Analysis of the Stability of the Riemann Solutions for the Suliciu Relaxation System*

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***Abstract*—In this study, we investigate the perturbed Riemann problem which contains the delta shock wave for the simplified Suliciu system. We observe that the stability of the Riemann solutions under the small perturbation. Furthermore, we investigate the perturbed initial problem and obtain the instability of the Riemann solutions under the perturbation. The above results both reveal the internal mechanism of the Suliciu relaxation system which is of the Temple class type and shows that the solution of the Temple class type has much more simpler structure.**

***Index Terms*—Wave interaction, Riemann problem, hyperbolic conservation laws, generalized Riemann problem, delta shock, Suliciu system.**

I. Introduction

We consider the following Suliciu relaxation equations ([1], [2], [3], [4] and the references cited therein) which describes the viscoelastic shallow fluid as follows

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + s^2 v)_x = 0, \\ (\rho v)_t + (\rho u v + u)_x = 0, \end{cases} \tag{1}$$

where $\rho \geq 0$, u, $s > 0$, $v = \frac{\pi}{s^2}$ denotes respectively the layer depth of fluid, the horizontal velocity, the related to the stress tensor, the new variable in connection with the pressure, and π is the relaxed pressure.

(1) is viewed the relaxation for the following isentropic Chaplygin gas equations ([5], [6])

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = 0, \end{cases} \tag{2}$$

where ρ, u, P is respectively the density, the velocity and the pressure, the state equation is given by $P(\rho) = -\frac{s^2}{\rho}$, where $s > 0$ is a constant.

Many authors ([7], [8] and the references cited therein) studied the Suliciu system (1). In [8], we studied the wave interactions containing no S_δ for (1) and found some interesting phenomena.

In this study, we will study the elementary wave interactions containing S_δ solution for (1) with

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho^l, u^l, v^l), & x \in (-\infty, -\theta), \\ (\rho^m, u^m, v^m), & x \in -(\theta, \theta), \\ (\rho^r, u^r, v^r), & x \in (\theta, +\infty), \end{cases} \tag{3}$$

and the small perturbation parameter $\theta > 0$. (3) can be viewed by the local perturbation on

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho^r, u^r, v^r), & x > 0, \\ (\rho^l, u^l, v^l), & x < 0. \end{cases} \tag{4}$$

We have detailed analysis in the phase plane (u, v), and see that S_δ may disappear after the small perturbation. The perturbed Riemann problem and Riemann problem were widely studied([9], [10], [11], [12]). Through detailed analysis and discussions, we obtain the instability of the initial problem (1) and (4) after the above small perturbation (3).

Furthermore, we study the perturbed initial problem (1) with the following initial values

$$(\rho, u, v)(x, 0) = (\check{\rho}_{l,r}, \check{u}_{l,r}, \check{v}_{l,r})(x), \quad x > 0, or x < 0, \tag{5}$$

where $\check{\rho}_{l,r}(x)$, $\check{u}_{l,r}(x)$, $\check{v}_{l,r}(x)$ are the arbitrary smooth functions satisfying

$$\lim_{x \to 0_{l,r}} (\check{\rho}_{l,r}, \check{u}_{l,r}, \check{v}_{l,r})(x) = (\rho_{l,r}, u_{l,r}, v_{l,r}).$$

(1) and (5) is viewed as the perturbation of the (1) and (4). We should solve the problem that is whether the solution structures of the perturbed initial problem (1) and (5) are similar with the original initial problem (1) and (4).

For the most cases, we find that the stability of the initial problem (1) and (4). However, for some other cases, S_δ may disappear after the local small perturbation and it shows the instability of the initial problem (1) and (4)

The arrangement of the article is as follows. The initial problem (1) and (4) is given briefly for convenience in II. In Section III, we study mainly the elementary wave interactions containing the delta shock wave for the perturbed initial problem (1) and (3). In Section IV, we construct the solutions of the perturbation for (1) and (5). Our main conclusions are given in Section V.

II. Preliminaries

Now give curtly the initial problem (1) and (4). For the detailed discussions, please refer to [7], [8].

The characteristic roots of (1) are $\kappa_1 = u - \frac{s}{\rho}$, $\kappa_2 = u$ $\kappa_3 = u + \frac{s}{\rho}$, it tells the strictly hyperbolic of (1). Since

$$\overrightarrow{\gamma}_1 = (\rho^2, -s, 1)^T, \quad \overrightarrow{\gamma}_2 = (1, 0, 0)^T, \quad \overrightarrow{\gamma}_3 = (\rho^2, s, 1)^T,$$

which imply that

$$\nabla \kappa_1 \cdot \overrightarrow{\gamma}_1 = 0, \quad \nabla \kappa_2 \cdot \overrightarrow{\gamma}_2 = 0, \quad \nabla \kappa_2 \cdot \overrightarrow{\gamma}_3 = 0,$$

it follows that the every characteristic region is linearly degenerate.

