Outer Theta and Theta Outer Inverses

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***Abstract*—In this paper, a new set of matrices - outer theta matrices and theta outer matrices are defined. These classes of matrices serve as outer inverses of $(A^{\dagger_s})^\theta$ where $(A^{\dagger_s})^\theta$ is the s-conjugate transpose of s-g inverse of A provided it exists. Certain characterizations of outer theta matrices and theta outer matrices are given here. Also, the integral and limit representations of both outer theta and theta outer matrices are explained. To demonstrate their application, some system of linear equations are solved using these matrices.**

***Index Terms*—Moore Penrose inverse, outer inverse, transpose, secondary conjugate transpose**

I. Introduction

The purpose of this paper is to define a new set of matrices - outer theta matrices and theta outer matrices. In fact, the class of matrices defined here serve as outer inverse for $(A^{\dagger_s})^\theta$. The recent developments in the characterization of DMP inverse [7] and CMP inverse [8] is the main motivation of this study. The DMP inverse is a new generalized inverse defined using the drazin inverse and Moore Penrose inverse. Meng [9] extended the notion of DMP inverse to rectangular matrices. For more characterizations of DMP inverse one can refer [17], [5] and [1]. In [10] Dijana Mosic defined a new class of matrices known as Drazin star and star Drazin Matrices. Along with these matrices, the outer star and star outer matrices [11] help in solving certain types of matrix equations. For more characterizations of CMP inverse one can refer [6], [12], [13].

Motivated by these studies, two classes of matrices are defined here with the help of outer inverses with specified column space and nullspace, and secondary conjugate transpose of a matrix. Also, certain characterizations and the integral and limit representations of both outer theta and theta outer matrices are explained. Additionally, using these matrices some system of linear equations are solved.

A. Preliminaries and Notations

The set of all $m \times n$ matrices is denoted by $\mathbb{C}^{m\times n}$. The notations $\mathcal{C}(A), \mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the column space, row space and null space of the matrix A respectively. $A^{(2)}_{T,S}$ is the unique outer inverse of the matrix A with column space T and null space S.

In 1976, Anna Lee [3] introduced the concept of secondary symmetric matrices. The secondary transpose or s-transpose of a matrix $A \in \mathbb{C}^{n\times n}$ is defined by reflecting the entries through the second diagonal.

Definition 1: [3] Let $A \in \mathbb{C}^{n\times n}$. Then the secondary transpose of A denoted by A^s and is defined as $A^s = (b_{ij})$ where $b_{ij} = a_{n-j+1,n-i+1}$ where $i, j = 1, 2, ...n$.

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The s-transpose A^s of A is orthogonally similar to the ordinary transpose A^T of A. i.e.;

$$A^s = VA^TV$$

where V is a permutation matrix containing units in the secondary diagonal. Vijayakumar [14] defined s-g inverse of a complex square matrix using the concept of secondary conjugate transpose.

Definition 2: [2] Let $A \in \mathbb{C}^{n\times n}$. Then the conjugate secondary transpose of A denoted by A^θ and is defined as $A^\theta = \overline{A}^s = (c_{ij})$ where $c_{ij} = \overline{a}_{n-j+1,n-i+1}$.

Definition 3: [14] Let $A \in \mathbb{C}^{n\times n}$. The unique matrix X satisfying the conditions

$$\begin{array}{llll} (1) & AXA = A & (2) & XAX = X \\ (3) & (AX)^\theta = AX & (4) & (XA)^\theta = XA. \end{array}$$

is called the s-g inverse of A and is denoted as $A^{\dagger_s}$. Without loss of generality this definition can be extended to any rectangular matrix. The concept of s-g inverse is analogous to the well known Moore Penrose inverse. However, these are two different inverses as clarified in the following example.

