On Picture Fuzzy (m,n)-ideals of Semigroups

Warud Nakkhasen

***Abstract*—In this paper, the concepts of picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups, where m and n are positive integers, are introduced and some basic properties are investigated. These concepts generalize the notions of picture fuzzy bi-ideals, picture fuzzy right ideals and picture fuzzy left ideals of semigroups. Then, some characterizations of different classes of semigroups, that is, (m,n)-regular, $(m,0)$-regular and $(0,n)$-regular, by properties of picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups are obtained.**

***Index Terms*—picture fuzzy set, picture fuzzy (m,n)-ideal, picture fuzzy $(m,0)$-ideal, picture fuzzy $(0,n)$-ideal, (m,n)-regular semigroup.**

I. Introduction

ZADEH [33] defined the concept of fuzzy subsets or fuzzy sets as a function from a nonempty set X to the unit interval $[0,1]$ in 1965. Then, Rosenfeld [28] proposed the concept of fuzzy groups, which was the first inspired application to various algebraic structures. Later, the concept of fuzzy subsemigroups was also suggested by Kuroki [18], [21]. He also studied the concept of fuzzy generalized bi-ideals in semigroups, which is an extension of the concept of fuzzy bi-ideals and used fuzzy left ideals and fuzzy right ideals to characterize particular classes of semigroups (see, [19], [20]). Yairayong [31] characterized regularities of semigroups by the properties of their hesitant fuzzy ideals. Subsequently, Gatetem and Khamrot [10] studied the characterizations of regular, left (resp. right) regular, intra-regular, weakly regular and quasi-regular semigroups by using bipolar fuzzy weakly interior ideals. As a generalization of the notion of fuzzy sets, Atanassov [5], [6] created the concept of intuitionistic fuzzy sets as follows. An intuitionistic fuzzy set $\mathcal{A}$ on a universe X is an object of the form $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y)\rangle \mid x \in X\}$ where $\mu_{\mathcal{A}}(x) \in [0,1]$ and $\eta_{\mathcal{A}}(x) \in [0,1]$ denote the degree of membership and the degree of non-membership, respectively, for each $x \in X$ to the set $\mathcal{A}$, and so, $0 \leq \mu_{\mathcal{A}}(x)+\eta_{\mathcal{A}}(x) \leq 1$ for all $x \in X$. We can see that the fuzzy sets define the degree of membership of an element in a given set, while the intuitionistic fuzzy sets provide both membership and non-membership degrees. The concept of intuitionistic fuzzy sets was applied to semigroups by Kim and Jun [16] in 2002. Later, theorems characterizing intra-regular semigroups in terms of intuitionistic fuzzy left ideals, intuitionistic fuzzy right ideals and intuitionistic fuzzy bi-ideals were presented by Hong and Fang [12]. In addition, the class of regular semigroups was described by Hur et al. [14] using intuitionistic fuzzy left, right, two-sided ideals and intuitionistic fuzzy bi-ideals of semigroups. Moreover, Shabir and Khan [29] have characterized intra-regular ordered semigroups by properties of their intuitionistic fuzzy interior ideals. Subsequently, Cuong and Kreinovich [8] first introduced the notion of picture fuzzy sets in 2013, as direct generalizations of the concepts of fuzzy sets and intuitionistic fuzzy sets. This concept has been investigated by various of mathematicians, see, e.g., [1], [13], [25], [26], [34].

The concept of (m,n)-ideals in semigroups, where m and n are non-negative integers, was first introduced and discussed by Lajos [22]. Many authors have examined theory in other structures, see, e.g., [2], [4], [7], [11], [15], [27]. Krgović [17] has characterized different classes of semigroups (i.e., (m,n)-regular, $(m,0)$-regular, $(0,n)$-regular) by the notions of (m,n)-ideals, $(m,0)$-ideals and $(0,n)$-ideals of semigroups. Now in 2019, Mahboob et al. [23] introduced the concepts of fuzzy (m,n)-ideals, fuzzy $(m,0)$-ideals and fuzzy $(0,n)$-ideals of semigroups as generalizes the concepts of fuzzy bi-ideals, fuzzy right ideals and fuzzy left ideals of semigroups, respectively. Also, they characterized (m,n)-regular, $(m,0)$-regular and $(0,n)$-regular semigroups by the properties of fuzzy (m,n)-ideals, fuzzy $(m,0)$-ideals and fuzzy $(0,n)$-ideals of semigroups.

Recently, Yiarayong [30], [32] has applied the notion of picture fuzzy sets to semigroup theory and used picture fuzzy left (resp. right) ideals and picture fuzzy bi-ideals of semigroups to characterize different classes regular and intra-regular semigroups. In late 2021, Nakkhasen [24] studied some further characterizations of regular and intra-regular semigroups in terms of picture fuzzy left (resp. right) ideals, picture fuzzy quasi-ideals and picture fuzzy (resp. generalized) bi-ideals of semigroups. In this article, as generalizations of picture fuzzy bi-ideals and picture fuzzy right (resp. left) ideals of semigroups, we introduce the concepts of picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups where m and n are positive integers. After that we discuss some properties of picture fuzzy (m,n)-ideals of semigroups. Finally, we characterize different classes (m,n)-regular, $(m,0)$-regular and $(0,n)$-regular semigroups using picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups.

II. Preliminaries

First, we will review some fundamental concepts and properties that are needed for this study. Let S be a semigroup and A and B be any two nonempty subsets of S. Then, the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Let A be a nonempty subset of a semigroup S. Then: A is called a *subsemigroup* of S if $AA \subseteq A$; A is called a *left* (resp. *right*) *ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$); A is called a *two-sided ideal* or an *ideal* of S if it is both a left

Manuscript received April 4, 2022; revised October 31, 2022. This research project was financially supported by Thailand Science Research and Innovation (TSRI).

Warud Nakkhasen is an Assistant Professor at Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham 44150, Thailand (corresponding author to provide e-mail: warud.n@msu.ac.th)

ideal and a right ideal of S; A is called a *bi-ideal* of S if A is a subsemigroup of S and $ASA \subseteq A$.

A *picture fuzzy set* [8] $\mathcal{A}$ on a universe X is defined as the form

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y), \nu_{\mathcal{A}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathcal{A}}(x) \in [0,1]$ is called the degree of positive membership of x in $\mathcal{A}$, $\eta_{\mathcal{A}}(x) \in [0,1]$ is called the degree of neutral membership of x in $\mathcal{A}$ and $\nu_{\mathcal{A}}(x) \in [0,1]$ is called the degree of negative membership of x in $\mathcal{A}$ such that $\mu_{\mathcal{A}}, \eta_{\mathcal{A}}$ and $\nu_{\mathcal{A}}$ satisfy the following condition: $0 \leq \mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leq 1$ for all $x \in X$. For any $a \in X$, $1 - (\mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x))$ is called the degree of refusal membership of x in $\mathcal{A}$.

Throughout this paper, we shall use symbol $\mathcal{A}$ instead of the picture fuzzy set $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y), \nu_{\mathcal{A}}(x) \mid x \in X\}$.

Let $\{a_i \mid i \in \Lambda\}$ be a family of real numbers. Then, we have

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite;} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite;} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For every two real numbers a and b, we write $a \vee b$ instead of $\vee\{a,b\}$ and $a \wedge b$ instead of $\wedge\{a,b\}$. The notions listed below will be considered in this paper, as they occurred in [9]. Let $\mathcal{A}$ and $\mathcal{B}$ be any two picture fuzzy sets on a universe X. Then:

(i) $\mathcal{A} \subseteq \mathcal{B}$ iff $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$, $\eta_{\mathcal{A}}(x) \geq \eta_{\mathcal{B}}(x)$ and $\nu_{\mathcal{A}}(x) \geq \nu_{\mathcal{B}}(x)$ for all $x \in X$;

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$;

(iii) $\mathcal{A} \cap \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \wedge \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \vee \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \vee \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$;

(iv) $\mathcal{A} \cup \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \vee \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \wedge \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \wedge \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$.

Next, we denote by $\mathcal{PFS}(S)$ the collection of picture fuzzy sets of a semigroup S with $\mathcal{S} = \{\langle x, 1, 0, 0\rangle \mid x \in S\}$ and $\emptyset = \{\langle x, 0, 0, 1\rangle \mid x \in S\}$, see [30]. Let X be any subset of a semigroup S. The *picture characteristic function* [30] of X is defined by $\mathcal{C}_X = \{\langle x, \mu_{\mathcal{C}_X}(x), \eta_{\mathcal{C}_X}(x), \nu_{\mathcal{C}_X}(x)\rangle \mid x \in S\}$, where

$$\mu_{\mathcal{C}_X}(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise,} \end{cases} \quad \eta_{\mathcal{C}_X}(x) = \begin{cases} 0 & \text{if } x \in X, \\ 1 & \text{otherwise,} \end{cases}$$

$$\nu_{\mathcal{C}_X}(x) = \begin{cases} 0 & \text{if } x \in X, \\ 1 & \text{otherwise.} \end{cases}$$

We observe that if $X = S$ (resp. $X = \emptyset$), then $\mathcal{C}_X = \mathcal{S}$ (resp. $\mathcal{C}_X = \emptyset$).

For any two picture fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ on a semigroup S, the *picture fuzzy product* [30] of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} \circ \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \circ \nu_{\mathcal{B}})(x)\rangle \mid x \in S\}$$

where

$$(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x) = \begin{cases} \bigvee_{(y,z) \in A_x} \min\{\mu_{\mathcal{A}}(y), \mu_{\mathcal{B}}(z)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

$$(\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(x) = \begin{cases} \bigwedge_{(y,z) \in A_x} \max\{\eta_{\mathcal{A}}(y), \eta_{\mathcal{B}}(z)\} & \text{if if } A_x \neq \emptyset, \\ 1 & \text{if } A_x = \emptyset, \end{cases}$$

$$(\nu_{\mathcal{A}} \circ \nu_{\mathcal{B}})(x) = \begin{cases} \bigwedge_{(y,z) \in A_x} \max\{\nu_{\mathcal{A}}(y), \nu_{\mathcal{B}}(z)\} & \text{if if } A_x \neq \emptyset, \\ 1 & \text{if } A_x = \emptyset, \end{cases}$$

where A_x is a relation on a semigroup S defined as $A_x = \{(y,z) \in S \times S \mid x = yz\}$.

Lemma 2.1: [30] Let $\mathcal{C}_X$ and $\mathcal{C}_Y$ be any two picture fuzzy sets on a semigroup S. Then, the following properties hold:

(i) $\mathcal{C}_{X \cap Y} = \mathcal{C}_X \cap \mathcal{C}_Y$;

(ii) $\mathcal{C}_{XY} = \mathcal{C}_X \circ \mathcal{C}_Y$.

Now, we will review some concepts of many types of picture fuzzy sets on a semigroup S to be used in the next sections.

A picture fuzzy set $\mathcal{A}$ on a semigroup S is called a *picture fuzzy subsemigroup* [30] of S if it satisfies the following conditions:

(i) $\mu_{\mathcal{A}}(xy) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}$;

(ii) $\eta_{\mathcal{A}}(xy) \leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y)\}$;

(iii) $\nu_{\mathcal{A}}(xy) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y)\}$,

for all $x, y \in S$.

A picture fuzzy set $\mathcal{A}$ on a semigroup S is called a *picture fuzzy left* (resp. *right*) *ideal* [30] of S if satisfies the following conditions:

(i) $\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(y)$ (resp. $\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(x)$);

(ii) $\eta_{\mathcal{A}}(xy) \leq \eta_{\mathcal{A}}(y)$ (resp. $\eta_{\mathcal{A}}(xy) \leq \eta_{\mathcal{A}}(x)$);

(iii) $\nu_{\mathcal{A}}(xy) \leq \nu_{\mathcal{A}}(y)$ (resp. $\nu_{\mathcal{A}}(xy) \leq \nu_{\mathcal{A}}(x)$),

for all $x, y \in S$.