Manuscript received April 27, 2022; revised October 10, 2022. This work is supported by the Foundation for Young Scholars of Shandong University of Technology (No. 115024).

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We study the self-similar solution $(\rho,u,v)(x,t) = (\rho,u,v)(\varsigma)$, $\varsigma=\frac{x}{t}$. (1) and (4) turns into the problem

$$\begin{cases} -\varsigma\rho_\varsigma+(\rho u)_\varsigma=0, \\ -\varsigma(\rho u)_\varsigma+(\rho u^2+s^2v)_\varsigma=0, \\ -\varsigma(\rho v)_\varsigma+(\rho uv+u)_\varsigma=0, \end{cases} \tag{6}$$

and $(\rho,u,v)(\pm\infty)=(\rho_\pm,u_\pm,v_\pm)$. For the smooth solutions, we denote $U=(\rho,u,v)^T$ and convert (6) to

$$B(H)H_\varsigma=0, \tag{7}$$

where

$$B(H)=\begin{pmatrix} u-\varsigma & \rho & 0 \\ 0 & \rho(u-\varsigma) & 0 \\ 0 & 1 & \rho(u-\varsigma) \end{pmatrix}.$$

Besides $(\rho,u,v)=constant$, for the given left state (ρ^l,u^l,v^l), the backward rarefaction wave curve is described by

$$\overleftarrow{R}(\rho^l,u^l,v^l):\begin{cases} \xi=\kappa_1=u-\frac{s}{\rho}, \\ u=u^l-s(v-v^l), \\ v<v^l,\quad u>u^l. \end{cases} \tag{8}$$

For the bounded discontinuity at $\varsigma=\omega$, it holds the Rankine-Hugoniot relations

$$\begin{cases} -\omega[\rho]+[\rho u]=0, \\ -\omega[\rho u]+[\rho u^2+s^2v]=0, \\ -\omega[\rho v]+[\rho uv+u]=0, \end{cases} \tag{9}$$

where $[\rho]=\rho^r-\rho^l$, $\rho^l=\rho(\omega-0)$, $\rho^r=\rho(\omega+0)$, etc.

For the given left state (ρ^l,u^l,v^l), the backward shock wave curve is described by

$$\overleftarrow{S}(\rho^l,u^l,v^l):\begin{cases} \tau=u-\frac{s}{\rho}, \\ u=u^l-s(v-v^l), \\ v>v^l,\quad u<u^l. \end{cases} \tag{10}$$

From (9) we know the expression of the contact discontinuity is given by

$$J:\quad u=u^l,\quad v=v^l,\quad [\rho]\neq 0. \tag{11}$$

R, S, and J are called the elementary waves of the system (1). Because the shock curves coincide with the rarefaction curves in (u,v) ([13]), we know that it is the classical Temple type.

The elementary wave curves in (u,v) from the state (l) are given in the plane (u,v) (Fig. 1), and the line l_1 is described by $v=\frac{1}{s}[u-(u_l+sv_l)]$, and $M(u_l+sv_l,\ 0)$.

In the similar way, the elementary wave curves in (u,v) from the state (r) are given in the plane (u,v) (Fig. 2), where the line l_2 is given by $v=-\frac{1}{s}[u-(u_r-sv_r)]$, and $N(u_r-sv_r,\ 0)$.

When $(u_r,v_r)\in$ I, we know that the Riemann solution is expressed by is $\overleftarrow{S}+J+\overrightarrow{R}$; when $(u_r,v_r)\in$ II, the Riemann solution is expressed by is $\overleftarrow{S}+J+\overrightarrow{S}$; when $(u_r,v_r)\in$ III, the Riemann solution is expressed by is $\overleftarrow{R}+J+\overrightarrow{S}$; when $(u_r,v_r)\in$ IV, the Riemann solution is expressed by is $\overleftarrow{R}+J+\overrightarrow{R}$.

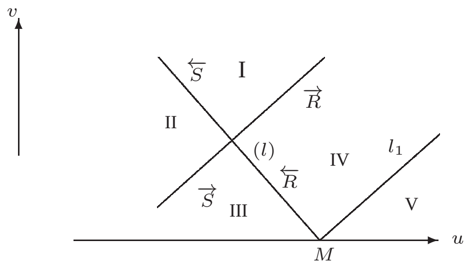

Fig. 1 Elementary wave curves from (l) in (u,v).

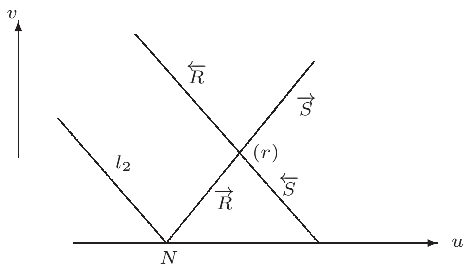

Fig. 2 Elementary wave curves from (r) in (u,v).