Example:- Consider a square matrix $A \in \mathbb{C}^{2\times 2}$, such that $A = \begin{bmatrix} 2+i & 5 \\ 1 & 2-i \end{bmatrix}$.
Here, A is a singular matrix of rank 1.
The Moore Penrose inverse of A is
$A^\dagger = \dfrac{1}{36}\begin{bmatrix} 2-i & 1 \\ 5 & 2+i \end{bmatrix}$.
The s-g inverse of A is $A^{\dagger_s} = \dfrac{1}{16}\begin{bmatrix} 2+i & 5 \\ 1 & 2-i \end{bmatrix}$.
It can be observed that the s-g inverse does not exists for all matrices unlike the Moore Penrose inverse since $rank(AA^\theta) \neq rank(A^\theta A) \neq rank(A)$ in general. However, the existence of s-g inverse is assured whenever this condition holds.

II. Outer Theta and Theta outer Matrices

Theorem 1: Consider $A \in \mathbb{C}^{m\times n}$ of rank r. Assume that AA^θ and $A^\theta A$ exist such that $rank(A^\theta A) = rank(AA^\theta) = r$.
(i) Then the set of equations

$$G(A^{\dagger_s})^\theta G = G, \quad AG = AA^{(2)}_{T,S}AA^\theta, \quad G(A^{\dagger_s})^\theta = A^{(2)}_{T,S}A \tag{1}$$

are consistent. Also $G = A^{(2)}_{T,S}AA^\theta$ is the unique solution of equation 1.
(ii) The set of equations

$$G(A^{\dagger_s})^\theta G = G, \quad GA = A^\theta AA^{(2)}_{T,S}A, \quad (A^{\dagger_s})^\theta G = AA^{(2)}_{T,S} \tag{2}$$

are consistent. Also $G = A^\theta AA^{(2)}_{T,S}$ is the unique solution of equation 2.

Proof: (i) Consider

$G(A^{\dagger_s})^\theta G = A^{(2)}_{T,S}AA^\theta(A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta$
$= A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^\theta = G$

Also,

$AG = AA^{(2)}_{T,S}AA^\theta$ and
$G(A^{\dagger_s})^\theta = A^{(2)}_{T,S}AA^\theta(A^{\dagger_s})^\theta = A^{(2)}_{T,S}AA^\theta A.$

Hence $G = A^{(2)}_{T,S}AA^\theta$ satisfies Equation (1).
Let G_1 and G_2 be two matrices satisfying (1).

$G_1 = G_1(A^{\dagger_s})^\theta G_1 = A^{(2)}_{T,S}(AG_1)$
$= (A^{(2)}_{T,S}A)G_2 = G_2(A^{\dagger_s})^\theta G_2 = G_2$

which proves the uniqueness.
Part (ii) of the theorem can be proved in a similar way. ■

Definition 4: Consider $A \in \mathbb{C}^{m\times n}$ of rank r. Assume that AA^θ and $A^\theta A$ exist such that $rank(A^\theta A) = rank(AA^\theta) = r$.
(i) The (T, S) outer theta matrix of A is defined as

$$A^{(2,\theta)}_{T,S} = A^{(2)}_{T,S}AA^\theta.$$

(ii) The theta (T, S) outer matrix of A is defined as

$$A^{(\theta,2)}_{T,S} = A^\theta AA^{(2)}_{T,S}.$$

It can be noted that both $A^{(2,\theta)}_{T,S}$ and $A^{(\theta,2)}_{T,S}$ are outer inverses of $(A^{\dagger_s})^\theta$.

The following lemma discusses the range space and null spaces of outer theta and theta outer inverses of $(A^{\dagger_s})^\theta$.

Lemma 1: Consider $A \in \mathbb{C}^{m\times n}$. Then the following statements holds:

(i) $(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}$ is a projector on $\mathcal{C}((A^{\dagger_s})^\theta A^{(2)}_{T,S})$ along $\mathcal{N}(A^{(2)}_{T,S}AA^\theta)$.
(ii) $A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta$ is a projector on T along $\mathcal{N}(A^{(2)}_{T,S}A)$.
(iii) $A^{(2,\theta)}_{T,S} = [(A^{\dagger_s})^\theta]^{(2)}_{T,\mathcal{N}(A^{(2)}_{T,S}AA^\theta)}$.
(iv) $(A^{\dagger_s})^\theta A^{(\theta,2)}_{T,S}$ is a projector on $\mathcal{C}(AA^{(2)}_{T,S})$ along S.
(v) $A^{(\theta,2)}_{T,S}(A^{\dagger_s})^\theta$ is a projector on $\mathcal{C}(A^\theta AA^{(2)}_{T,S})$ along $\mathcal{N}(A^{(2)}_{T,S}(A^{\dagger_s})^\theta)$.
(vi) $A^{(\theta,2)}_{T,S} = [(A^{\dagger_s})^\theta]^{(2)}_{\mathcal{C}(A^\theta AA^{(2)}_{T,S}),S}$.