A picture fuzzy set $\mathcal{A}$ on a semigroup S is said to be a *picture fuzzy ideal* of S if it is both a picture fuzzy left ideal and a picture fuzzy right ideal of S.

A picture fuzzy subsemigroup $\mathcal{A}$ on a semigroup S is called a *picture fuzzy bi-ideal* [24] of S if it satisfies the following conditions:

(i) $\mu_{\mathcal{A}}(xyz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}$;

(ii) $\eta_{\mathcal{A}}(xyz) \leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(z)\}$;

(iii) $\nu_{\mathcal{A}}(xyz) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(z)\}$,

for all $x, y, z \in S$.

It is known that every picture fuzzy left (resp. right) ideal of a semigroup S is a picture fuzzy bi-ideal of S. In general, the converse of this statement is not true, see [24].

Lemma 2.2: [24] Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then the following properties hold:

(i) $\mathcal{A}$ is a picture fuzzy subsemigroup of S if and only if $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$;

(ii) $\mathcal{A}$ is a picture fuzzy right ideal of S if and only if $\mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$;

(iii) $\mathcal{A}$ is a picture fuzzy left ideal of S if and only if $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

Lemma 2.3: [30] Let A be a nonempty subset of a semigroup S. Then, the following properties hold:

(i) A is a subsemigroup of S if and only if $\mathcal{C}_A$ is a picture fuzzy subsemigroup of S;

(ii) A is a left ideal of S if and only if $\mathcal{C}_A$ is a picture fuzzy left ideal of S;

(iii) A is a right ideal of S if and only if $\mathcal{C}_A$ is a picture fuzzy right ideal of S;

(iv) A is an ideal of S if and only if $\mathcal{C}_A$ is a picture fuzzy ideal of S.

For any nonempty subset A of a semigroup S, we denote $A^m = AA\cdots A$ (m times). Let m and n be non-negative integers. A subsemigroup A of a semigroup S is called an (m,n)-*ideal* [22] of S if $A^mSA^n \subseteq A$. Here, $A^0S = SA^0 = S$. In a semigroup S, we observe that every bi-ideal of S is a $(1,1)$-ideal of S, and every left (resp. right) ideal of S is a $(0,1)$-ideal (resp. $(1,0)$-ideal) of S.

Let A be a nonempty subset of a semigroup S and m,n be positive integers. Then, we denote $[A]_{(m,n)}, [A]_{(m,0)}$ and $[A]_{(0,n)}$ as the principle (m,n)-ideal, the principle $(m,0)$-ideal and the principle $(0,n)$-ideal of S generated by A, respectively, that is, $[A]_{(m,n)}$ is the smallest (m,n)-ideal, $[A]_{(m,0)}$ is the smallest $(m,0)$-ideal and $[A]_{(0,n)}$ is the smallest $(0,n)$-ideal of S containing A. In particular, if $A=\{a\}$, then $[\{a\}]_{(m,n)} = [a]_{(m,n)}, [\{a\}]_{(m,0)} = [a]_{(m,0)}$ and $[\{a\}]_{(0,n)} = [a]_{(0,n)}$. It is well known that

$$[A]_{(m,n)} = \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n, [A]_{(m,0)} = \bigcup_{i=1}^{m} A^i \cup A^mS$$
$$\text{and } [A]_{(0,n)} = \bigcup_{i=1}^{n} A^i \cup SA^n,$$

which they were shown by Krgović [17].

Lemma 2.4: [17] Let S be a semigroup, m,n be positive integers and $[a]_{(m,n)}$ be the principal (m,n)-ideal of S generated by the element $a \in S$. Then:

(i) $\left([a]_{(m,0)}\right)^m S = a^mS$;

(ii) $S\left([a]_{(0,n)}\right)^n = Sa^n$;

(iii) $\left([a]_{(m,n)}\right)^m S \left([a]_{(m,n)}\right)^n = a^mSa^n$.

III. Picture fuzzy (m,n)-ideals

In this section, we introduce the concept of picture fuzzy (m,n)-ideals of semigroups where m and n are positive integers. This generalizes the concept of picture fuzzy bi-ideals of semigroups. Then, we investigate some properties of picture fuzzy (m,n)-ideals of semigroups. Moreover, we give some characterizations of (m,n)-regular semigroups in terms of picture fuzzy (m,n)-ideals of semigroups.

Definition 3.1: Let S be a semigroup and m,n be positive integers. A picture fuzzy subsemigroup $\mathcal{A}$ of S is called a *picture fuzzy* (m,n)-*ideal* of S if it satisfies the following conditions:

(i) $\mu_{\mathcal{A}}(x_1x_2\cdots x_mzy_1y_2\cdots y_n) \geq \min\{\mu_{\mathcal{A}}(x_1), \mu_{\mathcal{A}}(x_2), \ldots, \mu_{\mathcal{A}}(x_m), \mu_{\mathcal{A}}(y_1), \mu_{\mathcal{A}}(y_2), \ldots, \mu_{\mathcal{A}}(y_n)\}$;

(ii) $\eta_{\mathcal{A}}(x_1x_2\cdots x_mzy_1y_2\cdots y_n) \leq \max\{\eta_{\mathcal{A}}(x_1), \eta_{\mathcal{A}}(x_2), \ldots, \eta_{\mathcal{A}}(x_m), \eta_{\mathcal{A}}(y_1), \eta_{\mathcal{A}}(y_2), \ldots, \eta_{\mathcal{A}}(y_n)\}$;

(iii) $\nu_{\mathcal{A}}(x_1x_2\cdots x_mzy_1y_2\cdots y_n) \leq \max\{\nu_{\mathcal{A}}(x_1), \nu_{\mathcal{A}}(x_2), \ldots, \nu_{\mathcal{A}}(x_m), \nu_{\mathcal{A}}(y_1), \nu_{\mathcal{A}}(y_2), \ldots, \nu_{\mathcal{A}}(y_n)\}$,

for all $x_1, x_2, \ldots, x_m, z, y_1, y_2, \ldots, y_n \in S$.

Proposition 3.2: Let S be a semigroup and m,n be positive integers. Then, every picture fuzzy bi-ideal of S is a picture fuzzy (m,n)-ideal of S.

Proof: The proof is straightforward. ■

In general, the converse of Proposition 3.2 is not true. We can show this by the following example.

Example 3.3: Consider $S = \{a,b,c,d\}$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	b	a

Then $(S,\cdot)$ is a semigroup [23]. Define the picture fuzzy set $\mathcal{A}$ on S as follows:

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	0.8	0	0.1
b	0.1	0.5	0.4
c	0.1	0.5	0.4
d	0.6	0.2	0.2

By routine calculations, we have that $\mathcal{A}$ is a picture fuzzy $(2,2)$-ideal of S, but it is not a picture fuzzy bi-ideal of S because

$$\mu_{\mathcal{A}}(dca) = \mu_{\mathcal{A}}(b) = 0.1 < 0.6 = \min\{\mu_{\mathcal{A}}(d), \mu_{\mathcal{A}}(a)\}.$$

Similarly, $\eta_{\mathcal{A}}(dca) > \max\{\eta_{\mathcal{A}}(d), \eta_{\mathcal{A}}(a)\}$ and $\nu_{\mathcal{A}}(dca) > \max\{\nu_{\mathcal{A}}(d), \nu_{\mathcal{A}}(a)\}$.

Theorem 3.4: Let S be a semigroup and $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of picture fuzzy (m,n)-ideals of S. Then, $\bigcap_{i\in\Lambda} \mathcal{A}_i$ is also a picture fuzzy (m,n)-ideal of S.

Proof: Let $x,y \in S$. Then, we have

$$\begin{aligned}\bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(xy) &\geq \bigwedge_{i\in\Lambda} \min\{\mu_{\mathcal{A}_i}(x), \mu_{\mathcal{A}_i}(y)\} \\ &= \min\left\{\bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(x), \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(y)\right\}\end{aligned}$$

and

$$\begin{aligned}\bigvee_{i\in\Lambda} \eta_{\mathcal{A}_i}(xy) &\leq \bigvee_{i\in\Lambda} \max\{\eta_{\mathcal{A}_i}(x), \eta_{\mathcal{A}_i}(y)\} \\ &= \max\left\{\bigvee_{i\in\Lambda} \eta_{\mathcal{A}_i}(x), \bigvee_{i\in\Lambda} \eta_{\mathcal{A}_i}(y)\right\}.\end{aligned}$$

Similarly, we have that

$$\bigvee_{i\in\Lambda} \nu_{\mathcal{A}_i}(xy) \leq \max\left\{\bigvee_{i\in\Lambda} \nu_{\mathcal{A}_i}(x), \bigvee_{i\in\Lambda} \nu_{\mathcal{A}_i}(y)\right\}.$$

Hence, $\bigcap_{i\in\Lambda} \mathcal{A}_i$ is a picture fuzzy subsemigroup of S. Let $x_1, x_2, \ldots, x_m, z, y_1, y_2, \ldots, y_n \in S$. Thus, we have

$$\begin{aligned}&\bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(x_1x_2\cdots x_mzy_1y_2\cdots y_n) \\ &\geq \bigwedge_{i\in\Lambda} \min\{\mu_{\mathcal{A}_i}(x_1), \mu_{\mathcal{A}_i}(x_2), \ldots, \mu_{\mathcal{A}_i}(x_m), \\ &\qquad \mu_{\mathcal{A}_i}(y_1), \mu_{\mathcal{A}_i}(y_2), \ldots, \mu_{\mathcal{A}_i}(y_n)\} \\ &= \min\left\{\bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(x_1), \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(x_2), \ldots, \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(x_m), \right. \\ &\qquad \left. \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(y_1), \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(y_2), \ldots, \bigwedge_{i\in\Lambda} \mu_{\mathcal{A}_i}(y_n)\right\}\end{aligned}$$

and

$$\begin{aligned}&\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\le\bigvee_{i\in\Lambda}\max\{\eta_{\mathcal{A}_i}(x_1),\eta_{\mathcal{A}_i}(x_2),\ldots,\eta_{\mathcal{A}_i}(x_m),\\ &\qquad\qquad\eta_{\mathcal{A}_i}(y_1),\eta_{\mathcal{A}_i}(y_2),\ldots,\eta_{\mathcal{A}_i}(y_n)\}\\ &\quad=\max\left\{\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(x_1),\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(x_2),\ldots,\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(x_m),\right.\\ &\qquad\left.\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(y_1),\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(y_2),\ldots,\bigvee_{i\in\Lambda}\eta_{\mathcal{A}_i}(y_n)\right\}.\end{aligned}$$

Also, we obtain that

$$\begin{aligned}&\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\le\max\left\{\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(x_1),\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(x_2),\ldots,\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(x_m),\right.\\ &\qquad\left.\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(y_1),\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(y_2),\ldots,\bigvee_{i\in\Lambda}\nu_{\mathcal{A}_i}(y_n)\right\}.\end{aligned}$$

Therefore, $\bigcap_{i\in\Lambda}\mathcal{A}_i$ is a picture fuzzy (m,n)-ideal of S. ■

In the following example, we present that the union of any two picture fuzzy (m,n)-ideals of a semigroup S need not to be a picture fuzzy (m,n)-ideal of S.