When $(u_r,v_r)\in$ V, that is to say, $u_r+\frac{s}{\rho_r}\le u_l-\frac{s}{\rho_l}$, in the follows we seek the delta shock wave solution.

For the piecewise smooth solution of the system (1), we have

$$(\rho,u,v)(x,t)=\begin{cases} (\rho^l,u^l,v^l), & as\ x<x(t), \\ (\sigma(t)\delta(x-x(t)),u_\delta(t),f(t)), & x=x(t), \\ (\rho^r,u^r,v^r), & as\ x>x(t), \end{cases} \tag{12}$$

where $\delta(x-x(t))$, $\sigma(t)$ denotes the $\delta-$ measure and the weight respectively. Furthermore, we have the generalized Rankine-Hugoniot condition as follows

$$\begin{cases} \frac{dx(t)}{dt}=u_\delta(t), \\ \frac{d\sigma(t)}{dt}=-[\rho]u_\delta(t)+[\rho u], \\ \frac{d(\sigma(t)u_\delta(t))}{dt}=-[\rho u]u_\delta(t)+[\rho u^2+s^2v)], \\ \frac{d(\sigma(t)g(t))}{dt}=-[\rho v]u_\delta(t)+[\rho uv+u)]. \end{cases} \tag{13}$$

The δ-entropy condition is given by

$$\kappa_3(\rho^r,u^r,v^r)\le u_\delta(t)\le\kappa_1(\rho^l,u^l,v^l), \tag{14}$$

i.e.,

$$u^r+\frac{s}{\rho^r}\le u_\delta(t)\le u^l-\frac{s}{\rho^l}. \tag{15}$$

When $[\rho]=0$, it follows that

$$\begin{cases} u_\delta=\frac{u^r+u^l}{2}+s^2\frac{v^r-v^l}{2\rho_l(u^r-u^l)}, \\ x(t)=u_\delta t, \\ \sigma(t)=\rho^l(u^l-u^r)t, \\ f(t)=\frac{(\rho^ru^rv^r+u^r-\rho^lu^lv^l-u^l)-u_\delta}{\rho^ru^r-\rho^lu^l}t, \end{cases} \tag{16}$$

When $[\rho] \neq 0$, it follows that

$$\begin{cases} u_\delta = \frac{([\rho u]) - \sqrt{([\rho u])^2 - ([\rho])([\rho u^2 + s^2 v])}}{[\rho]}, \\ x(t) = u_\delta t, \\ \sigma(t) = \sqrt{([\rho u])^2 - ([\rho])([\rho u^2 + s^2 v]} \cdot t, \\ f(t) = \frac{-[\rho v][\rho u] + [\rho v]\sqrt{([\rho u])^2 - ([\rho])([\rho u^2 + s^2 v])} + [\rho][\rho u v + u]}{([\rho])\sqrt{([\rho u])^2 - ([\rho])([\rho u^2 + s^2 v])}}. \end{cases} \tag{17}$$

According to the analysis, we obtain the result as follows.

Theorem 2.1 There is uniquely the Riemann solution of the initial value problem (1) and (4).

III. Wave interaction for (1) and (3)

In what follows, we study the elementary wave interactions for (1) and (3), and construct the unique solution in (u, v) (see [14], [15]). We view the initial data (3) as the small perturbation on (4) and want to know whether or not the initial problem (1) and (4) are the limits of the perturbed initial problem (1) and (3) when $\theta \to 0$. To discuss the all possible cases containing the delta shock, we should consider the four cases as follows: $\overrightarrow{W}$ and δ, $\overleftarrow{W}$ and δ, δ_1 and δ_2, J and δ, and notice that $\overrightarrow{W} = \overrightarrow{R} \cup \overrightarrow{S}$, $\overleftarrow{W} = \overleftarrow{R} \cup \overleftarrow{S}$.

Case 1: The wave interaction of $\overrightarrow{W}$ and δ. (Fig. 3)

Considering the delta entropy conditions

$$u_r + \frac{s}{\rho_r} \leq u_\delta(t) \leq u_m - \frac{s}{\rho_m}, \tag{18}$$

and the slope $\frac{dx}{dt} = u_m + \frac{s}{\rho_m}$ of the forward wave $\overrightarrow{W}$, we know that the forward wave $\overrightarrow{W}$ will overtake the delta shock wave at $t = t_0$, and should solve the new Riemann problem.

From the following conditions

$$\begin{cases} \overrightarrow{W}(Q_l Q_m): \; u_m = u_l + s(v_m - v_l), \\ \overleftarrow{W}(Q_m): \; u = u_m - s(v - v_m), \\ \overleftarrow{W}(Q_l Q_*): \; u_* = u_l - s(v_* - v_l), \\ \overrightarrow{W}(Q_r Q_*): \; u_* = u_r + s(v_* - v_r), \end{cases} \tag{19}$$

we conclude that there are three subcases for this case.