Proof:

(i) $(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}$
$= (A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta(A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta = (A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}$

implies that $(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}$ is a projector.
Since $(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S} = (A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta$,

$\mathcal{C}((A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}) \subseteq \mathcal{C}((A^{\dagger_s})^\theta A^{(2)}_{T,S})$
$= \mathcal{C}((A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^{\dagger_s}AA^{(2)}_{T,S})$
$= \mathcal{C}((A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta(A^{\dagger_s})^\theta A^{(2)}_{T,S})$
$\subseteq \mathcal{C}((A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}).$

Hence $\mathcal{C}((A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}) = \mathcal{C}((A^{\dagger_s})^\theta A^{(2)}_{T,S})$.
Also

$\mathcal{N}(A^{(2,\theta)}_{T,S}) \subseteq \mathcal{N}((A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S})$
$\subseteq \mathcal{N}(A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}) = \mathcal{N}(A^{(2,\theta)}_{T,S}).$

(ii) Clearly $A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta$ is a projector.
Since $A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta = A^{(2)}_{T,S}A$ and $\mathcal{C}(A^{(2)}_{T,S}A) = \mathcal{C}(A^{(2)}_{T,S})$, the condition $\mathcal{C}(A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta) = T$ follows directly and hence $\mathcal{N}(A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta) = \mathcal{N}(A^{(2)}_{T,S}A)$.
(iii) It follows by $\mathcal{C}(A^{(2,\theta)}_{T,S}) = \mathcal{C}(A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta) = T$ and $\mathcal{N}(A^{(2,\theta)}_{T,S}) = \mathcal{N}((A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}) = \mathcal{N}(A^{(2)}AA^\theta)$.
The remaining part of the theorem can be proved in an analogous way. ■

Unlike the conjugate transpose, in the case of secondary conjugate transpose $rank(AA^\theta) \neq rank(A^\theta A) \neq rank(A)$ which is clear from the following example.

Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R^{3\times 3}$.

The secondary conjugate transpose of A is $A^\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

and $AA^\theta = \begin{bmatrix} 0 & 16 & 4 \\ 0 & 64 & 16 \\ 0 & 0 & 0 \end{bmatrix}$.
Here $rank(AA^\theta) \neq rank(A)$.
So all these conditions holds only when $rank(A) = rank(AA^\theta) = rank(A^\theta A)$.
The following theorem gives necessary and sufficient conditions for a matrix to be the outer theta matrix.

Theorem 2: Consider $A \in \mathbb{C}^{m\times n}$. Then the following statements are equivalent:

(i) G is (T, S) outer - theta matrix of A.
(ii) G satisfies the conditions

$$G(A^{\dagger_s})^\theta G = G, \quad (A^{\dagger_s})^\theta G(A^{\dagger_s})^\theta = (A^{\dagger_s})^\theta A^{(2)}_{T,S}A$$
$$AG = AA^{(2)}_{T,S}AA^\theta, \quad G(A^{\dagger_s})^\theta = A^{(2)}_{T,S}A.$$

(iii) G satisfies the conditions

$$A^{(2)}_{T,S}AG = G \ \ and \ \ AG = AA^{(2)}_{T,S}AA^\theta.$$

(iv) G satisfies the conditions

$$A^{(2)}_{T,S}AGAA^{\dagger_s} = G \ \ and \ \ AG(A^{\dagger_s})^\theta = AA^{(2)}_{T,S}A.$$

(v) G satisfies the conditions

$$A^{(2)}_{T,S}AG = G \ \ and \ \ (A^{\dagger_s})^\theta G = (A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta.$$

