, we adopt for the problem (P) a closed-loop control approach. The corresponding discrete-time algebraic Riccati equations require the inversion of many large matrices, which makes the approach expensive and increases the computational complexity of each iteration. We use reduced-order Kalman filters to approximate the optimal cost with less computational complexity. In section 4, we present numerical examples to illustrate the method. We end this article with a brief conclusion in section 5.

2 Free end-point optimal control

In this Section, we adopt an open-loop strategy. We introduce an adjoint vector $\lambda(t)$, and we form the Lagrangian

$$J+\sum_{t=i}^{f-1}\lambda(t+1)^T\left[A_\varepsilon(t)x(t)+B_\varepsilon(t)u(t)-x(t+1)\right], \quad (4)$$

which has the same extrema as that J when the state equality constraint (1) is satisfied. The discrete version of Pontriyagin's maximum principle [5] gives the following first-order conditions of optimality

$$\begin{aligned}\lambda(t) =&Q(t)x(t)+A_\varepsilon^T(t)\lambda(t+1), \quad t\in[i,f-1],\\ \lambda(f) =&Q(f)x(f)\end{aligned} \quad (5)$$

$$0=R(t)u(t)+B_\varepsilon^T(t)\lambda(t+1), \quad t\in[i,f-1], \quad (6)$$

in addition to (1). These conditions are necessary and sufficient since J is convex. Condition (6) fix the optimal control

$$u(t)=-R^{-1}(t)B_\varepsilon^T(t)\lambda(t+1), \quad t\in[i,f-1]; \quad (7)$$

by substituting (7) in (1) results the corresponding optimal state

$$\begin{aligned}x(t+1) =&A_\varepsilon(t)x(t)-B_\varepsilon(t)R^{-1}(t)B_\varepsilon^T(t)\lambda(t+1),\\ &t\in[i,f-1],\\ x(i) =&\bar{x}.\end{aligned} \quad (8)$$

Minimizing the cost (3) involves solving the two-point boundary value problem (TPBVP) (5)–(8).

2.1 Optimization Algorithm

We restructure the TPBVP (5)–(8) in the block matrix form :

$$\begin{bmatrix}y(t+1)\\z(t+1)\\p(t)\\q(t)\end{bmatrix}=\mathbf{M}(t)\begin{bmatrix}y(t)\\z(t)\\p(t+1)\\q(t+1)\end{bmatrix}, \ t\in[i,f-1], \quad (9)$$

such that

$$\mathbf{M}(t)=\begin{bmatrix}A_{11}(t) & \varepsilon A_{12}(t) & \varepsilon^2E_{11}(t) & \varepsilon E_{12}(t)\\ A_{21}(t) & \varepsilon A_{22}(t) & \varepsilon E_{21}(t) & E_{22}(t)\\ Q_{11}(t) & Q_{12}(t) & A_{11}^T(t) & A_{21}^T(t)\\ Q_{12}^T(t) & Q_{22}(t) & \varepsilon A_{12}^T(t) & \varepsilon A_{22}^T(t)\end{bmatrix}. \quad (10)$$

System (9) is solved with the boundary values

$$\begin{aligned}y(i)=&\bar{y},\ z(i)=\bar{z},\\ \begin{bmatrix}p(f)\\q(f)\end{bmatrix}=&\begin{bmatrix}Q_{11}(f) & Q_{12}(f)\\ Q_{12}^T(f) & Q_{22}(f)\end{bmatrix}\begin{bmatrix}y(f)\\z(f)\end{bmatrix},\end{aligned} \quad (11)$$

where

$$\lambda(t)=\begin{bmatrix}p(t)\\q(t)\end{bmatrix}, \quad Q(t)=\begin{bmatrix}Q_{11}(t) & Q_{12}(t)\\ Q_{21}(t) & Q_{22}(t)\end{bmatrix},$$

$$E(t)=\begin{bmatrix}E_{11}(t) & E_{12}(t)\\ E_{21}(t) & E_{22}(t)\end{bmatrix}, \ E_{ij}(t)=-B_i(t)R^{-1}(t)B_j^T(t).$$