Example 3.5: Let $S=\{a,b,c,d\}$ and " $\cdot$ " be a binary operation defined on S in the following table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	a	a	a
d	a	a	b	d

Then, $(S,\cdot)$ is a semigroup [3]. Define two picture fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ on S as follows:

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	0.7	0	0
b	0	0.5	0.5
c	0.4	0.2	0.3
d	0	0.5	0.5

$\mathcal{B}$	$\mu_{\mathcal{B}}$	$\eta_{\mathcal{B}}$	$\nu_{\mathcal{B}}$
a	0.9	0	0
b	0	0.4	0.6
c	0	0.4	0.6
d	0.5	0.3	0.2

By routine computations, we obtain that $\mathcal{A}$ and $\mathcal{B}$ are picture fuzzy (m,n)-ideals of S, but $\mathcal{A}\cup\mathcal{B}$ is not a picture fuzzy (m,n)-ideal of S because

$$\begin{aligned}&(\mu_{\mathcal{A}}\vee\mu_{\mathcal{B}})(dc)\\ &\quad=(\mu_{\mathcal{A}}\vee\mu_{\mathcal{B}})(b)\\ &\quad=\max\{\mu_{\mathcal{A}}(b),\mu_{\mathcal{B}}(b)\}=0\\ &\quad<0.4=\min\{\mu_{\mathcal{B}}(d),\mu_{\mathcal{A}}(c)\}\\ &\quad=\min\{\max\{\mu_{\mathcal{A}}(d),\mu_{\mathcal{B}}(d)\},\max\{\mu_{\mathcal{A}}(c),\mu_{\mathcal{B}}(c)\}\}\\ &\quad=\min\{(\mu_{\mathcal{A}}\vee\mu_{\mathcal{B}})(d),(\mu_{\mathcal{A}}\vee\mu_{\mathcal{B}})(c)\}.\end{aligned}$$

This shows that $\mathcal{A}\cup\mathcal{B}$ is not a picture fuzzy subsemigroup of S.

Lemma 3.6: Let S be a nonempty subset of a semigroup S and m,n be positive integers. Then, A is an (m,n)-ideal of S if and only if $\mathcal{C}_A$ is a picture fuzzy (m,n)-ideal of S.

Proof: Assume that A is an (m,n)-ideal of S. Then, A is a subsemigroup of S. By Lemma 2.3, $\mathcal{C}_A$ is a picture fuzzy subsemigroup of S. Let $x_1,x_2,\ldots,x_m,z,y_1,y_2,\ldots,y_n\in S$. Thus, there are two cases to consider:

Case 1: If $x_i\notin A$ or $y_j\notin A$ for some $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$, then

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\ge 0=\min\{\mu_{\mathcal{C}_A}(x_1),\mu_{\mathcal{C}_A}(x_2),\ldots,\mu_{\mathcal{C}_A}(x_m),\\ &\qquad\mu_{\mathcal{C}_A}(y_1),\mu_{\mathcal{C}_A}(y_2),\ldots,\mu_{\mathcal{C}_A}(y_n)\},\end{aligned}$$

$$\begin{aligned}&\eta_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\le 1=\max\{\eta_{\mathcal{C}_A}(x_1),\eta_{\mathcal{C}_A}(x_2),\ldots,\eta_{\mathcal{C}_A}(x_m),\\ &\qquad\eta_{\mathcal{C}_A}(y_1),\eta_{\mathcal{C}_A}(y_2),\ldots,\eta_{\mathcal{C}_A}(y_n)\},\end{aligned}$$

$$\begin{aligned}&\nu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\le 1=\max\{\nu_{\mathcal{C}_A}(x_1),\nu_{\mathcal{C}_A}(x_2),\ldots,\nu_{\mathcal{C}_A}(x_m),\\ &\qquad\nu_{\mathcal{C}_A}(y_1),\nu_{\mathcal{C}_A}(y_2),\ldots,\nu_{\mathcal{C}_A}(y_n)\}.\end{aligned}$$

Case 2: If $x_i,y_j\in A$ for all $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$, then $x_1x_2\cdots x_mzy_1y_2\cdots y_n\in A^mSA^n\subseteq A$. Thus, we have

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)=1\\ &\quad\ge\min\{\mu_{\mathcal{C}_A}(x_1),\mu_{\mathcal{C}_A}(x_2),\ldots,\mu_{\mathcal{C}_A}(x_m),\\ &\qquad\mu_{\mathcal{C}_A}(y_1),\mu_{\mathcal{C}_A}(y_2),\ldots,\mu_{\mathcal{C}_A}(y_n)\},\end{aligned}$$

$$\begin{aligned}&\eta_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)=0\\ &\quad\le\max\{\eta_{\mathcal{C}_A}(x_1),\eta_{\mathcal{C}_A}(x_2),\ldots,\eta_{\mathcal{C}_A}(x_m),\\ &\qquad\eta_{\mathcal{C}_A}(y_1),\eta_{\mathcal{C}_A}(y_2),\ldots,\eta_{\mathcal{C}_A}(y_n)\},\end{aligned}$$

$$\begin{aligned}&\nu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)=0\\ &\quad\le\max\{\nu_{\mathcal{C}_A}(x_1),\nu_{\mathcal{C}_A}(x_2),\ldots,\nu_{\mathcal{C}_A}(x_m),\\ &\qquad\nu_{\mathcal{C}_A}(y_1),\nu_{\mathcal{C}_A}(y_2),\ldots,\nu_{\mathcal{C}_A}(y_n)\}.\end{aligned}$$

This means that $\mathcal{C}_A$ is a picture fuzzy (m,n)-ideal of S.

Conversely, assume that $\mathcal{C}_A$ is a picture fuzzy (m,n)-ideal of S. Also, $\mathcal{C}_A$ is a picture fuzzy subsemigroup of S. By Lemma 2.3, A is a subsemigroup of S. Let $x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_n\in A$ and $z\in S$. Then,

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)\\ &\quad\ge\min\{\mu_{\mathcal{C}_A}(x_1),\mu_{\mathcal{C}_A}(x_2),\ldots,\mu_{\mathcal{C}_A}(x_m),\\ &\qquad\mu_{\mathcal{C}_A}(y_1),\mu_{\mathcal{C}_A}(y_2),\ldots,\mu_{\mathcal{C}_A}(y_n)\}=1.\end{aligned}$$

It follows that $\mu_{\mathcal{C}_A}(x_1x_2\cdots x_mzy_1y_2\cdots y_n)=1$. Hence, $x_1x_2\cdots x_mzy_1y_2\cdots y_n\in A$. This implies that $A^mSA^n\subseteq A$. Consequently, A is an (m,n)-ideal of S. ■

Theorem 3.7: Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S and m,n be a positive integers. Then, $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S if and only if $\mathcal{A}\circ\mathcal{A}\subseteq\mathcal{A}$ and $\mathcal{A}^m\circ\mathcal{S}\circ\mathcal{A}^n\subseteq\mathcal{A}$.

Proof: Assume that $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S. By Lemma 2.2(i), $\mathcal{A}\circ\mathcal{A}\subseteq\mathcal{A}$. Let $a\in S$. If a is not to be expressible $a=x'y'$ for all $x',y'\in S$, then $(\mu_{\mathcal{A}}^m\circ\mu_{\mathcal{S}}\circ\mu_{\mathcal{A}}^n)(a)\le\mu_{\mathcal{A}}(a)$. On the other hand, suppose that $(\mu_{\mathcal{A}}^m\circ\mu_{\mathcal{S}}\circ\mu_{\mathcal{A}}^n)(a)\neq 0$. Then, there exist $x,y\in S$ such that $a=xy$, $(\mu_{\mathcal{A}}^m\circ\mu_{\mathcal{S}})(x)\neq 0$ and $\mu_{\mathcal{A}}^n(y)\neq 0$. Since $a=xy$, $(\mu_{\mathcal{A}}^m\circ\mu_{\mathcal{S}})(x)\neq 0$, there exist $u_1,v_1\in S$ such that $x=u_1v_1$, $\mu_{\mathcal{A}}^m(u_1)\neq 0$ and $\mu_{\mathcal{S}}(v_1)\neq 0$. By induction, we have that there exist $u_2,v_2,\ldots,u_m,v_m\in S$ such that $u_{i-1}=u_iv_i$ for each $i\in\{2,\ldots,m\}$, $\mu_{\mathcal{A}}^{m-i+1}(u_i)\neq 0$ and $\mu_{\mathcal{A}}(v_i)\neq 0$. Similarly, for case $\mu_{\mathcal{A}}^n(y)\neq 0$, we have that there exist $u'_2,v'_2,\ldots,u'_{n-1},v'_{n-1}\in S$ such that $u'_{j-1}=u'_jv'_j$ for all

$j \in \{2, \ldots, n-1\}$, $mu_{\mathcal{A}}^{n-j}(u'_j) \neq 0$ and $\mu_{\mathcal{A}}(v'_j) \neq 0$. Thus, we have

$$\begin{aligned}
&(\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{A}}^n)(a) \\
&= \bigvee_{a=xy} \min\{(\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}})(x), \mu_{\mathcal{A}}^n(y)\} \\
&= \bigvee_{a=xy} \min\left\{ \bigvee_{x=u_1v_1} \min\{\mu_{\mathcal{A}}^m(u_1), \mu_{\mathcal{S}}(v_1)\}, \bigvee_{y=u'_1v'_1} \min\{\mu_{\mathcal{A}}^{n-1}(u'_1), \mu_{\mathcal{A}}(v'_1)\} \right\} \\
&= \bigvee_{a=xy} \bigvee_{x=u_1v_1} \bigvee_{y=u'_1v'_1} \min\{\mu_{\mathcal{A}}^m(u_1), \mu_{\mathcal{S}}(v_1), \mu_{\mathcal{A}}^{n-1}(u'_1), \mu_{\mathcal{A}}(v'_1)\} \\
&= \bigvee_{a=xy} \bigvee_{x=u_1v_1} \bigvee_{y=u'_1v'_1} \min\{\mu_{\mathcal{A}}^m(u_1), \mu_{\mathcal{A}}^{n-1}(u'_1), \mu_{\mathcal{A}}(v'_1)\} \\
&= \bigvee_{a=xy} \bigvee_{x=u_1v_1} \bigvee_{y=u'_1v'_1} \min\left\{ \bigvee_{u_1=u_2v_2} \min\{\mu_{\mathcal{A}}^{m-1}(u_2), \mu_{\mathcal{A}}(v_2)\}, \bigvee_{u'_1=u'_2v'_2} \min\{\mu_{\mathcal{A}}^{n-2}(u'_2), \mu_{\mathcal{A}}(v'_2)\}, \mu_{\mathcal{A}}(v'_1) \right\} \\
&= \bigvee_{a=xy} \bigvee_{x=u_1v_1} \bigvee_{y=u'_1v'_1} \bigvee_{u_1=u_2v_2} \bigvee_{u'_1=u'_2v'_2} \min\{\mu_{\mathcal{A}}^{m-1}(u_2), \mu_{\mathcal{A}}(v_2), \mu_{\mathcal{A}}^{n-2}(u'_2), \mu_{\mathcal{A}}(v'_2), \mu_{\mathcal{A}}(v'_1)\} \\
&\ \vdots \\
&= \bigvee_{a=xy} \bigvee_{x=u_1v_1} \bigvee_{y=u'_1v'_1} \bigvee_{u_1=u_2v_2} \bigvee_{u'_1=u'_2v'_2} \cdots \bigvee_{u_{m-1}=u_mv_m} \bigvee_{u'_{n-2}=u'_{n-1}v'_{n-1}} \min\{\mu_{\mathcal{A}}(u_m), \mu_{\mathcal{A}}(v_m), \mu_{\mathcal{A}}(v_{m-1}), \ldots, \mu_{\mathcal{A}}(v_2), \\
&\qquad \mu_{\mathcal{A}}(u'_{n-1}), \mu_{\mathcal{A}}(v'_{n-1}), \mu_{\mathcal{A}}(v'_{n-2}), \ldots, \mu_{\mathcal{A}}(v_1)'\} \\
&\leq \bigvee_{a=xy} \mu_{\mathcal{A}}(u_mv_mv_{m-1} \cdots v_2v-1 \; u'_{n-1}v'_{n-1}v'_{n-2} \cdots v'_1) \\
&= \bigvee_{a=xy} \mu_{\mathcal{A}}(xy) = \mu_{\mathcal{A}}(a).
\end{aligned}$$

Similarly, we obtain that $(\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{A}}^n)(a) \geq \eta_{\mathcal{A}}(a)$ and $(\nu_{\mathcal{A}}^m \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{A}}^n)(a) \geq \nu_{\mathcal{A}}(a)$. This implies that $\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \subseteq \mathcal{A}$.