(vi) G satisfies the conditions

$$GAA^{\dagger_s} = G \ \ and \ \ G(A^{\dagger_s})^\theta = A^{(2)}_{T,S}A.$$

(vii) G satisfies the conditions

$$G(A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta = G \ \ and \ \ G(A^{\dagger_s})^\theta A^{(2)}_{T,S} = A^{(2)}_{T,S}$$

(viii) G satisfies the conditions

$$GAA^{\dagger_s} = G \ \ and \ \ GA = A^{(2)}_{T,S}AA^\theta A.$$

(ix) G satisfies the conditions
$G(A^{\dagger_s})^\theta A^{(2)}_{T,S}AG = G, (A^{\dagger_s})^\theta A^{(2)}_{T,S}AG =$
$(A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta$ and
$G(A^{\dagger_s})^\theta A^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$
(x) G satisfies the conditions
$G(A^{\dagger_s})^\theta A^{(2)}_{T,S}AG = G$
$(A^{\dagger_s})^\theta A^{(2)}_{T,S}AG(A^{\dagger_s})^\theta A^{(2)}_{T,S}A = (A^{\dagger_s})^\theta A^{(2)}_{T,S}A,$
$(A^{\dagger_s})^\theta A^{(2)}_{T,S}AG = (A^{\dagger_s})^\theta A^{(2)}_{T,S}AA^\theta$ and
$G(A^{\dagger_s})^\theta A^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$

(xi) G satisfies the conditions
$A^{(2)}_{T,S}AG = G,\ (A^{\dagger_s})^{\theta}A^{(2)}_{T,S}AG = (A^{\dagger_s})^{\theta}A^{(2)}_{T,S}AA^{\theta}.$

(xii) G satisfies the conditions
$G(A^{\dagger_s})^{\theta}A^{(2)}_{T,S}AA^{\theta} = G,\ G(A^{\dagger_s})^{\theta}A^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$

Proof: (i) $\Longrightarrow$ (ii):
By definition of $G = A^{(2)}_{T,S}AA^{\theta}$, we get

$$(A^{\dagger_s})^{\theta}G(A^{\dagger_s})^{\theta} = (A^{\dagger_s})^{\theta}A^{(2)}_{T,S}AA^{\theta}(A^{\dagger_s})^{\theta} = (A^{\dagger_s})^{\theta}A^{(2)}_{T,S}AA^{\dagger_s}A = (A^{\dagger_s})^{\theta}A^{(2)}_{T,S}A$$

By Theorem 1 the remaining equations hold true.
(ii) $\Longrightarrow$ (iii): It is clear from the defintion that $A^{(2)}_{T,S}AG = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^{\theta} = G$
(iii) $\Longrightarrow$ (iv):

$$A^{(2)}_{T,S}AGAA^{\dagger_s} = GAA^{\dagger_s} = A^{(2)}_{T,s}AA^{\theta}AA^{\dagger_s} = A^{(2)}_{T,S}AA^{\theta}(AA^{\dagger_s})^{\theta} = G$$

Also
$AG(A^{\dagger_s})^{\theta} = AA^{(2)}_{T,S}AA^{\theta}(A^{\dagger_s})^{\theta} = AA^{(2)}_{T,S}A$
(iv) $\Longrightarrow$ (i):

$$G = A^{(2)}_{T,S}AGAA^{\dagger_s} = A^{(2)}_{T,S}(AG(A^{\dagger_s})^{\theta})A^{\theta} = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^{\theta} = A^{(2)}_{T,S}AA^{\theta}$$

(i) $\Longrightarrow$ (v) - (xii): The rest of the theorem can be verified with $G = A^{(2)}_{T,S}AA^{\theta}$. ■

The definition of (T,S) - outer theta matrix can be generalized for any rectangular matrix U and V with column space T and null space S. The next theorem gives the characterization of such matrices.

Theorem 3: Consider $A \in \mathbb{C}^{m\times n}_r$. Let $U \in \mathbb{C}^{n\times p}$ and $V \in \mathbb{C}^{m\times q}$. Also assume that AA^{θ}, $A^{\theta}A$ and $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}$ exist. Then the following statements for $G \in \mathbb{C}^{n\times m}$ are equivalent.