Subcase 1.1 $\overrightarrow{S} + \delta$ and $v_* \leq 0$. (Fig. 4)

For this subcase, Q_l and Q_l in the (u, v) plane are given in Fig. 4. From Q_l we draw $\overleftarrow{W}(Q_l)$, and from Q_r we draw $\overrightarrow{W}(Q_r)$, therefore we should consider the δ solution. The strength of the new δ solution is superimposed on the original δ curve, and the new δ solution of the new initial problem is constructed by as follows.

When $\rho_l = \rho_r$, we obtain that

$$\begin{cases} u_\delta = \frac{u^r + u^l}{2} + s^2 \frac{v^r - v^l}{2\rho^l(u^r - u^l)}, \\ \sigma(t) = \rho_l(u^l - u^r)(t - t_0) + \rho^m(u^m - u^r)t_0, \end{cases} \tag{20}$$

when $\rho_l \neq \rho_r$, we know that

$$\begin{cases} u_\delta = \frac{([\rho u]_1) - \sqrt{([\rho u]_1)^2 - ([\rho]_1)([\rho u^2 + s^2 v]_1)}}{[\rho]_1}, \\ \sigma(t) = \sqrt{([\rho u]_1)^2 - ([\rho]_1)([\rho u^2 + s^2 v]_1)}(t - t_0) \\ + \sqrt{([\rho u]_2)^2 - ([\rho u]_2)([\rho u^2 + s^2 v]_2)}t_0, \end{cases} \tag{21}$$

where $[\cdot]_1$ denotes the jump between l and r, $[\cdot]_2$ denotes the jump between m and r.

In this case, it follows that the unique solution after perturbation is shown by $\overrightarrow{S} + \delta \to \delta$.

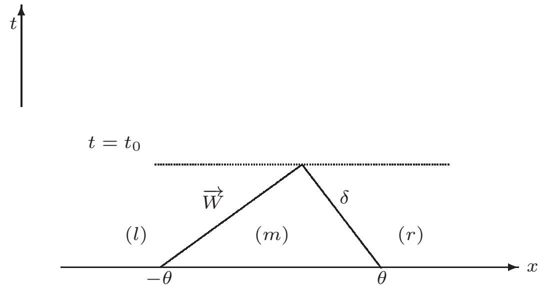

Fig. 3 Interaction of $\overrightarrow{W}$ and δ.

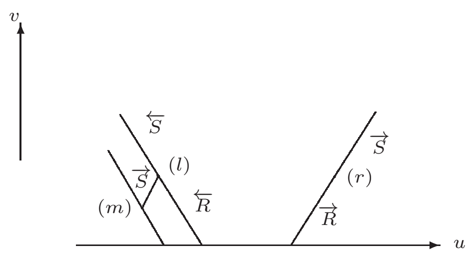

Fig. 4 Overtaking of $\overrightarrow{S}$ and δ, $v_* \leq 0$.

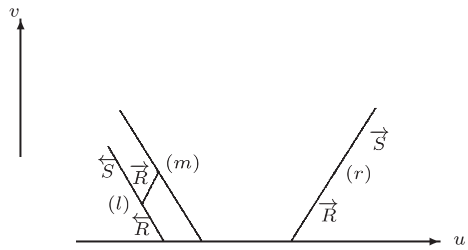

Fig. 5 Overtaking of $\overrightarrow{R}$ and δ, $v_* \leq 0$.

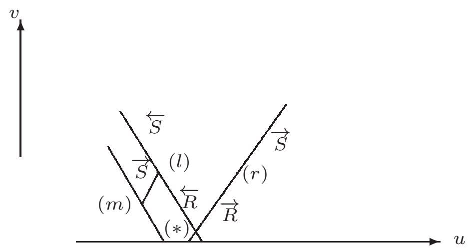

Fig. 6 Overtaking of $\overrightarrow{S}$ and δ, $v_* > 0$.

Subcase 1.2 $\overrightarrow{R} + \delta$ and $v_* \leq 0$. (Fig. 5)

Similar discussions with the above subcase, the relations of Q_l and Q_l are given in Fig. 5. In this case, we just consider δ wave solution. The strength and the slope of the new delta shock wave is given as (20) and (21). It follows that the unique solution of the perturbed initial problem is $\overrightarrow{R} + \delta \to \delta$.

Subcase 1.3 $\overrightarrow{S} + \delta$ and $v_* > 0$. (Fig. 6)

For this subcase, Q_l and Q_l in the (u, v) plane are given in Fig. 6. $\overleftarrow{W}(Q_l)$ intersects with $\overrightarrow{W}(Q_r)$ at the point Q_* and $v_* > 0$. It reveals that the unique solution after the perturbation is shown as $\overrightarrow{S} + \delta \to \overleftarrow{R} + J + \overrightarrow{R}$ *or* $\overrightarrow{S}$.

Theorem 3.1 When $\overrightarrow{S}$ intersects with δ, we find that, the new δ wave solution occurs, or $\overleftarrow{R}$ occurs. Furthermore, $\overrightarrow{R}$ may occur. When $\overrightarrow{R}$ intersects with δ, the result is the new δ wave solution. This reveals the stability of (1) and (4).