Conversely, by Lemma 2.2(i), we have that $\mathcal{A}$ is a picture fuzzy subsemisemigroup of S. Let $x_1, x_2, \ldots, x_m, z, y_1, y_2, \ldots, y_n \in S$ and take $a = x_1x_2 \cdots x_mzy_1y_2 \cdots y_n$. By assumption, we have

$$\begin{aligned}
&\mu_{\mathcal{A}}(x_1x_2 \cdots x_mzy_1y_2 \cdots y_n) = \mu_{\mathcal{A}}(a) \\
&\geq (\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{A}}^n)(a) \\
&= \bigvee_{a=bc} \min\{(\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}})(b), \mu_{\mathcal{A}}^n(c)\} \\
&\geq \min\{(\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}})(x_1x_2 \cdots x_mz), \mu_{\mathcal{A}}^n(y_1y_2 \cdots y_n)\} \\
&= \min\left\{ \bigvee_{x_1x_2\cdots x_mz=pq} \min\{\mu_{\mathcal{A}}^m(p), \mu_{\mathcal{S}}(q)\}, \bigvee_{y_1y_2\cdots y_n=uv} \min\{\mu_{\mathcal{A}}^{n-1}(u), \mu_{\mathcal{A}}(v)\} \right\} \\
&\geq \min\{\min\{\mu_{\mathcal{A}}^m(x_1x_2 \cdots x_m), \mu_{\mathcal{S}}(z)\}, \min\{\mu_{\mathcal{A}}^{n-1}(y_1y_2 \cdots y_{n-1}), \mu_{\mathcal{A}}(y_n)\}\} \\
&= \min\{\mu_{\mathcal{A}}^m(x_1x_2 \cdots x_m), \mu_{\mathcal{A}}^{n-1}(y_1y_2 \cdots y_{n-1}), \mu_{\mathcal{A}}(y_n)\} \\
&\ \vdots \\
&\geq \min\{\mu_{\mathcal{A}}(x_1), \mu_{\mathcal{A}}(x_2), \ldots, \mu_{\mathcal{A}}(x_m), \mu_{\mathcal{A}}(y_1), \mu_{\mathcal{A}}(y_2), \ldots, \mu_{\mathcal{A}}(y_n)\}
\end{aligned}$$

and

$$\begin{aligned}
&\eta_{\mathcal{A}}(x_1x_2 \cdots x_mzy_1y_2 \cdots y_n) = \eta_{\mathcal{A}}(a) \\
&\leq (\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{A}}^n)(a) \\
&= \bigwedge_{a=bc} \max\{(\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}})(b), \eta_{\mathcal{A}}^n(c)\} \\
&\leq \max\{(\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}})(x_1x_2 \cdots x_mz), \eta_{\mathcal{A}}^n(y_1y_2 \cdots y_n)\} \\
&= \max\left\{ \bigwedge_{x_1x_2\cdots x_mz=pq} \max\{\eta_{\mathcal{A}}^m(p), \eta_{\mathcal{S}}(q)\}, \bigwedge_{y_1y_2\cdots y_n=uv} \max\{\eta_{\mathcal{A}}^{n-1}(u), \eta_{\mathcal{A}}(v)\} \right\} \\
&\leq \max\{\max\{\eta_{\mathcal{A}}^m(x_1x_2 \cdots x_m), \eta_{\mathcal{S}}(z)\}, \max\{\eta_{\mathcal{A}}^{n-1}(y_1y_2 \cdots y_{n-1}), \eta_{\mathcal{A}}(y_n)\}\} \\
&= \max\{\eta_{\mathcal{A}}^m(x_1x_2 \cdots x_m), \eta_{\mathcal{A}}^{n-1}(y_1y_2 \cdots y_{n-1}), \eta_{\mathcal{A}}(y_n)\} \\
&\ \vdots \\
&\leq \max\{\eta_{\mathcal{A}}(x_1), \eta_{\mathcal{A}}(x_2), \ldots, \eta_{\mathcal{A}}(x_m), \eta_{\mathcal{A}}(y_1), \eta_{\mathcal{A}}(y_2), \ldots, \eta_{\mathcal{A}}(y_n)\}.
\end{aligned}$$

Also, we can show that

$$\nu_{\mathcal{A}}(x_1x_2 \cdots x_mzy_1y_2 \cdots y_n) \leq \max\{\nu_{\mathcal{A}}(x_1), \nu_{\mathcal{A}}(x_2), \ldots, \nu_{\mathcal{A}}(x_m), \nu_{\mathcal{A}}(y_1), \nu_{\mathcal{A}}(y_2), \ldots, \nu_{\mathcal{A}}(y_n)\}.$$

Therefore, $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S. ■

Proposition 3.8: Let S be a semigroup and m, n be positive integers. If $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S and $\mathcal{B}$ is a picture fuzzy subsemigroup of S such that $\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \subseteq \mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ is also a picture fuzzy (m,n)-ideal of S.

Proof: Assume that $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S and $\mathcal{B}$ is a picture fuzzy subsemigroup of S such that $\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \subseteq \mathcal{B} \subseteq \mathcal{A}$. By Lemma 2.2($i$), $\mathcal{B} \circ \mathcal{B} \subseteq \mathcal{B}$. Let $a \in S$. Then,

$$(\mu_{\mathcal{B}}^m \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{B}}^n)(a) \leq (\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{A}}^n)(a) \leq \mu_{\mathcal{B}}(a)$$

and

$$(\eta_{\mathcal{B}}^m \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{B}}^n)(a) \geq (\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{A}}^n)(a) \geq \eta_{\mathcal{B}}(a).$$

Also, we can prove that $(\nu_{\mathcal{B}}^m \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{B}}^n)(a) \geq \nu_{\mathcal{B}}(a)$. This implies that $\mathcal{B}^m \circ \mathcal{S} \circ \mathcal{B}^n \subseteq \mathcal{B}$. By Theorem 3.7, $\mathcal{B}$ is a picture fuzzy (m,n)-ideal of S. ■

Definition 3.9: [17] Let S be a semigroup and m, n be positive integers. Then, S is called (m, n)-*regular* if for any $a \in S$, there exists $x \in S$ such that $a = a^m x a^n$. This is equivalent to saying that $a \in a^m S a^n$, for every $a \in S$ or $A \subseteq A^m S A^n$, for each $A \subseteq S$.

Lemma 3.10: Let S be an (m, n)-regular semigroup where m, n are positive integers. Then, every picture fuzzy (m, n)-ideal of S is also a picture fuzzy bi-ideal of S.

Proof: Assume that $\mathcal{A}$ is a picture fuzzy (m, n)-ideal of S. Also, $\mathcal{A}$ is a picture fuzzy subsemigroup of S. Let $x, y, z \in S$. Since S is (m, n)-regular, there exist $a, b \in S$ such that $xzy = x^m a x^n z y^m b y^n$. Thus, we have

$$\begin{aligned}\mu_{\mathcal{A}}(xzy) &= \mu_{\mathcal{A}}(x^m(ax^n zy^m b)y^n)\\ &= \mu_{\mathcal{A}}(\underbrace{xx\cdots x}_{m \text{ times}}(ax^n zy^m b)\underbrace{yy\cdots y}_{n \text{ times}})\\ &\geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x), \ldots, \mu_{\mathcal{A}}(x),\\ &\qquad \mu_{\mathcal{A}}(y), \mu_{\mathcal{A}}(y), \ldots, \mu_{\mathcal{A}}(y)\}\\ &= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}\end{aligned}$$

and

$$\begin{aligned}\eta_{\mathcal{A}}(xzy) &= \eta_{\mathcal{A}}(x^m(ax^n zy^m b)y^n)\\ &= \eta_{\mathcal{A}}(\underbrace{xx\cdots x}_{m \text{ times}}(ax^n zy^m b)\underbrace{yy\cdots y}_{n \text{ times}})\\ &\leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \ldots, \eta_{\mathcal{A}}(x),\\ &\qquad \eta_{\mathcal{A}}(y), \eta_{\mathcal{A}}(y), \ldots, \eta_{\mathcal{A}}(y)\}\\ &= \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y)\}.\end{aligned}$$

Similarly, we have that $\nu_{\mathcal{A}}(xzy) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y)\}$. Hence, $\mathcal{A}$ is a picture fuzzy bi-ideal of S. ∎

Lemma 3.11: Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, $\mu_{\mathcal{A}}(x) \leq \mu^k_{\mathcal{A}}(x^k), \eta_{\mathcal{A}}(x) \geq \eta^k_{\mathcal{A}}(x^k)$ and $\nu_{\mathcal{A}}(x) \geq \nu^k_{\mathcal{A}}(x^k)$ for each positive integer k and for every $x \in S$.

Proof: Let $x \in S$ and k be any positive integer. Since $x^k = x^{k-1}x$, we have

$$\begin{aligned}\mu^k_{\mathcal{A}}(x^k) &= \bigvee_{x^k=ab} \min\{\mu^{k-1}_{\mathcal{A}}(a), \mu_{\mathcal{A}}(b)\}\\ &\geq \min\{\mu^{k-1}_{\mathcal{A}}(x^{k-1}), \mu_{\mathcal{A}}(x)\}\\ &= \min\left\{\bigvee_{x^{k-1}=pq} \min\{\mu^{k-2}_{\mathcal{A}}(p), \mu_{\mathcal{A}}(q)\}, \mu_{\mathcal{A}}(x)\right\}\\ &\geq \min\{\min\{\mu^{k-2}_{\mathcal{A}}(x^{k-2}), \mu_{\mathcal{A}}(x)\}, \mu_{\mathcal{A}}(x)\}\\ &= \min\{\mu^{k-2}_{\mathcal{A}}(x^{k-2}), \mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x)\}\\ &\vdots\\ &\geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x), \ldots, \mu_{\mathcal{A}}(x)\}\\ &= \mu_{\mathcal{A}}(x)\end{aligned}$$

and

$$\begin{aligned}\eta^k_{\mathcal{A}}(x^k) &= \bigwedge_{x^k=ab} \max\{\eta^{k-1}_{\mathcal{A}}(a), \eta_{\mathcal{A}}(b)\}\\ &\leq \max\{\eta^{k-1}_{\mathcal{A}}(x^{k-1}), \eta_{\mathcal{A}}(x)\}\\ &= \max\left\{\bigwedge_{x^{k-1}=pq} \max\{\eta^{k-2}_{\mathcal{A}}(p), \eta_{\mathcal{A}}(q)\}, \eta_{\mathcal{A}}(x)\right\}\\ &\leq \max\{\max\{\eta^{k-2}_{\mathcal{A}}(x^{k-2}), \eta_{\mathcal{A}}(x)\}, \eta_{\mathcal{A}}(x)\}\\ &= \max\{\eta^{k-2}_{\mathcal{A}}(x^{k-2}), \eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x)\}\\ &\vdots\\ &\leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \ldots, \eta_{\mathcal{A}}(x)\}\\ &= \eta_{\mathcal{A}}(x).\end{aligned}$$

Also, we obtain that $\nu^k_{\mathcal{A}}(x^k) \leq \nu_{\mathcal{A}}$. This completed the proof. ∎

Theorem 3.12: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$, for every picture fuzzy set $\mathcal{A}$ on S.