(i) G is the $(\mathcal{C}(U),\mathcal{N}(V))$ outer theta matrix of A.

(ii) G satisfies the conditions
$VAG = VAA^{\theta}$ *and* $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG = G.$

(iii) G satisfies the conditions
$VAG(A^{\dagger_s})^{\theta} = VA$ *and* $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AGAA^{\dagger_s} = G.$

(iv) G satisfies the conditions
$VAGAA^{\dagger_s} = VAA^{\theta}$ *and* $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AGAA^{\dagger_s} = G.$

(v) G satisfies the conditions
$G(A^{\dagger_s})^{\theta}U = U$ and $G(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$

(vi) G satisfies the conditions
$AG(A^{\dagger_s})U = AU$ and
$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$

(vii) G satisfies the conditions
$A^{\theta}AG(A^{\dagger_s})^{\theta}U = A^{\theta}AU$ and
$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$

Proof: We refer Theorem 2.3 of [11] which is a pre requisite to prove these results.
$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AU = U$ and $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)} = UU^{(1)}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}$, for $U^{(1)} \in U\{1\}$.
Also $VAA^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)} = V$ and
$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}V$ for $V^{(1)} \in V\{1\}$.

We use the above mentioned result while proving our theorem.
(i) $\Longrightarrow$ (ii) Given that $G = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}$.
So, $VAG = VAA^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = VAA^{\theta}$. Also

$$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$$

(ii) $\Longrightarrow$ (i) Since
$VAG = VAA^{\theta}$ and $A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}V$,
we have

$$G = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}VAG = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}V = (A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}V)AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}.$$

(i) $\Longrightarrow$ (iii) $VAG(A^{\dagger_s})^{\theta} = VAA^{\theta}(A^{\dagger_s})^{\theta} = VA.$

$$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AGA^{\dagger_s} = GAA^{\dagger_s} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}AA^{\dagger_s} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$$

(iii) $\Longrightarrow$ (iv) Since $VAG(A^{\dagger_s})^{\theta} = VA$,
$VAG(A^{\dagger_s})^{\theta}A^{\theta} = VAA^{\theta} \implies VAGA(A^{\dagger_s}) = VAA^{\theta}$.
(iv) $\Longrightarrow$ (i) Consider

$$G = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AGAA^{\dagger_s} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}(VAGAA^{\dagger_s}) = (A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}V^{(1)}V)AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}$$

(i) $\Longrightarrow$ (v) Consider

$$G(A^{\dagger_s})^{\theta}U = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}(A^{\dagger_s})^{\theta}U = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AU = U$$

and

$$G(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$$

(v) $\Longrightarrow$ (i)

$$G = G(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G(A^{\dagger_s})^{\theta}UU^{(1)}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = UU^{(1)}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}$$

Hence G is the $(\mathcal{C}(U),\mathcal{N}(V))$ outer theta matrix of A.
(i) $\Longrightarrow$ (vi) Consider

$$A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = G.$$

(vi) $\Longrightarrow$ (vii) The result is obvious.
(vii) $\Longrightarrow$ (i) Consider

$$G = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AG(A^{\dagger_s})^{\theta}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}(A^{\dagger_s})^{\theta}(A^{\theta}AG(A^{\dagger_s})^{\theta}U)U^{(1)}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}(A^{\dagger_s})^{\theta}(A^{\theta}A(UU^{(1)}A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)})AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta} = A^{(2)}_{\mathcal{C}(U),\mathcal{N}(V)}AA^{\theta}.$$

■

Remark:- An analogous result can be obtained for theta - outer matrix also.
The following lemma gives the relationship of outer theta inverses with various other generalized inverses.