2.1.1 Reduced TPBVP

Setting the parameter value to zero in (9), we find the set of recurrence equations :

$$\begin{aligned}y^{(0)}(i)&=\bar{y},\\ y^{(0)}(t+1)&=A_{11}(t)y^{(0)}(t),\ t\in[i,f-1];\end{aligned} \quad (12)$$

$$\begin{aligned}z^{(0)}(i)&=\bar{z},\\ z^{(0)}(t+1)&=A_{21}(t)y^{(0)}(t)+E_{22}(t)q^{(0)}(t+1),\\ &t\in[i,f-1];\end{aligned} \quad (13)$$

$$\begin{aligned}q^{(0)}(t)&=Q_{12}^T(t)y^{(0)}(t)+Q_{22}(t)z^{(0)}(t),\\ &t\in[i,f-1],\\ q^{(0)}(f)&=q(f);\end{aligned} \quad (14)$$

$$\begin{aligned}p^{(0)}(t)=&Q_{11}(t)y^{(0)}(t)+Q_{12}(t)z^{(0)}(t)+A_{11}^T(t)p^{(0)}(t+1)\\ &+A_{21}^T(t)q^{(0)}(t+1),\ t\in[i,f-1],\ p^{(0)}(f)=p(f).\end{aligned} \quad (15)$$

Obviously, the closed-form solution of (12) is

$$y^{(0)}(i)=\bar{y}, \quad y^{(0)}(t+1)=\prod_{l=i}^{t}A_{11}(l)\bar{y}, t\in[i,f-1]. \quad (16)$$

We suppose that the matrices :

$$[I - Q_{22}(t+1)E_{22}(t)]\,,\; t \in [i, f-1], \tag{17}$$

are nonsingular, and we use the notations :

$$\begin{aligned} F(t) &:= [I - Q_{22}(t)E_{22}(t-1)]^{-1}\,, \\ G(t) &:= Q_{12}^T(t)\prod_{l=i}^{t-1} A_{11}(l) + Q_{22}(t)A_{21}(t-1)\prod_{l=i}^{t-2} A_{11}(l). \end{aligned} \tag{18}$$

By combining (16), (13) and (14), we find the closed-form solution of (13) :

$$\begin{aligned} z^{(0)}(i) &= \bar{z}, \\ z^{(0)}(i+1) &= A_{21}(i)\bar{y} + E_{22}(i)F(i+1)G(i+1)\bar{y}, \\ z^{(0)}(t+1) &= A_{21}(t)\prod_{l=i}^{t-1} A_{11}(l)\bar{y} + \\ &\quad E_{22}(t)F(t+1)G(t+1)\bar{y}, \\ &\quad t \in [i+1, f-1]. \end{aligned} \tag{19}$$

Solving (14), we get :

$$\begin{aligned} q^{(0)}(i) &= Q_{12}^T(i)\bar{y} + Q_{22}(i)\bar{z}, \\ q^{(0)}(t) &= F(t)G(t)\bar{y},\; t \in [i+1, f-1], \\ q^{(0)}(f) &= q(f). \end{aligned} \tag{20}$$

It is to be noted that the values $p(f)$ and $q(f)$ are automatically fixed from the terminal condition in (11) once $y^{(0)}(f)$ and $z^{(0)}(f)$ are calculated from formulas (16) and (19), respectively. In order to emphasize the recursion in equation (15), we use the notation :

$$\begin{aligned} H(t) := Q_{11}(t)y^{(0)}(t) + Q_{12}(t)z^{(0)}(t) + A_{21}^T(t)q^{(0)}(t+1), \\ t \in [i, f-1]. \end{aligned} \tag{21}$$