Case 2: The wave interaction of $\overleftarrow{W}$ and δ.

For this subcase, Q_l and Q_l in the (u, v) plane are given in Fig. 8 and Fig. 10.

It can be seen that the state Q_l is located in the wave curve $\overleftarrow{W}(Q_m)$, and δ wave solution should be considered. The unique solution is listed by $\overleftarrow{R} + \delta \to \delta$ (see Fig. 7) and $\overleftarrow{S} + \delta \to \delta$ (see Fig. 9). The strength and the slope of the new δ are given by (20) and (21) respectively.

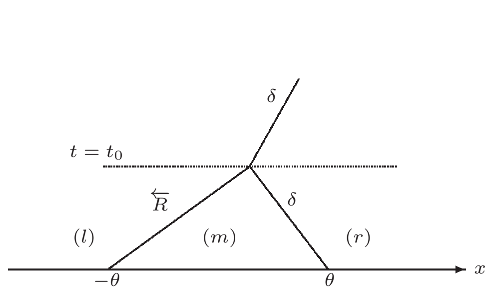

Fig. 7 Interaction of $\overleftarrow{R}$ and δ.

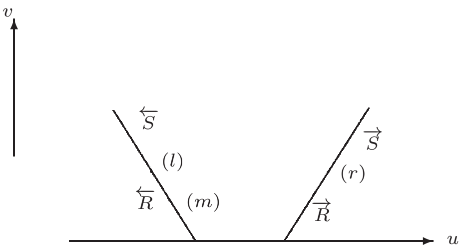

Fig. 8 Interaction of $\overleftarrow{R}$ and δ.

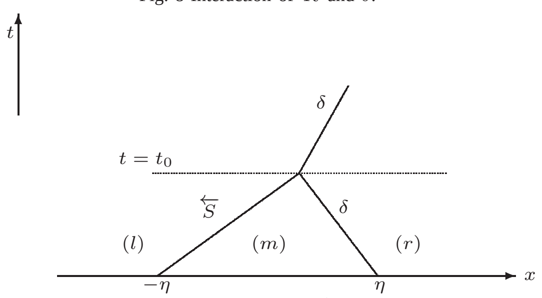

Fig. 9 Interaction of $\overleftarrow{S}$ and δ.

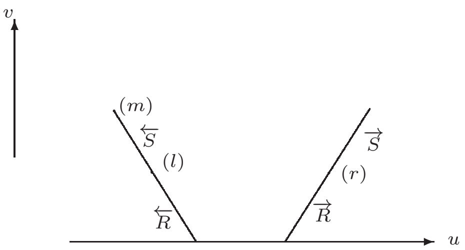

Fig. 10 Interaction of $\overleftarrow{S}$ and δ.

Theorem 3.2 When $\overleftarrow{S}$ or $\overleftarrow{R}$ intersects with δ, we find the unique solution is the δ wave solution which reveals reveals the stability of (1) and (4).

Case 3: The wave interaction of δ_1 and δ_2.

From the delta entropy conditions

$$\begin{cases} \delta_1: \quad u^m + \frac{s}{\rho^m} \le u_\delta(t) \le u^l - \frac{s}{\rho^l}, \\ \delta_2: \quad u^r + \frac{s}{\rho^r} \le u_\delta(t) \le u^m - \frac{s}{\rho^m}, \end{cases} \tag{22}$$

we know that δ_1 will overtake δ_2 at $t = t_0$. The delta shock wave δ_1 is formed by the backward wave $\overleftarrow{W}(Q_l)$ with the forward wave $\overrightarrow{W}(Q_m)$, and the delta shock wave δ_2 is formed by the backward wave $\overleftarrow{W}(Q_m)$ with the forward wave $\overrightarrow{W}(Q_r)$. The relations of Q_l and Q_l are given in Fig. 12. For the the backward wave $\overleftarrow{W}(Q_l)$ with the forward wave $\overrightarrow{W}(Q_r)$, we just consider δ_3, that is to say, $\delta_1 + \delta_2 \to \delta_3$ (see Fig. 11). The strength and the slope of δ_3 are given by respectively as follows.

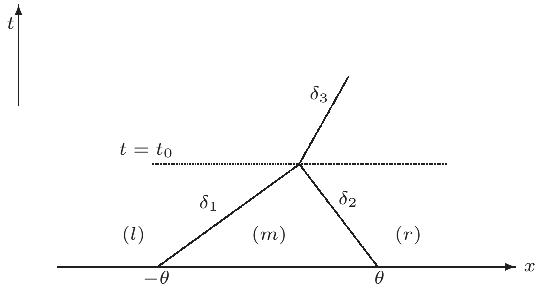

Fig. 11 Interaction of δ_1 and δ_2.