Lemma 2: Consider $A \in \mathbb{C}^{m\times n}$. Then the following conditions hold:

(i) $\mathcal{N}(A) = \mathcal{N}(A^{(2)}_{T,S}A) \Leftrightarrow (A^{\dagger_s})^\theta A^{(2,\theta)}_{T,S}(A^{\dagger_s})^\theta = (A^{\dagger_s})^\theta$
$\Leftrightarrow A^\dagger AA^{(2)}_{T,S}A = A^\dagger AA^{(2)}_{T,S}A = A^{\dagger_s}A$
$\Leftrightarrow AA^{(2)}_{T,S}A = A \Leftrightarrow AA^{(2)}_{T,S}AA^{\dagger_s} = AA^{\dagger_s}$
$\Leftrightarrow (A^{\dagger_s})^\theta A^{(\theta,2)}_{T,S}(A^\dagger_s)^\theta = (A^\dagger_s)^\theta$
$\Leftrightarrow \mathcal{C}(A) = \mathcal{C}(AA^{(2)}_{T,S})$.
(ii) $AA^{(2,\theta)}_{T,S} = AA^{(2)}_{T,S} \Leftrightarrow A^{(2,\theta)}_{T,S} = A^{(2)}_{T,S}$.
(iii) $A^{(2,\theta)}_{T,S}A = A^{(2)}_{T,S}A \Leftrightarrow A^{(2,\theta)}_{T,S} = A^{(2)}_{T,S}AA^{\dagger_s}$.
(iv) $A^{(2,\theta)}_{T,S}A = A^{\dagger_s}A \Leftrightarrow A^{(2,\theta)}_{T,S} = A^{\dagger_s}$.
(v) $AA^{(2,\theta)}_{T,S} = AA^{\dagger_s} \Leftrightarrow AA^{(2,\theta)}_{T,S}A = A$.
(vi) $A^{(2)}_{T,S}AA^{\dagger_s} = A^{\dagger_s} \Leftrightarrow A^{(2)}_{T,S}A = A^{\dagger_s}A$
$\Longrightarrow A^{(2,\theta)}_{T,S} = A^\theta$
(vii) $A^{(\theta,2)}_{T,S}A = A^{(2)}_{T,S}A \Leftrightarrow A^{\theta,2}_{T,S} = A^{\dagger_s}AA^{(2)}_{T,S}$.
(viii) $AA^{(\theta,2)}_{T,S} = AA^{(2)}_{T,S} \Leftrightarrow A^{(\theta,2)}_{T,S} = A^{\dagger_s}AA^{(2)}_{T,S}$.
(ix) $AA^{(\theta,2)}_{T,S} = AA^{\dagger_s} \Leftrightarrow A^{(\theta,2)}_{T,S} = A^{\dagger_s}$.
(x) $A^{(\theta,2)}_{T,S}A = A^{\dagger_s}A \Leftrightarrow AA^{(\theta,2)}_{T,S}A = A$.
(xi) $A^{\theta,2}_{T,S} = A^\theta \Leftrightarrow AA^{(2)}_{T,S} = AA^{\dagger_s} \Leftrightarrow A^{\dagger_s,(2)}_{T,S} = A^{\dagger_s}$.

III. Representation of outer - theta and theta - outer inverses

The following lemma is useful in representing outer theta inverses.

Lemma 3: [15] Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Suppose $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If A has an outer inverse $A^{(2)}_{T,S}$, then $ind(AG) = ind(GA) = 1$ and

$$A^{(2)}_{T,S} = (GA)^\sharp G = G(AG)^\sharp.$$

Theorem 4: If A and G satisfy the conditions of Lemma 3, then

$$A^{2,\theta}_{T,S} = (GA)^\sharp GAA^\theta = P_{T,\mathcal{N}(GA)}A^\theta$$

and

$$A^{\theta,2}_{T,S} = A^\theta AG(AG)^\sharp = A^\theta P_{\mathcal{R}(AG),S}.$$

Proof: Since $\mathcal{R}(G) = T = \mathcal{R}(A^{(2)}_{T,S})$ and $\mathcal{N}(G) = S = \mathcal{N}(A^{(2)}_{T,S})$, $G = A^{(2)}_{T,S}AG$ and $G = GAA^{(2)}_{T,S}$, which implies $\mathcal{N}(AG) = \mathcal{N}(G) = S$ and $\mathcal{R}(GA) = \mathcal{R}(G) = T$. Referring lemma 3 and properties of group inverse, we obtain

$$A^{2,\theta}_{T,S} = (GA)^\sharp GAA^\theta = P_{\mathcal{R}(GA),\mathcal{N}(GA)}A^\theta = P_{T,\mathcal{N}(GA)A^\theta}$$

and

$$A^{\theta,2}_{T,S} = A^\theta AG(AG)^\sharp = A^\theta P_{\mathcal{R}(AG),\mathcal{N}(AG)} = A^\theta P_{\mathcal{R}(AG),S}.$$