Accordingly, (15) can be written in the form :

$$p^{(0)}(t) = A_{11}^T(t)p^{(0)}(t+1) + H(t),\; t \in [i, f-1]. \tag{22}$$

It is easy to verify by induction that the closed form solution of (22) is given below.

$$\begin{aligned} p^{(0)}(f-t) &= \prod_{j=f-1}^{f-t} A_{11}^T(j)p^{(0)}(t_f) + \\ &\quad \sum_{j=0}^{t-2}\prod_{j=f-j-2}^{f-t} A_{11}^T(j)H(f-t), \\ &\quad t \in [i, f-1], \\ p^{(0)}(f) &= p(f). \end{aligned} \tag{23}$$

Consequently, we have at our disposal the unique solution of the reduced TPBVP where the small parameter has been eliminated.

2.1.2 Formal expansion

For approximations of higher order $l \geq 1$, the TPBVP will be solved in terms of $\varepsilon-$ expansion. We substitute the formal power series

$$\begin{aligned} y(t) &= \sum_{l=0}^{\infty} y^{(l)}(t), \quad z(t) = \sum_{l=0}^{\infty} z^{(l)}(t), \\ p(t) &= \sum_{l=0}^{\infty} p^{(l)}(t), \quad q(t) = \sum_{l=0}^{\infty} q^{(l)}(t), \end{aligned} \tag{24}$$

into (9)−(11) and we collect similar power coefficients of ε on each side. If we agree that any negative order coefficient is zero, the following set of recurrences define the coefficients of all approximation orders.

$$\begin{aligned} y^{(l)}(t+1) &= A_{11}(t)y^{(l)}(t) + A_{12}(t)z^{(l-1)}(t) + \\ &\quad E_{11}(t)p^{(l-2)}(t+1) + E_{11}(t)q^{(l-1)}(t), \end{aligned} \tag{25}$$

$$\begin{aligned} z^{(l)}(t+1) &= A_{21}(t)y^{(l)}(t) + A_{22}(t)z^{(l-1)}(t) + \\ &\quad E_{21}(t)p^{(l-1)}(t+1) + E_{22}(t)q^{(l)}(t+1), \end{aligned} \tag{26}$$

$$\begin{aligned} p^{(l)}(t) &= Q_{11}(t)y^{(l)}(t) + Q_{12}(t)z^{(l)}(t) + \\ &\quad A_{11}^T(t)p^{(l)}(t+1) + A_{21}^T(t)q^{(l)}(t+1), \end{aligned} \tag{27}$$

$$\begin{aligned} q^{(l)}(t) &= Q_{12}^T(t)y^{(l)}(t) + Q_{22}(t)z^{(l)}(t) + \\ &\quad A_{12}^T(t)p^{(l-1)}(t+1) + A_{22}^T(t)q^{(l-1)}(t+1). \end{aligned} \tag{28}$$

Under condition (17), the combination of (26) and (28) gives

$$\begin{aligned} z^{(l)}(t+1) &= F(t+1)\Big[A_{22}(t)z^{(l-1)}(t) + \\ &\quad E_{22}(t)A_{12}^T(t+1)p^{(l-1)}(t+2) + \\ &\quad E_{22}(t)A_{22}^T(t+1)q^{(l-1)}(t+2) + \\ &\quad E_{22}(t)Q_{12}^T(t+1)y^{(l)}(t+1) + \\ &\quad E_{21}(t)p^{(l-1)}(t+1) + A_{21}(t)y^{(l)}(t)\Big], \\ &\quad i \leq t \leq f-2, \\ z^{(l)}(f) &= A_{21}(f-1)y^{(l)}(f-1) + \\ &\quad A_{22}(f-1)z^{(l-1)}(f-1) + \\ &\quad E_{21}(f-1)p^{(l-1)}(f) + E_{22}(f-1)q^{(l)}(f). \end{aligned} \tag{29}$$

To calculate the coefficients which will be used in writing the asymptotic development (24), we solve the equations above with the boundary conditions

$$y^{(l)}(i) = 0,\; z^{(l)}(i) = 0,\; p^{(l)}(f) = 0,\; q^{(l)}(f) = 0,\; l \geq 1; \tag{30}$$

thus emerges the following algorithm.