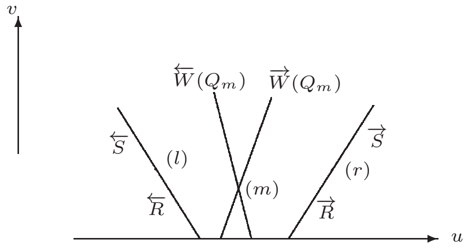

Fig. 12 Interaction of δ_1 and δ_2.

When $\rho_l = \rho_r$, we have that

$$\begin{cases} u_\delta = \frac{u^r + u^l}{2} + s^2 \frac{v^r - v^l}{2\rho^l (u^r - u^l)}, \\ \sigma(t) = \rho^l([u]_1)(t - t_0) + \rho_m([u]_2)t_0 + \rho^l([u]_3)t_0, \end{cases} \tag{23}$$

when $\rho_l \neq \rho_r$, we obtain that

$$\begin{cases} u_\delta = \frac{([\rho u]_1) - \sqrt{([\rho u]_1)^2 - ([\rho]_1)([\rho u^2 + s^2 v]_1)}}{[\rho]_1}, \\ \sigma(t) = \sqrt{([\rho u]_1)^2 - ([\rho]_1)([\rho u^2 + s^2 v]_1)}(t - t_0) \\ + \sqrt{([\rho u]_2)^2 - ([\rho]_2)([\rho u^2 + s^2 v]_2)}t_0 \\ + \sqrt{([\rho u]_3)^2 - ([\rho]_3)([\rho u^2 + s^2 v]_3)}t_0, \end{cases} \tag{24}$$

where $[\cdot]_1$ denotes the jump between l and r, $[\cdot]_2$ denotes the jump between m and r, $[\cdot]_3$ denotes the jump between l and m.

Theorem 3.3 When δ intersects with another δ, we find that after the perturbation, the new solution is still δ which tells the stability of (1) and (4).

Case 4: The wave interaction of J and δ.

From

$$\begin{cases} \delta_1: \quad u^r + \frac{s}{\rho^r} \le u_\delta(t) \le u^m - \frac{s}{\rho^m}, \\ J: \quad \frac{dx}{dt} = u^m, \end{cases} \tag{25}$$

and since $u^m - \frac{s}{\rho^m} < u^m$, J will overtake δ_1 at $t = t_0$ (Fig. 13), and we should solve the new initial problem. Q_l and Q_l are given respectively in the (u, v) plane in Fig. 14 if $v_* \le 0$ and in Fig. 16 if $v_* > 0$.

Due to

$$\begin{cases} \overleftarrow{W}(Q_l Q_*): \ u_* = u^l - s(v_* - v^l), \\ \overrightarrow{W}(Q_r Q_*): \ u_* = u^r + s(v_* - v^r), \end{cases} \tag{26}$$

for $\overleftarrow{W}(Q_l)$ and $\overrightarrow{W}(Q_r)$, when $v_* \le 0$ we find that it is just to consider δ_2 (see Fig. 13), and the strength and the slope of the new δ_2 are given respectively by (20) and (21). When $v_* > 0$, we get $\overleftarrow{R} + J + \overrightarrow{R}$ *or* $\overrightarrow{S}$ (see Fig. 15).

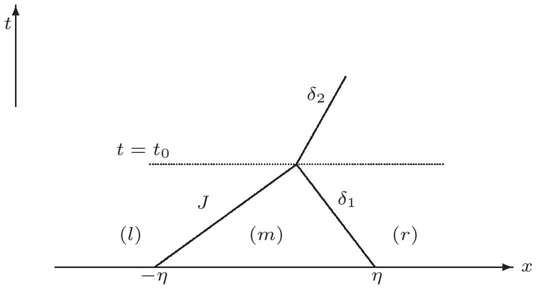

Fig. 13 Interaction of J and δ.

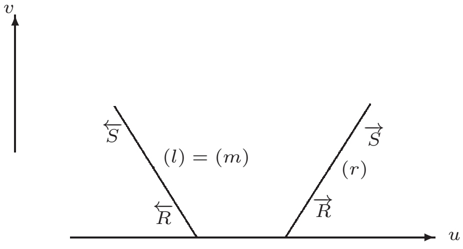

Fig. 14 Interaction of J and δ, $v_* \le 0$.

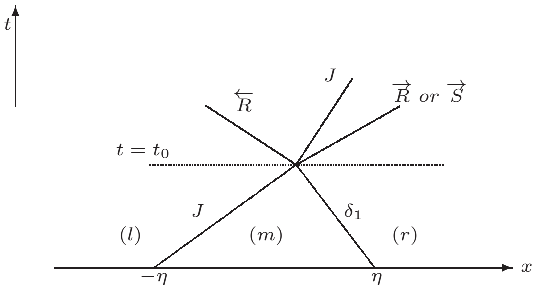

Fig. 15 Interaction of J and δ.