Proof: Assume that S is (m, n)-regular. Let $a \in S$. Then, there exists $x \in S$ such that $a = a^m x a^n$. So, we have

$$\begin{aligned}&(\mu^m_{\mathcal{A}} \circ \mu_{\mathcal{S}} \circ \mu^n_{\mathcal{A}})(a)\\ &= \bigvee_{a=bc} \min\{(\mu^m_{\mathcal{A}} \circ \mu_{\mathcal{S}})(b), \mu^n_{\mathcal{A}}(c)\}\\ &\geq \min\{\mu^m_{\mathcal{A}} \circ \mu_{\mathcal{S}})(a^m x), \mu^n_{\mathcal{A}}(a^n)\}\\ &= \min\left\{\bigvee_{a^m x=uv} \min\{\mu^m_{\mathcal{A}}(u), \mu_{\mathcal{S}}(v)\}, \mu^n_{\mathcal{A}}(a^n)\right\}\\ &\geq \min\{\min\{\mu^m_{\mathcal{A}}(a^m), \mu_{\mathcal{S}}(x)\}, \mu^n_{\mathcal{A}}(a^n)\}\\ &= \min\{\mu^m_{\mathcal{A}}(a^m), \mu^n_{\mathcal{A}}(a^n)\}\\ &\geq \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{A}}(a)\}\\ &= \mu_{\mathcal{A}}(a)\end{aligned}$$

and

$$\begin{aligned}&(\eta^m_{\mathcal{A}} \circ \eta_{\mathcal{S}} \circ \eta^n_{\mathcal{A}})(a)\\ &= \bigwedge_{a=bc} \max\{(\eta^m_{\mathcal{A}} \circ \eta_{\mathcal{S}})(b), \eta^n_{\mathcal{A}}(c)\}\\ &\leq \max\{\eta^m_{\mathcal{A}} \circ \eta_{\mathcal{S}})(a^m x), \eta^n_{\mathcal{A}}(a^n)\}\\ &= \max\left\{\bigwedge_{a^m x=uv} \max\{\eta^m_{\mathcal{A}}(u), \eta_{\mathcal{S}}(v)\}, \eta^n_{\mathcal{A}}(a^n)\right\}\\ &\leq \max\{\max\{\eta^m_{\mathcal{A}}(a^m), \eta_{\mathcal{S}}(x)\}, \eta^n_{\mathcal{A}}(a^n)\}\\ &= \max\{\eta^m_{\mathcal{A}}(a^m), \eta^n_{\mathcal{A}}(a^n)\}\\ &\leq \max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{A}}(a)\}\\ &= \eta_{\mathcal{A}}(a).\end{aligned}$$

Similarly, we can prove that $(\nu^m_{\mathcal{A}} \circ \nu_{\mathcal{S}} \circ \nu^n_{\mathcal{A}})(a) \leq \nu_{\mathcal{A}}(a)$. Hence, $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$.

Conversely, let $a \in S$. Then, $\mathcal{C}_a$ is a picture fuzzy set on S. By assumption and Lemma 2.1, we have $\mathcal{C}_a \subseteq \mathcal{C}^m_a \circ \mathcal{S} \circ \mathcal{C}^n_a = \mathcal{C}_{a^m S a^n}$. It follows that $a \in a^m S a^n$. Therefore, S is (m, n)-regular. ∎

Theorem 3.13: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$, for every picture fuzzy (m, n)-ideal $\mathcal{A}$ of S.

Proof: Assume that S is (m,n)-regular. Let $\mathcal{A}$ be a picture fuzzy (m,n)-ideal of S. By Theorem 3.7 and Theorem 3.12, we obtain that $\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$. That is, $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$.

Conversely, assume that $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$, for every picture fuzzy (m,n)-ideal $\mathcal{A}$ of S. Let $a \in S$. Then, $[a]_{(m,n)}$ is the smallest (m,n)-ideal of S containing a. By Lemma 3.6, $\mathcal{C}_{[a]_{(m,n)}}$ is a picture fuzzy (m,n)-ideal of S. By using the given assumption, we have

$$\mathcal{C}_{[a]_{(m,n)}} = \mathcal{C}^m_{[a]_{(m,n)}} \circ \mathcal{S} \circ \mathcal{C}^n_{[a]_{(m,n)}} = \mathcal{C}_{\left([a]_{(m,n)}\right)^m S \left([a]_{(m,n)}\right)^n}.$$

It follows that $[a]_{(m,n)} = \left([a]_{(m,n)}\right)^m S \left([a]_{(m,n)}\right)^n$. By Lemma 2.4, $a \in a^m S a^n$. Therefore, S is (m,n)-regular. ■

Lemma 3.14: [23] Let S be a semigroup and m,n be positive integers with $m \geq 2$ or $n \geq 2$. Then, S is (m,n)-regular if and only if $A = A^2$, for each (m,n)-ideal A of S.

Theorem 3.15: Let S be a semigroup and m,n be positive integers where $m \geq 2$ or $n \geq 2$. Then, S is (m,n)-regular if and only if $\mathcal{A} = \mathcal{A} \circ \mathcal{A}$, for every picture fuzzy (m,n)-ideal $\mathcal{A}$ of S.

Proof: Assume that S is (m,n)-regular. Let $\mathcal{A}$ be a picture fuzzy (m,n)-ideal of S. By Theorem 3.13, we have

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \\
&= \mathcal{A}^m \circ \mathcal{S} \circ (\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n)^n \\
&= \mathcal{A}^m \circ \mathcal{S} \circ ((\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n) \circ (\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n) \\
&\qquad \circ \cdots \circ (\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n))\ (n \text{ terms}) \\
&\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \circ \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \\
&= \mathcal{A} \circ \mathcal{A}.
\end{aligned}$$

Since $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S, $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$. This implies that $\mathcal{A} = \mathcal{A} \circ \mathcal{A}$.

Conversely, assume that $\mathcal{A} = \mathcal{A} \circ \mathcal{A}$, for every picture fuzzy (m,n)-ideal of S. Let A be an (m,n)-ideal of S. Then, $AA \subseteq A$. Let $a \in A$. By the hypothesis, we have $\mu_{\mathcal{C}_A}(a) = (\mu_{\mathcal{C}_A} \circ \mu_{\mathcal{C}_A})(a) = \mu_{\mathcal{C}_{A^2}}(a)$. Since $\mu_{\mathcal{C}_A}(a) = 1$, $\mu_{\mathcal{C}_{A^2}}(a) = 1$. It turns out that $a \in A^2$. So, $A \subseteq A^2$. Hence, $A = A^2$. By Lemma 3.14, S is (m,n)-regular.

For case $m \geq 2$, we can prove similarly. ■

The following example shows that the condition $m \geq 2$ or $n \geq 2$ in Theorem 3.15 is neccessary.

Example 3.16: Let $S = \{a,b,c,d,e\}$. Define the binary operation "$\cdot$" on S by the following table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

Then, $(S,\cdot)$ is a regular semigroup [23]. Next, we define the picture fuzzy set $\mathcal{A}$ on S as follows:

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	1	0	0
b	0.2	0.4	0.4
c	0	0.5	0.4
d	0	0.5	0.5
e	0.6	0.2	0.1

It turns out that $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S. Moreover, $\mathcal{A} \neq \mathcal{A} \circ \mathcal{A}$, since

$$\begin{aligned}
\mu_{\mathcal{A}}(e) &= 0.6 \neq 0.2 \\
&= \bigvee_{e=yz} \min\{\mu_{\mathcal{A}}(y), \mu_{\mathcal{A}}(z)\} = (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(e),
\end{aligned}$$

$$\begin{aligned}
\eta_{\mathcal{A}}(e) &= 0.2 \neq 0.4 \\
&= \bigwedge_{e=yz} \max\{\eta_{\mathcal{A}}(y), \eta_{\mathcal{A}}(z)\} = (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(e)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\mathcal{A}}(e) &= 0.1 \neq 0.4 \\
&= \bigwedge_{e=yz} \max\{\nu_{\mathcal{A}}(y), \nu_{\mathcal{A}}(z)\} = (\nu_{\mathcal{A}} \circ \nu_{\mathcal{A}})(e).
\end{aligned}$$

IV. Picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals

In this section, we present the concepts of picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups for any positive integers m and n which are generalizations of picture fuzzy right ideals and picture fuzzy left ideals of semigroups, respectively, and we consider some properties of them. Finally, we characterize different classes of semigroups, that is, (m,n)-regular, $(m,0)$-regular and $(0,n)$-regular, using the concepts of picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups.

Definition 4.1: Let S be a semigroup and m be a positive integer. A picture fuzzy subsemigroup $\mathcal{A}$ of S is called a *picture fuzzy* $(m,0)$-*ideal* of S if it satisfies the following conditions:

(i) $\mu_{\mathcal{A}}(x_1x_2\cdots x_m z) \geq \min\{\mu_{\mathcal{A}}(x_1), \mu_{\mathcal{A}}(x_2), \ldots, \mu_{\mathcal{A}}(x_m)\}$;

(ii) $\eta_{\mathcal{A}}(x_1x_2\cdots x_m z) \leq \max\{\eta_{\mathcal{A}}(x_1), \eta_{\mathcal{A}}(x_2), \ldots, \eta_{\mathcal{A}}(x_m)\}$;

(iii) $\nu_{\mathcal{A}}(x_1x_2\cdots x_m z) \leq \max\{\nu_{\mathcal{A}}(x_1), \nu_{\mathcal{A}}(x_2), \ldots, \nu_{\mathcal{A}}(x_m)\}$,

for all $x_1, x_2, \ldots, x_m, z \in S$.

Dually, a *picture fuzzy* $(0,n)$-*ideal* of S is defined.

Lemma 4.2: Every picture fuzzy right (resp. left) ideal of a semigroup S is a picture fuzzy $(m,0)$-ideal (resp. $(0,n)$-ideal) for any positive integer m (resp. n).

Proof: The proof is straightforward. ■

The converse of Lemma 4.2 is not true in general.

Example 4.3: In Example 3.3, we have that $\mathcal{A}$ is a picture fuzzy $(m,0)$-ideal and a picture fuzzy $(0,n)$-ideal of S for every positive integers $m,n \geq 2$, but is not a picture fuzzy right ideal of S because

$$\begin{aligned}
\mu_{\mathcal{A}}(dc) &= \mu_{\mathcal{A}}(b) = 0.1 < 0.6 = \mu_{\mathcal{A}}(d), \\
\eta_{\mathcal{A}}(dc) &= \eta_{\mathcal{A}}(b) = 0.5 > 0.2 = \eta_{\mathcal{A}}(d), \\
\nu_{\mathcal{A}}(dc) &= \nu_{\mathcal{A}}(b) = 0.4 > 0.2 = \nu_{\mathcal{A}}(d).
\end{aligned}$$

Also, $\mathcal{A}$ is not a picture fuzzy left ideal of S because

$$\begin{aligned}
\mu_{\mathcal{A}}(ca) &= \mu_{\mathcal{A}}(c) = 0.1 < 0.8 = \mu_{\mathcal{A}}(a), \\
\eta_{\mathcal{A}}(ca) &= \eta_{\mathcal{A}}(c) = 0.5 > 0 = \eta_{\mathcal{A}}(a), \\
\nu_{\mathcal{A}}(ca) &= \nu_{\mathcal{A}}(c) = 0.4 > 0 = \nu_{\mathcal{A}}(a).
\end{aligned}$$

Definition 4.4: [23] Let S be a semigroup and m,n be positive integers. Then, S is called $(m,0)$-*regular* (resp.

$(0,n)$-*regular*) if for any $a \in S$, there exists $x \in S$ such that $a = a^m x$ (resp. $a = xa^n$).

Lemma 4.5: Let S be a semigroup and m, n be positive integers. Then, the following statements hold:

(i) if S is $(m,0)$-regular, then every picture fuzzy $(m,0)$-ideal of S is also a picture fuzzy right ideal of S;

(ii) if S is $(0,n)$-regular, then every picture fuzzy $(0,n)$-ideal of S is also a picture fuzzy left ideal of S.

Proof: (i) Assume that S is $(m,0)$-regular. Let $\mathcal{A}$ be a picture fuzzy $(m,0)$-ideal of S and let $x, y \in S$. Then, there exists $z \in S$ such that $xy = x^m zy$. Thus, we have

$$\begin{aligned}\mu_{\mathcal{A}}(xy) &= \mu_{\mathcal{A}}(x^m(zy)) \\ &\geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x), \ldots, \mu_{\mathcal{A}}(x)\} = \mu_{\mathcal{A}}(x), \\ \eta_{\mathcal{A}}(xy) &= \eta_{\mathcal{A}}(x^m(zy)) \\ &\leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \ldots, \eta_{\mathcal{A}}(x)\} = \eta_{\mathcal{A}}(x), \\ \nu_{\mathcal{A}}(xy) &= \nu_{\mathcal{A}}(x^m(zy)) \\ &\leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x), \ldots, \nu_{\mathcal{A}}(x)\} = \nu_{\mathcal{A}}(x).\end{aligned}$$

Hence, $\mathcal{A}$ is a picture fuzzy right ideal of S.