2.1.3 Algorithm

***Zero*-order coefficients**
Step 1. From (16)**, compute** $y^{(0)}(t)$**,** $t \in [i, f]$**.**
Step 2. From (19)**, compute** $z^{(0)}(t)$**,** $t \in [i, f]$**.**
Step 3. From (20)**, compute** $q^{(0)}(t)$**,** $t \in [i, f]$**.**
Step 4. From (23)**, compute** $p^{(0)}(t)$**,** $t \in [i, f]$**.**
***l*-order coefficients**
Step 5. Fix $y^{(l)}(i) = 0$**;**
compute $y^{(l)}(t)$**,** $t \in [i+1, f]$ **from** (25)**.**
Step 6. Fix $z^{(l)}(i) = 0$**;**
compute $z^{(l)}(t)$**,** $t \in [i+1, f]$ **from** (29)**.**
Step 7. Fix $q^{(l)}(f) = 0$**;**
compute $q^{(l)}(t)$**,** $t \in [i, f-1]$ **from** (28)**.**
Step 8. Fix $p^{(l)}(i) = 0$**;**
compute $p^{(l)}(t)$**,** $t \in [i, f-1]$ **from** (27)**.**
Step 9. Return to step 5.

2.1.4 Convergence of the asymptotic solution

The following theorem gives the main result of this Section. We prove the convergence of the algorithm above, i.e., the convergence of the power-expansion series (24) **for some domain of the small parameter** ε**; the maximum norm is used.**

Theorem 1. *Suppose assumption* (17) *satisfied, there exists a positive real number* ε_0*, s. t. for all* ε*,* $|\varepsilon| < \varepsilon_0$*, the optimal control of the TPBVP* (9)$-$(11)*, that is* $u(t)$*,* $i \le t \le f$*, exists, is unique, and satisfies*

$$\begin{aligned} u(t) &= \sum_{l=0}^{\infty} \varepsilon^l u^{(l)}(t), \quad t \in [i, f-1], \\ u^{(0)}(t) &= -R^{-1}(t) B_2^T(t) F(t+1) G(t+1) \bar{y}, \\ u^{(l)}(t) &= -R^{-1}(t)[B_1^T(t) p^{(l-1)}(t+1) + B_2^T(t) q^{(l)}(t+1)], \end{aligned} \tag{31}$$

moreover, we have the following estimates

$$|u(t,\varepsilon) - \sum_{l=0}^{n} \varepsilon^l u(t)^{(l)}| \le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \tag{32}$$

where C *is a positive constant.*

Proof. Setting

$$\tilde{p}(t) = \varepsilon p(t), \quad t \in [i+1, f], \tag{33}$$

and replacing the first equation of system (9) by

$$\begin{aligned} y(t+1) =& A_{11}(t) y(t) + \varepsilon A_{12}(t) z(t) + \varepsilon E_{11}(t) \tilde{p}(t+1) \\ &+ \varepsilon E_{12}(t) q(t+1), \quad t \in [i, f-1], \end{aligned} \tag{34}$$

we transform the TPBVP (9)$-$(11) in the vectorial form

$$\begin{aligned} \hat{\mathcal{A}}_\varepsilon v = f; \quad f := (\bar{y}, \bar{z}, 0, \cdots, 0)^T, \\ v := (y(i), z(i), p(i), , \cdots, q(f), \tilde{p}(i+1), \cdots, \tilde{p}(f))^T, \end{aligned} \tag{35}$$

and the matrix $\hat{\mathcal{A}}_\varepsilon$ is written as $\hat{\mathcal{A}}_\varepsilon = \hat{\mathcal{A}}_0 + \varepsilon U$, thus $\hat{\mathcal{A}}_0$ is the block matrix given below