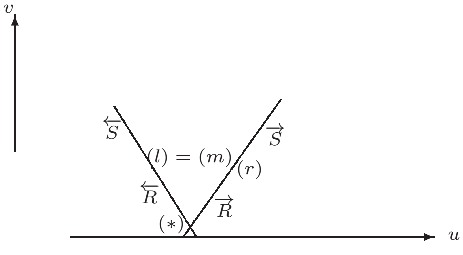

Fig. 16 Overtaking of J and δ, $v_* > 0$.

Theorem 3.4 When J intersects with δ, we observe that the unique solution is still δ, or $\overleftarrow{R}$ will occur propagating in the opposite direction, and $\overrightarrow{R}$ (or $\overrightarrow{S}$) will occur. The above results show the instability of (1) and (4).

Now we summarize the following conclusions for the Riemann problem (1) and (4).

Theorem 3.5 The Riemann solution of (1) and (4) is stable for the most case, while for few cases, the Riemann solution is unstable which shows the instability of the system (1).

IV. The perturbed initial problem (1) and (5)

Next we discuss the construction of the perturbed initial problem of (1) and (5). From the conclusions in [16] and [17], we know that the conventional solution $(\rho^l, u^l, v^l)(x,t)$ $((\rho^r, u^r, v^r)(x,t))$ is respectively expressed in $G_l(G_r)$ when $t > 0$ is small. The characteristic curve OP is given by $x = \kappa_- t$, and the characteristic curve OQ is given by $x = \kappa_+ t$ (see Fig. 17).

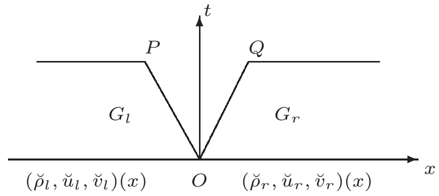

Fig. 17 The region in (x,t) plane.

In what follows, we study the perturbed solution of (1) and (5), and use the same symbols in our next discussions after the small perturbation because there will be no confusion.

Case 1. $u^r + \frac{s}{\rho^r} > u^l - \frac{s}{\rho^l}$. After perturbation, we still have $u^r + \frac{s}{\rho^r} > u^l - \frac{s}{\rho^l}$.

Subcase 1.1. When $(u^r, v^r) \in$ I (see Fig. 1), i.e., the initial solution is $\overleftarrow{S} + J + \overrightarrow{R}$.

By the small perturbation on the initial data, it follows that $\overleftarrow{S} + J + \overrightarrow{R}$.

Subcase 1.2. When $(u^r, v^r) \in$ II (see Fig. 1), i.e., the initial solution is $\overleftarrow{S} + J + \overrightarrow{S}$.

By the small perturbation on the initial data, it follows that $\overleftarrow{S} + J + \overrightarrow{S}$.

Subcase 1.3. When $(u^r, v^r) \in$ III (see Fig. 1), i.e., the initial solution is $\overleftarrow{R} + J + \overrightarrow{S}$.

By the small perturbation on the initial data, it follows that $\overleftarrow{R} + J + \overrightarrow{S}$.

Subcase 1.4. When $(u^r, v^r) \in$ IV (see Fig. 1), i.e., the initial solution is $\overleftarrow{R} + J + \overrightarrow{R}$.

By the small perturbation on the initial data, it follows that $\overleftarrow{R} + J + \overrightarrow{R}$.

Now, we know the following conclusions.

Theorem 4.1 When the Riemann solution contains no δ, we find that there is still no δ for the perturbed initial problem (1) and (5), which shows that the initial problem (1) and (4) are stable.

Case 2. $u^r + \frac{s}{\rho^r} \le u^l - \frac{s}{\rho^l}$, and the solution of (1) and (4) is δ. By the small perturbation on the initial data, it reveals two subcases as follows $u^r + \frac{s}{\rho^r} \le u^l - \frac{s}{\rho^l}$ or $u^r + \frac{s}{\rho^r} > u^l - \frac{s}{\rho^l}$.

Subcase 2.1. If $u^r + \frac{s}{\rho^r} \le u^l - \frac{s}{\rho^l}$, the perturbed initial problem has the δ solution.

Subcase 2.2. If $u_r + \frac{s}{\rho_r} > u_l - \frac{s}{\rho_l}$, we discuss as follows.

When $(u_r, v_r) \in$ I, the perturbed initial problem has the solution $\overleftarrow{S} + J + \overrightarrow{R}$; when $(u_r, v_r) \in$ II, the perturbed initial problem has the solution $\overleftarrow{S} + J + \overrightarrow{S}$; when $(u_r, v_r) \in$ III, the perturbed initial problem has the solution $\overleftarrow{R} + J + \overrightarrow{S}$; when $(u_r, v_r) \in$ IV, the perturbed initial problem has the solution $\overleftarrow{R} + J + \overrightarrow{R}$.

Theorem 4.2 When the initial problem (1) and (4) contains δ, we conclude that the perturbed initial problem (1) and (5) contains δ for subcase 2.1, while for subcase 2.2, δ does not occur for the new perturbed initial problem (1) and (5) which shows the stability of the initial problem (1) and (4).