(ii) The proof is similar to (i). ■

The following lemma, we will prove for picture fuzzy $(m,0)$-ideals of semigroups. For picture fuzzy $(0,n)$-ideals of semigroups, one can prove similarly.

Lemma 4.6: Let S be a semigroup and m, n be positive integers. Then, A is an $(m,0)$-ideal (resp. $(0,n)$-ideal) of S if and only if $\mathcal{C}_A$ is a picture fuzzy $(m,0)$-ideal (resp. $(0,n)$-ideal) of S.

Proof: Assume that A is an $(m,0)$-ideal of S. Then, A is a subsemigroup of S. By Lemma 2.3, $\mathcal{C}_A$ is a picture fuzzy subsemigroup of S. Let $x_1, x_2, \ldots, x_m, z \in S$. If $x_i \notin A$ for some $i \in \{1, 2, \ldots, m\}$, then

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad \geq 0 = \min\{\mu_{\mathcal{C}_A}(x_1), \mu_{\mathcal{C}_A}(x_2), \ldots, \mu_{\mathcal{C}_A}(x_m)\}, \\ &\eta_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad \leq 1 = \max\{\eta_{\mathcal{C}_A}(x_1), \eta_{\mathcal{C}_A}(x_2), \ldots, \eta_{\mathcal{C}_A}(x_m)\}.\end{aligned}$$

Also, we have that

$$\begin{aligned}&\nu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad \leq \max\{\nu_{\mathcal{C}_A}(x_1), \nu_{\mathcal{C}_A}(x_2), \ldots, \nu_{\mathcal{C}_A}(x_m)\}.\end{aligned}$$

If $x_i \in A$ for all $i \in \{1, 2, \ldots, m\}$, then $x_1x_2\cdots x_m z \in A^m S \subseteq A$. It follows that

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad = 1 \geq \min\{\mu_{\mathcal{C}_A}(x_1), \mu_{\mathcal{C}_A}(x_2), \ldots, \mu_{\mathcal{C}_A}(x_m)\}, \\ &\eta_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad = 0 \leq \max\{\eta_{\mathcal{C}_A}(x_1), \eta_{\mathcal{C}_A}(x_2), \ldots, \eta_{\mathcal{C}_A}(x_m)\}.\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}&\nu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad \leq \max\{\nu_{\mathcal{C}_A}(x_1), \nu_{\mathcal{C}_A}(x_2), \ldots, \nu_{\mathcal{C}_A}(x_m)\}.\end{aligned}$$

Hence, $\mathcal{C}_A$ is a picture fuzzy $(m,0)$-ideal of S.

Conversely, assume that $\mathcal{C}_A$ is a picture fuzzy $(m,0)$-ideal of S. Also, $\mathcal{C}_A$ is a picture fuzzy subsemigroup of S. Again by Lemma 2.3, A is a subsemigroup of S. Let $x_1, x_2, \ldots, x_m \in A$ and $z \in S$. Then,

$$\begin{aligned}&\mu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) \\ &\quad \geq \min\{\mu_{\mathcal{C}_A}(x_1), \mu_{\mathcal{C}_A}(x_2), \ldots, \mu_{\mathcal{C}_A}(x_m)\} = 1,\end{aligned}$$

and so, $\mu_{\mathcal{C}_A}(x_1x_2\cdots x_m z) = 1$. This implies that $x_1x_2\cdots x_m z \in A$. It turns out that $A^m S \subseteq A$. Therefore, A is an $(m,0)$-ideal of S. ■

Theorem 4.7: Let S be a semigroup, $\mathcal{A}$ be any picture fuzzy subsemigroup of S and m, n be positive integers. Then, $\mathcal{A}$ is a picture fuzzy $(m,0)$-ideal (resp. $(0,n)$-ideal) of S if and only if $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A}^m \circ \mathcal{S} \subseteq \mathcal{A}$ (resp. $\mathcal{S} \circ \mathcal{A}^n \subseteq \mathcal{A}$).

Proof: The proof is similar to Theorem 3.7. ■

A picture fuzzy set $\mathcal{A}$ on a semigroup S is called *idempotent* if $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$.

Lemma 4.8: If S is an (m,n)-regular semigroup, then the picture fuzzy $(m,0)$-ideals and the picture fuzzy $(0,n)$-ideals of S are idempotent.

Proof: Assume that S is an (m,n)-regular semigroup. Let $\mathcal{A}$ is a picture fuzzy $(m,0)$-ideal of S. Then, $\mathcal{A}$ is a picture fuzzy subsemigroup of S. It follows that $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ by Lemma 2.2(i). On the other hand, by assumption and Theorem 3.12, we have

$$\begin{aligned}\mathcal{A} &\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \\ &= \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^{n-1} \circ \mathcal{A} \\ &\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^{n-1} \circ \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n \\ &\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^m \circ \mathcal{S} \\ &\subseteq \mathcal{A} \circ \mathcal{A}.\end{aligned}$$

We obtain that $\mathcal{A} \subseteq \mathcal{A} \circ \mathcal{A}$. Thus, $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$. Hence, $\mathcal{A}$ is idempotent. Similarly, we can prove that the picture fuzzy $(0,n)$-ideals of S are idempotent. ■

Lemma 4.9: Let S be an (m,n)-regular semigroup and m, n be positive integers. Then, the following statements hold:

(i) if $\mathcal{A}$ is a picture fuzzy $(m,0)$-ideal of S, then $\mathcal{A}^m$ is also a picture fuzzy right ideal of S;

(ii) if $\mathcal{A}$ is a picture fuzzy $(0,n)$-ideal of S, then $\mathcal{A}^n$ is also a picture fuzzy left ideal of S.

Proof: (i) Let $\mathcal{A}$ be a picture fuzzy $(m,0)$-ideal of S. By Theorem 4.7 and Lemma 4.8, we have that $\mathcal{A}^m \circ \mathcal{S} \subseteq \mathcal{A} = \mathcal{A}^m$. By Lemma 2.2$(ii)$, we obtain that $\mathcal{A}^m$ is a picture fuzzy right ideal of S.

(ii) The proof is similar to the proof of (i). ■

Theorem 4.10: Let S be a semigroup and m, n be positive integers. Then, the following statements hold:

(i) S is $(m,0)$-regular if and only if $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S}$, for every picture fuzzy set $\mathcal{A}$ on S;

(ii) S is $(0,n)$-regular if and only if $\mathcal{A} \subseteq \mathcal{S} \circ \mathcal{A}^n$, for every picture fuzzy set $\mathcal{A}$ on S.

Proof: (i) Assume that S is $(m,0)$-regular. Let $a \in S$. Then, there exists $x \in S$ such that $a = a^m x$. So, we have

$$\begin{aligned}(\mu_{\mathcal{A}}^m \circ \mu_{\mathcal{S}})(a) &= \bigvee_{a=bc} \min\{\mu_{\mathcal{A}}^m(b), \mu_{\mathcal{S}}(c)\} \\ &\geq \min\{\mu_{\mathcal{A}}^m(a^m), \mu_{\mathcal{S}}(x)\} \\ &= \mu_{\mathcal{A}}^m(a^m) \\ &\geq \mu_{\mathcal{A}}(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{A}}^m \circ \eta_{\mathcal{S}})(a) &= \bigwedge_{a=bc} \max\{\eta_{\mathcal{A}}^m(b), \eta_{\mathcal{S}}(c)\} \\ &\leq \max\{\eta_{\mathcal{A}}^m(a^m), \eta_{\mathcal{S}}(x)\} \\ &= \eta_{\mathcal{A}}^m(a^m) \\ &\leq \eta_{\mathcal{A}}(a).\end{aligned}$$

Similarly, we can prove that $(\nu_{\mathcal{A}}^m \circ \nu_{\mathcal{S}})(a) \leq \nu_{\mathcal{A}}(a)$. Hence, $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S}$.

Conversely, let $a \in S$. Then, $\mathcal{C}_a$ is a picture fuzzy set on S. By the given assumption and Lemma 2.1, we have that $\mathcal{C}_a \subseteq \mathcal{C}_a^m \circ \mathcal{S} = \mathcal{C}_{a^m S}$, implies that $a \in a^m S$. Therefore S is $(m, 0)$-regular.

(ii) The proof is similar to (i). ■

Lemma 4.11: [17] Let S be a semigroup, m, n be positive integers. Then, the following statements hold:

(i) S is $(m, 0)$-regular if and only if $R = R^m S$, for every $(m, 0)$-ideal R of S;

(ii) S is $(0, n)$-regular if and only if $L = SL^n$, for every $(0, n)$-ideal L of S.

Theorem 4.12: Let S be a semigroup and m, n be positive integers. Then, the following statements hold:

(i) S is $(m, 0)$-regular if and only if $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S}$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ of S;

(ii) S is $(0, n)$-regular if and only if $\mathcal{A} = \mathcal{S} \circ \mathcal{A}^n$, for every picture fuzzy $(0, n)$-ideal $\mathcal{A}$ of S.

Proof: (i) Assume that S is $(m, 0)$-regular. Let A be a picture fuzzy $(m, 0)$-ideal of S. By Theorem 4.7 and Theorem 4.10, we have that $\mathcal{A}^m \circ \mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{A}^m \circ \mathcal{S}$. It follows that $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S}$.

Conversely, let R be any $(m, 0)$-ideal of S. Then, $R^2 \subseteq R$ and $R^m S \subseteq R$. By Lemma 2.1, we have $\mathcal{C}_R \circ \mathcal{C}_R = \mathcal{C}_{R^2} \subseteq \mathcal{C}_R$. Again by Lemma 2.1, $\mathcal{C}_R^m \circ \mathcal{S} = \mathcal{C}_{R^m S} \subseteq \mathcal{C}_R$. Thus, $\mathcal{C}_R$ is a picture fuzzy $(m, 0)$-ideal of S by Theorem 4.7. By the given assumption, we get that $\mathcal{C}_R = \mathcal{C}_R^m \circ \mathcal{S} = \mathcal{C}_{R^m S}$. It turns out that $R = R^m S$. By Lemma 4.11, S is $(m, 0)$-regular.

(ii) The proof is similar to (i). ■

Lemma 4.13: [23] Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $R \cap L = R^m L^n$, for each $(m, 0)$-ideal R of S and for each $(0, n)$-ideal L of S.

Theorem 4.14: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}^m \circ \mathcal{B}^n$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ of S and for every picture fuzzy $(0, n)$-ideal $\mathcal{B}$ of S.

Proof: Assume that S is (m, n)-regular. Let $\mathcal{A}$ be any picture fuzzy $(m, 0)$-ideal of S and $\mathcal{B}$ be any picture fuzzy $(0, n)$-ideal of S. By Theorem 3.12, we have

$$\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \cap \mathcal{B})^m \circ \mathcal{S} \circ (\mathcal{A} \cap \mathcal{B})^n \subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B}^n.$$

By Theorem 4.7, $\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B}^n \subseteq \mathcal{A}^m \circ \mathcal{B}$. By Lemma 4.8, $\mathcal{B} = \mathcal{B}^n$. It follows that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}^n$. On the other hand, $\mathcal{A}^m \circ \mathcal{B}^n \subseteq \mathcal{A}^m \circ \mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{A}^m \circ \mathcal{B}^n \subseteq \mathcal{S} \circ \mathcal{B}^n \subseteq \mathcal{B}$. This implies that $\mathcal{A}^m \circ \mathcal{B}^n \subseteq \mathcal{A} \cap \mathcal{B}$. Therefore, $\mathcal{A} \cap \mathcal{B} = \mathcal{A}^m \circ \mathcal{B}^n$.