$$\begin{pmatrix} I & 0 & \cdots & & & 0 \\ 0 & I & \cdots & & & 0 \\ C(i) & D(i) & & & & \\ 0 & C(i+1) & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & C(f-1) & D(f-1) & 0 \\ & & & 0 & F & 0 \\ 0 & & \cdots & 0 & 0 & I \end{pmatrix},$$

where

$$C(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) & -I & 0 \\ Q_{12}^T(t) & Q_{22}(t) & 0 & -I \\ A_{11}(t) & 0 & 0 & 0 \\ A_{21}(t) & 0 & 0 & 0 \end{pmatrix},$$

$$D(t) = \begin{pmatrix} 0 & 0 & A_{11}^T(t) & A_{21}^T(t) \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & E_{22}(t) \end{pmatrix},$$

$$F = \begin{pmatrix} Q_{11}(f) & Q_{12}(f) & -I & 0 \\ Q_{12}^T(f) & Q_{22}(f) & 0 & -I \end{pmatrix}, \tag{36}$$

and the matrix U can be deduced easily. By induction, using the formula based on a factorization involving the Schur complement, we have

$$\det \hat{\mathcal{A}}_0 = \prod_{t=i}^{f-1} \det[Q_{22}(t+1) E_{22}(t) - I]. \tag{37}$$

Since the matrices $[I - Q_{22}(t+1)E_{22}(t)]$ are supposed to be nonsingular for all $t \in [i, f-1]$, then $\det \hat{\mathcal{A}}_0 \neq 0$, which ensures that the reduced TPBVP has a unique solution; we can denote :

$$\varepsilon_0 := \frac{1}{\|U \hat{\mathcal{A}}_0^{-1}\|}, \qquad C := \|\hat{\mathcal{A}}_0^{-1}\| \|f\|. \tag{38}$$

By choosing ε such that $|\varepsilon| < \varepsilon_0$, we have $\|\varepsilon U A_0^{-1}\| < 1$ and the infinite sum

$$\hat{\mathcal{A}}_\varepsilon^{-1} = \hat{\mathcal{A}}_0^{-1} \left(I + \varepsilon U \hat{\mathcal{A}}_0^{-1} \right)^{-1} = \hat{\mathcal{A}}_0^{-1} \sum_{j=0}^{\infty} \left(-\varepsilon U \hat{\mathcal{A}}_0^{-1} \right)^j, \tag{39}$$

is well defined. Therefore, the solution $v(\varepsilon) = \hat{\mathcal{A}}_\varepsilon^{-1} f$ of the system (9)$-$(11) exists and is unique. Moreover, if we denote by $v^{(l)}$ the following vector

$$\left(y^{(l)}(i), z^{(l)}(i), \cdots, q^{(l)}(f), \tilde{p}^{(l)}(i+1), \cdots, \tilde{p}^{(l)}(f) \right)', \tag{40}$$

from (39), we verify that :

$$v(\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l v^{(l)}, \text{ where } \quad v^{(l)} = \hat{\mathcal{A}}_0^{-1} \left(-U \hat{\mathcal{A}}_0^{-1} \right)^l f, \tag{41}$$

and we can easily deduce that the components of $v^{(l)}$ are those described in Sections 2.1.1 and 2.1.2. Also, we have

$$\begin{aligned}\|\hat{\mathcal{A}}_\varepsilon^{-1} - \hat{\mathcal{A}}_0^{-1} \textstyle\sum_{j=0}^n \left(-\varepsilon U \hat{\mathcal{A}}_0^{-1}\right)^{(j)} \| \\ \leq \|\hat{\mathcal{A}}_0^{-1}\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1-|\varepsilon|/\varepsilon_0}.\end{aligned} \tag{42}$$