■

Corollary 1: If $A \in \mathbb{C}^{n\times n}$ and $ind(A) = k$, then

$$A^{D,\theta} = (A^l)^\sharp A^l A^\theta = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}A^\theta$$

and

$$A^{\theta,D} = A^\theta A^l(A^l)^\sharp = A^\theta P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$$

for $l \geq k$.

Proof: If $G = A^l$ for $l \geq k$ in Theorem 4, we get

$$A^{D,\theta} = (A^{l+1})^\sharp A^{l+1}A^\theta = P_{\mathcal{R}(A^{l+1},\mathcal{N}(A^{l+1}))}A^\theta$$
$$= P_{\mathcal{R}(A^l,\mathcal{N}(A^l))}A^\theta = (A^l)^\sharp A^l A^\theta$$

Similarly $A^{\theta,D} = A^\theta A^l(A^l)^\sharp = A^\theta P_{\mathcal{R}(A^l),\mathcal{N}(A^l))}$.

■

The integral representation and the representation of outer theta inverses in their limiting form are depicted in theorem 5 and theorem 6.

Theorem 5: If A and G satisfy the conditions of Lemma 3, then

$$A^{(2,\theta)}_{T,S} = \int_0^\infty exp[-G(GAG)^\theta GAt]G(GAG)^\theta GAA^\theta dt$$

and

$$A^{(\theta,2)}_{T,S} = \int_0^\infty A^\theta Aexp[-G(GAG)^\theta GAt]G(GAG)^\theta Gdt.$$

Proof: From the integral representations of outer inverse given in [16] the given result follows. ■

Theorem 6: Let $A \in \mathbb{C}^{m\times n}$ be of rank r, $B \in \mathbb{C}^{n\times s}$ be of rank s and $C \in \mathbb{C}^{s\times m}$ be of rank s. If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ exists, then

$$A^{(2,\theta)}_{\mathcal{R}(B),\mathcal{N}(C)} = \lim_{t\to 0} B(tI + CAB)^{-1}CAA^\theta$$

and

$$A^{(\theta,2)} = \lim_{t\to 0} A^\theta AB(tI + CAB)^{-1}C$$

Proof: According to [4]

$$A^{(2)}_{T,S} = \lim_{t\to 0} B(tI + CAB)^{-1}C.$$

So the result follows directly. ■

IV. Applications of outer theta and theta outer matrices

Applying outer theta and theta outer matrices, we can solve certain systems of linear equations.

Theorem 7: Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the equation

$$A^{(2)}_{T,S}Ax = A^{(2,\theta)}_{T,S}b \tag{3}$$

is consistent and its general solution is

$$x = A^{(2,\theta)}_{T,S}b + (I - A^{(2)}_{T,S}A)y \tag{4}$$

for arbitrary $y \in \mathbb{C}^{(n)}$.

Proof:

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}AA^{(2,\theta)}_{T,S}b = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^\theta b = A^{(2,\theta)}_{T,S}b.$$

Suppose x is a solution of (3). Then $A^{(2,\theta)}_{T,S}b = A^{(2)}_{T,S}Ax$ gives $x = A^{(2,\theta)}_{T,S}b + x - A^{(2)}_{T,S}Ax = A^{(2,\theta)}_{T,S}b + (I - A^{(2)}_{T,S}A)x$. Hence the solution x is of the form (4). ■

V. Conclusion

A new set of matrices - outer theta and theta outer matrices are defined, and a few characterizations and applications are obtained here. Further, we can study iterative methods for computing the outer theta inverse and theta outer inverses. These results can also be extended to Hilbert spaces to broaden research in those lines.

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