Conversely, assume that $\mathcal{A} \cap \mathcal{B} = \mathcal{A}^m \circ \mathcal{B}^n$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ and for every picture fuzzy $(0, n)$-ideal $\mathcal{B}$ of S. Let R and L be an $(m, 0)$-ideal and an $(0, n)$-ideal of S, respectively. By Lemma 4.6, $\mathcal{C}_R$ and $\mathcal{C}_L$ are a picture fuzzy $(m, 0)$-ideal and a picture fuzzy $(0, n)$-ideal of S, respectively. By hypothesis, we have

$$\mathcal{C}_{R \cap L} = \mathcal{C}_R \cap \mathcal{C}_L = \mathcal{C}_R^m \circ \mathcal{C}_L^n = \mathcal{C}_{R^m L^n}.$$

This implies that $R \cap L = R^m L^n$. By Lemma 4.13, S is (m, n)-regular. ■

By Lemma 4.8 and Theorem 4.14, we have the following corollary.

Corollary 4.15: Let S be a semigroup and m, n be positive integers. If S is (m, n)-regular, then $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \circ \mathcal{B}$ for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ of S and for every picture fuzzy $(0, n)$-ideal $\mathcal{B}$ of S.

Theorem 4.16: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}^n$, for each picture fuzzy (m, n)-ideal $\mathcal{A}$ and for each picture fuzzy $(0, n)$-ideal $\mathcal{B}$ of S.

Proof: Assume that S is (m, n)-regular. Let $\mathcal{A}$ and $\mathcal{B}$ be a picture fuzzy (m, n)-ideal and a picture fuzzy $(0, n)$-ideal of S, respectively. By Theorem 3.12, Theorem 4.7 and Lemma 4.8, we have

$$\begin{aligned}\mathcal{A} \cap \mathcal{B} &\subseteq (\mathcal{A} \cap \mathcal{B})^m \circ \mathcal{S} \circ (\mathcal{A} \cap \mathcal{B})^n \\ &\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B}^n \\ &\subseteq \mathcal{A}^m \circ \mathcal{B} \\ &= \mathcal{A}^m \circ \mathcal{B}^n.\end{aligned}$$

This shows that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}^n$.

Conversely, let $\mathcal{A}$ be a picture fuzzy $(m, 0)$-ideal and $\mathcal{B}$ be a picture fuzzy $(0, n)$-ideal of S. It is not difficult to verify that $\mathcal{A}$ is also a picture fuzzy (m, n)-ideal of S. By the given assumption, we get $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}^n$. On the other hand, $\mathcal{A}^m \circ \mathcal{B}^n \subseteq \mathcal{A} \cap \mathcal{B}$. Hence, $\mathcal{A} \cap \mathcal{B} = \mathcal{A}^m \circ \mathcal{B}^n$. By Theorem 4.14, we have that S is (m, n)-regular. ■

The following theorem can be proved similarly to Theorem 4.16.

Theorem 4.17: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}^n$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ and for every picture fuzzy (m, n)-ideal $\mathcal{B}$ of S.

Theorem 4.18: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{R} \cap \mathcal{A} \cap \mathcal{L} \subseteq \mathcal{R}^m \circ \mathcal{A} \circ \mathcal{L}^n$, for every picture fuzzy (m, n)-ideal $\mathcal{A}$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{R}$ and for every picture fuzzy $(0, n)$-ideal $\mathcal{L}$ of S.

Proof: Assume that S is (m, n)-regular. Let $\mathcal{A}, \mathcal{R}$ and $\mathcal{L}$ be a picture fuzzy (m, n)-ideal, a picture fuzzy $(m, 0)$-ideal and a picture fuzzy $(0, n)$-ideal of S, respectively. By Lemma 4.9, we have that $\mathcal{R}^m$ is a picture fuzzy right ideal and $\mathcal{L}^n$ is a picture fuzzy left ideal of S. Let $a \in S$. Then, there exists $x \in S$ such that

$$\begin{aligned}a &= a^m x a^n = a^m x (a^m x a^n)^n \\ &= a^m x \underbrace{(a^m x a^n)(a^m x a^n) \cdots (a^m x a^n)}_{n \text{ terms}}.\end{aligned}$$

Thus, $a = (a^m x)(a^m x a^n)(s a^n)$ for some $s =$

$a^m x a^n \cdots a^m x$. Using the Lemma 3.11, we have

$$\begin{aligned}
&(\mu_{\mathcal{R}}^m \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{L}}^n)(a)\\
&\quad= \bigvee_{a=bc} \min\{(\mu_{\mathcal{R}}^m \circ \mu_{\mathcal{A}})(b), \mu_{\mathcal{L}}^n(c)\}\\
&\quad= \bigvee_{a=bc} \min\left\{\bigvee_{b=pq} \min\{\mu_{\mathcal{R}}^m(p), \mu_{\mathcal{A}}(q)\}, \mu_{\mathcal{L}}^n(c)\right\}\\
&\quad\geq \min\{\min\{\mu_{\mathcal{R}}^m(a^m x), \mu_{\mathcal{A}}(a^m x a^n)\}, \mu_{\mathcal{L}}^n(sa^n)\}\\
&\quad\geq \min\{\min\{\mu_{\mathcal{R}}^m(a^m), \mu_{\mathcal{A}}(a)\}, \mu_{\mathcal{L}}^n(a^n)\}\\
&\quad\geq \min\{\min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{A}}(a)\}, \mu_{\mathcal{L}}(a)\}\\
&\quad= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{A}} \cap \mu_{\mathcal{L}})(a)
\end{aligned}$$

and

$$\begin{aligned}
&(\eta_{\mathcal{R}}^m \circ \eta_{\mathcal{A}} \circ \eta_{\mathcal{L}}^n)(a)\\
&\quad= \bigwedge_{a=bc} \max\{(\eta_{\mathcal{R}}^m \circ \eta_{\mathcal{A}})(b), \eta_{\mathcal{L}}^n(c)\}\\
&\quad= \bigwedge_{a=bc} \max\left\{\bigwedge_{b=pq} \max\{\eta_{\mathcal{R}}^m(p), \eta_{\mathcal{A}}(q)\}, \eta_{\mathcal{L}}^n(c)\right\}\\
&\quad\leq \max\{\max\{\eta_{\mathcal{R}}^m(a^m x), \eta_{\mathcal{A}}(a^m x a^n)\}, \eta_{\mathcal{L}}^n(sa^n)\}\\
&\quad\leq \max\{\max\{\eta_{\mathcal{R}}^m(a^m), \eta_{\mathcal{A}}(a)\}, \eta_{\mathcal{L}}^n(a^n)\}\\
&\quad\leq \max\{\max\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{A}}(a)\}, \eta_{\mathcal{L}}(a)\}\\
&\quad= (\eta_{\mathcal{R}} \cup \eta_{\mathcal{A}} \cup \eta_{\mathcal{L}})(a).
\end{aligned}$$

Similarly, we can show that $(\nu_{\mathcal{R}}^m \circ \nu_{\mathcal{A}} \circ \nu_{\mathcal{L}}^n)(a) \leq (\nu_{\mathcal{R}} \cup \nu_{\mathcal{A}} \cup \nu_{\mathcal{L}})(a)$. This implies that $\mathcal{R} \cap \mathcal{A} \cap \mathcal{L} \subseteq \mathcal{R}^m \circ \mathcal{A} \circ \mathcal{L}^n$.

Conversely, let $\mathcal{R}$ be a picture fuzzy $(m, 0)$-ideal and $\mathcal{L}$ be a picture fuzzy $(0, n)$-ideal of S. We can see that $\mathcal{S}$ itself is a picture fuzzy (m, n)-ideal of S. By the hypothesis, it follows that

$$\mathcal{R} \cap \mathcal{L} = \mathcal{R} \cap \mathcal{S} \cap \mathcal{L} \subseteq \mathcal{R}^m \circ \mathcal{S} \circ \mathcal{L}^n \subseteq \mathcal{R}^m \cap \mathcal{L}^n.$$

On the other hand, $\mathcal{R}^m \circ \mathcal{L}^n \subseteq \mathcal{R} \cap \mathcal{L}$. That is, $\mathcal{R} \cap \mathcal{L} = \mathcal{R}^m \circ \mathcal{L}^n$. By Theorem 4.14, S is (m, n)-regular. ■

Lemma 4.19: [17] Let S be a semigroup, m, n be positive integers. Then, S is (m, n)-regular if and only if $R \cap L = R^m L \cap R L^n$ for each $(m, 0)$-ideal R and for each $(0, n)$-ideal L of S.

Theorem 4.20: Let S be a semigroup and m, n be positive integers. Then, S is (m, n)-regular if and only if $\mathcal{A} \cap \mathcal{B} = (\mathcal{A}^m \circ \mathcal{B}) \cap (\mathcal{A} \circ \mathcal{B}^n)$, for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ of S and for every picture fuzzy $(0, n)$-ideal $\mathcal{B}$ of S.

Proof: Assume that S is (m, n)-regular. Let $\mathcal{A}$ be a picture fuzzy $(m, 0)$-ideal of S and $\mathcal{B}$ be a picture fuzzy $(0, n)$-ideal of S. By Theorem 3.12, we have

$$\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \cap \mathcal{B})^m \circ \mathcal{S} \circ (\mathcal{A} \cap \mathcal{B})^n \subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B}^n \subseteq \mathcal{A}^m \circ \mathcal{B},$$

that is, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^m \circ \mathcal{B}$. Similarly, we can show that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}^n$. It turns out that $\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A}^m \circ \mathcal{B}) \cap (\mathcal{A} \circ \mathcal{B}^n)$. On the other hand,

$$(\mathcal{A}^m \circ \mathcal{B}) \cap (\mathcal{A} \circ \mathcal{B}^n) \subseteq (\mathcal{A}^m \circ \mathcal{S}) \cap (\mathcal{S} \circ \mathcal{B}^n) \subseteq \mathcal{A} \cap \mathcal{B}.$$

Hence, $\mathcal{A} \cap \mathcal{B} = (\mathcal{A}^m \circ \mathcal{B}) \cap (\mathcal{A} \circ \mathcal{B}^n)$.

Conversely, let R and L be an $(m, 0)$-ideal and a $(0, n)$-ideal of S, respectively. By Lemma 4.6, $\mathcal{C}_R$ and $\mathcal{C}_L$ are a picture fuzzy $(m, 0)$-ideal and a picture fuzzy $(0, n)$-ideal of S, respectively. By the hypothesis, we have

$$\mathcal{C}_{R \cap L} = \mathcal{C}_R \cap \mathcal{C}_L = (\mathcal{C}_R^m \circ \mathcal{C}_L) \cap (\mathcal{C}_R \circ \mathcal{C}_L^n) = \mathcal{C}_{R^m L \cap R L^n}.$$

It follows that $R \cap L = R^m L \cap R L^n$. By Lemma 4.19, S is (m, n)-regular. ■

Lemma 4.21: Let S be a semigroup and m, n be positive integers. Then, the following statements hold:

(i) if S is $(m, 0)$-regular, then for any picture fuzzy set $\mathcal{A}$ on S, $\mathcal{A}^m \circ \mathcal{S}$ is a picture fuzzy $(m, 0)$-ideal of S;

(ii) if S is $(0, n)$-regular, then for any picture fuzzy set $\mathcal{A}$ on S, $\mathcal{S} \circ \mathcal{A}^n$ is a picture fuzzy $(0, n)$-ideal of S.

Proof: The proof is straightforward. ■

Lemma 4.22: Let S be a semigroup and m, n be positive integers. If S is (m, n)-regular, then for any picture (m, n)-ideal $\mathcal{A}$ of S, there exist a picture fuzzy $(m, 0)$-ideal $\mathcal{B}$ of S and a picture fuzzy $(0, n)$-ideal $\mathcal{C}$ of S such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$.