From (39), (24) and (42), follows

$$\|v(\varepsilon) - \textstyle\sum_{j=0}^n \varepsilon^j v^{(j)}\| \leq \|\hat{\mathcal{A}}_0^{-1}\| \|f\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1-|\varepsilon|/\varepsilon_0}, \tag{43}$$

what proves (32). This completes the proof. □

3 Fixed end-point optimal control

In this section, we apply closed-loop control laws to minimize (3), which is more useful in practice than the open-loop control, and we consider the following class of state weighting matrices

$$Q_\varepsilon(t) = \begin{bmatrix} Q_{11}(t) & \varepsilon Q_{12}(t) \\ \varepsilon Q_{21}(t) & Q_{22}(t) \end{bmatrix}, \quad t \in [i, f]. \tag{44}$$

One important attractive quality of the closed-loop strategy applied to LQOC models is its explicit control policy which can be derived by solving the corresponding MRDE

$$\begin{aligned} P_\varepsilon(t) = A_\varepsilon^T(t) &\left[P_\varepsilon(t+1)^{-1} + B_\varepsilon(t) R^{-1}(t) B_\varepsilon^T(t)\right] A_\varepsilon(t) \\ &+ Q_\varepsilon(t), \qquad t \in [i, f-1], \\ P_\varepsilon(f) = Q_\varepsilon(f),& \end{aligned} \tag{45}$$

what gives the minimizing closed-loop control :

$$u(t) = -K(t)x(t), \quad t \in [i, f-1], \tag{46}$$

where the Kalman gain matrices $K(t)$ are given below :

$$\left[B^T(t)P(t+1)B(t) + R(t)\right]^{-1} B^T(t)P(t+1)A(t). \tag{47}$$

We will study the behavior of the positive definite solution $P_\varepsilon(t)$ of MRDE (45) as ε tends to zero; we will suffice with an approximation of zeroth-order :

$$P_\varepsilon(t) = P_0(t) + O(\varepsilon), \tag{48}$$

to reduce the cost of each iteration and provide computational simplifications. By canceling the small parameter ε in (45), $P_0(t)$ verify the recurrence :

$$\begin{aligned} P_0(t) = A_0^T(t) &\left[P_0(t+1)^{-1} + B_0(t) R^{-1}(t) B_0^T(t)\right] A_0(t) \\ &+ Q_0(t), \quad t \in [i, f-1], \qquad P_0(f) = Q_0(f). \end{aligned} \tag{49}$$

Since the weighting matrices in (3) are positive definite, the existence of the inverses required in equations (45) and (49) is assured. The solution of (49) has a simplified block matrix structure. Induction backward in time can be used to obtain the following representation.

Proposition 2. *The blocks characterization of the solution of (49) is :*

$$P_0(t) = \begin{pmatrix} \bar{P}(t) & 0 \\ 0 & Q_{22}(t) \end{pmatrix}, \quad t \in [i, f] \tag{50}$$

where the matrix coefficient $\bar{P}(t)$ is solved backwards in time from the recurrence :

$$\begin{aligned} \bar{P}(t) = A_{21}^T(t)&[Q_{22}^{-1}(t+1) + E_{22}(t)]A_{21}(t) + Q_{11}(t) \\ &+ A_{11}^T(t)\bar{P}^{-1}(t+1)A_{11}(t), \quad t \in [i, f-1]; \\ \bar{P}(f) = Q_{11}(f).& \end{aligned} \tag{51}$$

The original MRDE (45) is brought back to solve the reduced-order recurrence (51), which facilitates and accelerates the calculations. The following theorem justifies this approximation.