Based on the above discussions, we get the following conclusions for the perturbed Riemann problem.

Theorem 4.3 For the perturbed initial problem (1) and (5), it follows that the initial problem (1) and (4) are stable under the such small local perturbation on the initial data. However, for some cases, the delta shock wave of the initial problem (1) and (4) may disappear which indicates the instability of the initial problem (1) and (4) under the such small perturbation.

V. Conclusion

We construct uniquely in detail the perturbed solution of the initial problem (1) and (3) by virtue of the characteristic analysis skills. We conclude that for some cases, the delta shock wave of the initial problem (1) and (4) may disappear after the perturbation which indicates that the initial problem (1) and (4) are unstable. For the perturbed initial problem (1) and (5), we observe that for some cases the delta shock wave may disappear which reveals the instabilty of the Riemann solutions for such small perturbation.

The elementary wave interactions and the generalized Riemann problem for system (1) play an important role in Suliciu relaxation system such as the comparison with the numerical analysis, and it is also important for mathematical theory of the Suliciu system.

In our coming works, we would like to discuss the numeral calculations analysis for the Suliciu system. In the future, we also consider to relax or extend some conditions and discuss the more general initial value problem.

References

[1] F. Bouchut, “Nonlinear stability of finite volume mehtods for hyperbolic conservation laws and well-balanced schemes for sources,” *Frontiers in Mathematics*, Birkhäuser, Basel, Switzerland, 2004.

[2] F. Bouchut and S. Boyaval, “A new model for shallow viscoelastic fluids,” *Mathematical Models and Methods in Applied Sciences*, vol. 23, no. 8, pp. 1479-1526, 2013.

[3] G. Carbou, B. Hanouzet and R. Natalini, Semiliear behaviour for totally linearly degenerate hyperbolic systems with relaxation, *Journal of Differential Equations*, vol. 246, no. 1, pp. 291-319, 2009.

[4] I. Suliciu, “On modelling phase transitions by means of ratetype constitutive equations, shock wave structure,” *International Journal of Engineering Science*, vol. 28, no. 8, pp. 829-841, 1990.

[5] T.T. Chen, A.F. Qu and Z. Wang, “The two-dimensional Riemann problem for isentropic Chaplygin gas,” *Acta Mathematica Scientia*, (Series A) vol. 37A, no. 6, pp. 1053-1061, 2017.

[6] Q. Wang, J.Q. Zhang and H.C. Yang, “Two dimensional Riemann-type problem and shock diffraction for the Chaplygin gas,” *Applied Mathematics Letters*, Volume 101, Article 106046, 2020.

[7] Richard De la cruz, Juan Galvis, Juan Carlos Juajibioy and Leonardo Rendón, “Delta shock wave for the Suliciu relaxation system,” *Advances in Mathematical Physics*, vol. 2014, Article ID 354349, 11 pages, 2014.

[8] Bingzhuang Liu and Yujin Liu, “Wave interactions for the Suliciu relaxation system,” *Journal of Applied Mathematics and Physics*, vol. 9, no. 7, pp. 1483-1496, 2021.

[9] Y.J. Liu and W.H. Sun, “The Riemann problem for the simplified combustion model in magnetogasdynamics,” *IAENG International Journal of Applied Mathematics*, vol. 49, No. 4, pp. 513-520, 2019.

[10] W.C. Sheng, M.N. Sun and T. Zhang, “The generalized Riemann problem for a scalar nonconvex Chapman-Jouguet combustion model,” *SIAM journal on Applied Mathematics*, vol. 68, No. 2, pp. 544-561, 2007.

[11] Yujin Liu and Wenhua Sun , “Elementary Wave Interactions for a Simplified Model in Magnetogasdynamics,” *Engineering Letters*, vol. 28, no. 4, pp. 1081-1087, 2020.

[12] Wenhua Sun and Yujin Liu, “The Generalized Riemann Problem for the Simplified Model in Magnetogasdynamics with Combustion,” *Engineering Letters*, vol. 29, no. 2, pp. 391-399, 2021.

[13] B. Temple, “Systems of conservation laws with coinciding shock and rarefaction curves,” *Contemporary Mathematics*, 1983, vol. 17, pp. 143-151.

[14] T. Chang and L. Hsiao, “The Riemann problem and interaction of waves in gas dynamics,” Pitman Monographs, No. 41, *Longman Scientific and technical*, Essex, 1989.

[15] L. Zhu, “Chaplygin gas dynamic system in adiabatic flow,” *Master thesis*, Shanghai, Shanghai University, 2010.

[16] T.T. Li, “Global classical solutions for quasilinear hyperbolic system,” *John Wiley and Sons*, New York, 1994.

[17] T.T. Li and W.C. Yu, “Boundary value problems for quasilinear hyperbolic systems,” *Duke University Mathematics*, 1985.