Theorem 3. *There exists a positive real number ε_0, for all ε, s. t. $|\varepsilon| < \varepsilon_0$, the optimal feedback control satisfies*

$$u(t) = -R^{-1}(t) B_2^T(t) Q_{22}(t+1) z(t+1) + O(\varepsilon). \tag{52}$$

Moreover, the minimum cost achieved using the above control is :

$$J = \frac{1}{2}\left(y^T(i)\bar{P}(i)y(i) + z^T(i)Q_{22}(i)z(i)\right) + O(\varepsilon). \tag{53}$$

Proof. By denoting $\mathcal{X}^{(\varepsilon)} := (P_\varepsilon(i), P_\varepsilon(i+1), \cdots, P_\varepsilon(f))$, the MRDE (45) has the form :

$$\begin{aligned} \mathcal{P}_t(\mathcal{X}^{(\varepsilon)}, \varepsilon) :=& A^T(t,\varepsilon)\left[P(t+1,\varepsilon)^{-1} + E(t,\varepsilon)\right] A(t,\varepsilon) \\ &+ Q(t,\varepsilon) - P_\varepsilon(t), \quad t \in [i, f-1], \\ \mathcal{P}_f(\mathcal{X}^{(\varepsilon)}) :=& Q(f,\varepsilon) - P(f,\varepsilon), \end{aligned} \tag{54}$$

or can be written as $\mathcal{P}(\mathcal{X}^{(\varepsilon)}, \varepsilon) = 0$, where the mapping $\mathcal{P} : \mathbf{R}^{n\times n\times(f-i+1)} \times (-1,1) \longrightarrow \mathbf{R}^{n\times n\times(f-i+1)}$,

$$\mathcal{P}(\mathcal{X}^{(\varepsilon)}, \varepsilon) = \mathcal{P}\left(\mathcal{P}_i(\mathcal{X}^{(\varepsilon)}, \varepsilon), \cdots, \mathcal{P}_f(\mathcal{X}^{(\varepsilon)}, \varepsilon), \varepsilon\right), \tag{55}$$

is well defined and continuously differentiable with respect to all arguments including the parameter ε, since all $P_\varepsilon(t)$ are positive definite. The components of $\mathcal{X}^{(0)}$ satisfy (49), i.e., $\mathcal{P}(\mathcal{X}^{(0)}, 0) = 0$, and the Jacobian matrix $J_{\mathcal{X}^{(\varepsilon)}}\mathcal{P}|_{(\mathcal{X}^{(0)},0)}$ below

$$\begin{pmatrix} -I & \partial_2\mathcal{P}_i|_{(\mathcal{X}^{(0)},0)} & \cdots & 0 \\ 0 & -I & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & \cdots & \partial_{f-i+1}\mathcal{P}_i|_{(\mathcal{X}^{(0)},0)} \\ 0 & & \cdots & -I \end{pmatrix}, \tag{56}$$

is obviously invertible. Thus, the conditions for applying the implicit function theorem are fulfilled, there exist $0 < \varepsilon_0 < 1$, and a unique continuously differentiable $g : (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbf{R}^{n\times(f-i+1)}$, s.t., $\mathcal{P}(g(\varepsilon), \varepsilon) = 0$ and $g(\varepsilon) = g(0) + \varepsilon g'(\bar{\varepsilon})$, zeroth-order Maclaurin approximation, where $g' = -\left[J_{\mathcal{X}^{(\varepsilon)}}\mathcal{P}\right]^{-1} \partial_\varepsilon \mathcal{P}$ which is continuous, and $g(0)$ corresponds to the solution of (51). □

4 Illustrative examples

4.1 Example I

Consider a second order discrete control system

$$\begin{bmatrix} y(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} 3 & 0.00002 \\ 2 & 0.00005 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0.00002 \\ 3 \end{bmatrix} u(t), \quad (57)$$

s.t. $\varepsilon = 10^{-5}$; $A_{12}(t) = 2$, $A_{22}(t) = 5$, $B_1(t) = 2$ **and the initial conditions are** $y(0) = 2$, $z(0) = 3$. **We aim to minimize the index** (3) **for** $Q(t) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $Q_f = \begin{bmatrix} 2.5 & 3 \\ 3 & 2.5 \end{bmatrix}$, $N = 5$, **and** $R = 1$. **From** (38), **we find** $10^{-6} \leq \varepsilon_0 \leq 10^{-5}$.

Exact and zeroth-order solutions are evaluated and compared as shown in Table 1. Their values are very close, even so the value of ε **must be chosen below the threshold of** ε_0. **Reduced problem solving is intuitive; before adopting reduced solution, it is first necessary to check that the value of** $|\varepsilon|$ **is lower than** ε_0, **which must be calculated precisely.**

Table 1: Example I

	Zeroth-order solution	Exact solution
$y(0)$	2	2
$z(0)$	3	3
$u(0)$	-24	-22.763
$y(1)$	6	6
$z(1)$	-3	-3
$u(1)$	-3	-2.587
$y(2)$	18	18
$z(2)$	-8	-8
$u(2)$	-6	-5.862
$y(3)$	54	54
$z(3)$	-24	-24
$u(3)$	-21	-20.954
$y(4)$	162	162
$z(4)$	-73	-71
$u(4)$	-60	-59.984
$y(5)$	486	486
$z(5)$	-545	-545

4.2 Example II

For the state system (57), **and the initial conditions** $y(0) = 2$, $z(0) = 3$, **we apply closed-loop control laws to minimize the cost** (3), **for** $N = 5$, $R = 1$, $\varepsilon = 10^{-4}$ **and**

$$Q(t) = Q_f = \begin{bmatrix} 2 & 0.0001 \\ 0.0001 & 2 \end{bmatrix} \quad (58)$$

Exact solutions of Riccati equation (45) **and the and reduced solutions of the reduced equation** (51) **are evaluated and compared as shown in Table 2. Their values are very close, and the reduced problem** (51) **solving is intuitive and provides simplicity and speed of calculation.**

Table 2: Example II

	Reduced solution	Exact Solution
$P(4)$	$\begin{bmatrix} 44.5012 & 0.0000 \\ 0.0000 & 2.0000 \end{bmatrix}$	$\begin{bmatrix} 44.5000 & 0.0011 \\ 0.0011 & 2.0000 \end{bmatrix}$
$P(3)$	$\begin{bmatrix} 40.2025 & 0.0000 \\ 0.0000 & 2.0000 \end{bmatrix}$	$\begin{bmatrix} 40.2037 & 0.0011 \\ 0.0011 & 2.0000 \end{bmatrix}$
$P(2)$	$\begin{bmatrix} 40.0248 & 0.0000 \\ 0.0000 & 2.0000 \end{bmatrix}$	$\begin{bmatrix} 40.2253 & 0.0011 \\ 0.0011 & 2.0000 \end{bmatrix}$
$P(1)$	$\begin{bmatrix} 40.0249 & 0.0000 \\ 0.0000 & 2.0000 \end{bmatrix}$	$\begin{bmatrix} 40.2253 & 0.0011 \\ 0.0011 & 2.0000 \end{bmatrix}$
$P(0)$	$\begin{bmatrix} 40.0249 & 0.0000 \\ 0.0000 & 2.0000 \end{bmatrix}$	$\begin{bmatrix} 40.2253 & 0.0011 \\ 0.0011 & 2.0000 \end{bmatrix}$

5 Conclusions and Future Work

In this paper, a general approach is suggested for solving a class of finite-horizon discrete-time linear-quadratic optimal control problems governed by time-variant two-time-scale state equations. First, the fixed end-point problem is investigated, it is equivalent to solving the corresponding TPBVP which is not exactly solvable. We use a natural perturbation method to emerge an asymptotic iterative algorithm converging to the exact solution. Second, we consider the free end-point problem. The solution of the linear-quadratic optimal control problem is obtained by solving the reduced-order discrete-time algebraic Riccati equations and implementing the reduced-order Kalman filters. Several cases of finite-horizon optimization problems will be considered separately. This will be indicated in other articles.

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