Proof: Let $\mathcal{A}$ be a picture fuzzy (m, n)-ideal of S. By Theorem 3.13, $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n$. As S is (m, n)-regular, we have that S is both $(m, 0)$-regular and $(0, n)$-regular. By Lemma 4.21, we get that $\mathcal{A}^m \circ \mathcal{S}$ is a picture fuzzy $(m, 0)$-ideal of S and $\mathcal{S} \circ \mathcal{A}^n$ is a picture fuzzy $(0, n)$-ideal of S. Now, consider $\mathcal{S} \subseteq \mathcal{S}^m \circ \mathcal{S} \circ \mathcal{S}^n = \mathcal{S}^{m+n+1} \subseteq \mathcal{S}^2 \subseteq \mathcal{S}$. So, $\mathcal{S} = \mathcal{S}^2$. We obtain that

$$(\mathcal{A}^m \circ \mathcal{S}) \circ (\mathcal{S} \circ \mathcal{A}^n) = \mathcal{A}^m \circ \mathcal{S}^2 \circ \mathcal{A}^n = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n = \mathcal{A}.$$

This completed the proof. ■

Lemma 4.23: Let S be a semigroup and m, n be positive integers. If S is (m, n)-regular, then for every picture fuzzy $(m, 0)$-ideal $\mathcal{A}$ of S and for every picture fuzzy set $\mathcal{B}$ on S, $\mathcal{A} \circ \mathcal{B}$ is a picture fuzzy (m, n)-ideal of S.

Proof: Assume that S is (m, n)-regular. Let $\mathcal{A}$ be a picture fuzzy $(m, 0)$-ideal of S and $\mathcal{B}$ be a picture fuzzy set on S. By Theorem 3.12 and Theorem 4.7, we have

$$\begin{aligned}
&(\mathcal{A} \circ \mathcal{B}) \circ (\mathcal{A} \circ \mathcal{B})\\
&\qquad\subseteq (\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n) \circ \mathcal{B} \circ (\mathcal{A}^m \circ \mathcal{S} \circ \mathcal{A}^n) \circ \mathcal{B}\\
&\qquad\subseteq \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B}\\
&\qquad\subseteq \mathcal{A} \circ \mathcal{B}.
\end{aligned}$$

Thus, $\mathcal{A} \circ \mathcal{B}$ is a picture fuzzy subsemigroup of S. Now, by Lemma 4.8 and Theorem 4.7, we get

$$\begin{aligned}
&(\mathcal{A} \circ \mathcal{B})^m \circ \mathcal{S} \circ (\mathcal{A} \circ \mathcal{B})^n\\
&\qquad= \underbrace{(\mathcal{A} \circ \mathcal{B}) \circ (\mathcal{A} \circ \mathcal{B}) \circ \cdots \circ (\mathcal{A} \circ \mathcal{B})}_{m \text{ terms}} \circ \mathcal{S}\\
&\qquad\quad \circ \underbrace{(\mathcal{A} \circ \mathcal{B}) \circ (\mathcal{A} \circ \mathcal{B}) \circ \cdots \circ (\mathcal{A} \circ \mathcal{B})}_{n \text{ terms}}\\
&\qquad\subseteq \mathcal{A} \circ \mathcal{S} \circ \mathcal{B} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}.
\end{aligned}$$

Therefore, $\mathcal{A} \circ \mathcal{B}$ is a picture fuzzy (m, n)-ideal of S. ■

The following theorem obtains from Lemma 4.22 and Lemma 4.23.

Theorem 4.24: Let S be an (m, n)-regular semigroup where m, n be positive integers and $\mathcal{A}$ be any picture fuzzy set on S. Then, $\mathcal{A}$ is a picture fuzzy (m, n)-ideal of S if and only if there exist a picture fuzzy $(m, 0)$-ideal $\mathcal{B}$ of S and a picture fuzzy $(0, n)$-ideal $\mathcal{C}$ of S such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$.

Let m, n are positive integers. Then, a picture fuzzy (m, n)-ideal $\mathcal{A}$ of a semigroup S is called *minimal* if for each picture fuzzy (m, n)-ideal $\mathcal{B}$ of S, $\mathcal{B} \subseteq \mathcal{A}$ implies $\mathcal{B} = \mathcal{A}$. Dually, a minimal picture fuzzy $(m, 0)$-ideal of S and a minimal picture fuzzy $(0, n)$-ideal of S are defined.

Theorem 4.25: Let S be an (m,n)-regular semigroup where m,n be positive integers and $\mathcal{A}$ be any picture fuzzy set on S. Then, $\mathcal{A}$ is a minimal picture fuzzy (m,n)-ideal of S if and only if there exist a minimal picture fuzzy $(m,0)$-ideal $\mathcal{B}$ of S and a minimal picture fuzzy $(0,n)$-ideal $\mathcal{C}$ of S such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$.

Proof: Assume that $\mathcal{A}$ is a minimal picture fuzzy (m,n)-ideal of S. Since S is (m,n)-regular, $\mathcal{A} = \mathcal{A}^m \circ \mathcal{S} \circ \mathcal{S} \circ \mathcal{A}^n$. By Lemma 4.21, $\mathcal{A}^m \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{A}^n$ are a picture fuzzy $(m,0)$-ideal and a picture fuzzy $(0,n)$-ideal of S, respectively. Next, we want to show that $\mathcal{A}^m \circ \mathcal{S}$ is minimal. Let $\mathcal{M}$ be any picture fuzzy $(m,0)$-ideal of S such that $\mathcal{M} \subseteq \mathcal{A}^m \circ \mathcal{S}$. By Corollary 4.15, we have that

$$\begin{aligned}\mathcal{M} \circ (\mathcal{S} \circ \mathcal{A}^n) &= \mathcal{M} \cap (\mathcal{S} \circ \mathcal{A}^n)\\ &\subseteq (\mathcal{A}^m \circ \mathcal{S}) \cap (\mathcal{S} \circ \mathcal{A}^n)\\ &= (\mathcal{A}^m \circ \mathcal{S}) \circ (\mathcal{S} \circ \mathcal{A}^n) = \mathcal{A}.\end{aligned}$$

By Lemma 4.23, $\mathcal{M} \circ (\mathcal{S} \circ \mathcal{A}^n)$ is a picture fuzzy (m,n)-ideal of S. By minimality of $\mathcal{A}$, we have that $\mathcal{M} \circ (\mathcal{S} \circ \mathcal{A}^n) = \mathcal{A}$. By Theorem 4.7 and Lemma 4.8, we have that $\mathcal{A} = \mathcal{M} \circ (\mathcal{S} \circ \mathcal{A}^n) \subseteq \mathcal{M}^m \circ \mathcal{S} \subseteq \mathcal{M}$. Thus, $\mathcal{A}^m \circ \mathcal{S} \subseteq \mathcal{M}^m \circ \mathcal{S} \subseteq \mathcal{M}$. Hence, $\mathcal{M} = \mathcal{A}^m \circ \mathcal{S}$. Therefore, $\mathcal{A}^m \circ \mathcal{S}$ is a minimal picture fuzzy $(m,0)$-ideal of S. Similary, we can show that $\mathcal{S} \circ \mathcal{A}^n$ is a minimal picture fuzzy $(0,n)$-ideal of S.

Conversely, assume that there exist a minimal picture fuzzy $(m,0)$-ideal $\mathcal{B}$ of S and a minimal picture fuzzy $(0,n)$-ideal $\mathcal{C}$ of S such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$. By Theorem 4.24, $\mathcal{A}$ is a picture fuzzy (m,n)-ideal of S. Let $\mathcal{M}$ be any picture fuzzy (m,n)-ideal of S such that $\mathcal{M} \subseteq \mathcal{A}$. Then,

$$\begin{aligned}\mathcal{M}^m \circ \mathcal{S} &\subseteq \mathcal{A}^m \circ \mathcal{S} = (\mathcal{B} \circ \mathcal{C})^m \circ \mathcal{S}\\ &= \underbrace{(\mathcal{B} \circ \mathcal{C}) \circ (\mathcal{B} \circ \mathcal{C}) \circ \cdots \circ (\mathcal{B} \circ \mathcal{C})}_{m \text{ terms}} \circ \mathcal{S}\\ &\subseteq \mathcal{B} \circ \mathcal{S} \subseteq (\mathcal{B}^m \circ \mathcal{S} \circ \mathcal{B}^n) \circ \mathcal{S}\\ &\subseteq \mathcal{B}^m \circ \mathcal{S} \subseteq \mathcal{B}.\end{aligned}$$

Since $\mathcal{M}^m \circ \mathcal{S}$ is a picture fuzzy $(m,0)$-ideal of S and $\mathcal{B}$ is a minimal picture fuzzy $(m,0)$-ideal of S, we have that $\mathcal{M}^m \circ \mathcal{S} = \mathcal{B}$. Similarly, we can show that $\mathcal{S} \circ \mathcal{M}^n = \mathcal{C}$. By Theorem 3.13, we have

$$\mathcal{A} = \mathcal{B} \circ \mathcal{C} = (\mathcal{M}^m \circ \mathcal{S}) \circ (\mathcal{S} \circ \mathcal{M}^n) = \mathcal{M}^m \circ \mathcal{S} \circ \mathcal{M}^n = \mathcal{M}.$$

That is, $\mathcal{M} = \mathcal{A}$. Consequently, $\mathcal{A}$ is a minimal picture fuzzy (m,n)-ideal of S. ■

V. Conclusion

As generalizations of picture fuzzy bi-ideals, picture fuzzy right ideals and picture fuzzy left ideals of semigroups, we have introduced the concepts of picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals of semigroups where m and n are positive ingeters. Then, we investigated some characterization of (m,n)-regular semigroups by their picture fuzzy (m,n)-ideals as shown in Theorem 3.13. Finally, we have characterized various classes (m,n)-regular, $(m,0)$-regular and $(0,n)$-regular semigroups by the properties of their picture fuzzy (m,n)-ideals, picture fuzzy $(m,0)$-ideals and picture fuzzy $(0,n)$-ideals were shown in Theorem 4.12, Theorem 4.14 and Theorem 4.20. In our future work, it will be possible to characterize many classes of regularities in semigroups and other algebraic structures using many types of their picture fuzzy (m,n)-ideals.

References

[1] T. C. Ahn, K. Hur and K. W. Jang, "Intuitionistic fuzzy subgroups and level subgroups," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 6, no. 3, pp. 240-246, 2006.

[2] M. Akram, N. Yaqoob and M. Khan, "On (m,n)-ideals in LA-semigroups," *Applied Mathematical Sciences*, vol. 7, no. 44, pp. 2187-2191, 2013.

[3] M. S. Ali Khan, K. Rahman, A. Fahmi and M. Shakeel, "Generalized $(\in, \in \vee q_k)$-fuzzy quasi-ideals in semigroups," *Punjab University Journal of Mathematics*, vol. 50, no. 1, , pp. 35-53, 2018.

[4] S. Al-Kaseasbeh, M. A. Tahan, B. Davvaz and M. Hariri, "Single valued neutrosophic (m,n)-ideals of ordered semirings," *AIMS Mathematics*, vol. 7, no. 1, pp. 1211-1223, 2021.

[5] K. T. Atanassov, *Intuitionistic fuzzy sets*. Bulgarian Academy Science, Bulgaria, 1983.

[6] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87-96, 1986.

[7] A. Basar, "A note on (m,n)-Γ-ideals of ordered LA-Γ-semigroups," *Konuralp Journal of Mathematics*, vol. 7, no. 1, pp. 107-111, 2019.

[8] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets - a new concept for computational intelligence problems," *Proceedings of the Third World Congress on Information and Communication Technologies WIICT*, pp. 1-6, 2013.

[9] P. Dutta and S. Ganju, "Some aspects of picture fuzzy set," *Transition of A. Razmadze Mathematical Institute*, vol. 172, pp. 164-175, 2018.

[10] T. Gaketem and P. Khamrot, "On some semigroups characterized in terms of bipolar fuzzy weakly interior ideals," *IAENG International Journal of Computer Science*, vol. 48, no. 2, pp. 250-256, 2021.

[11] K. Hila, B. Davvaz and K. Naka, "On quasi-hyperideals in semihypergroups," *Communications in Algebra*, vol. 39, pp. 4183-4194, 2011.

[12] Y. Hong, X. Fang, "Characterizing intraregular semigroups by intuitionistic fuzzy sets," *Mathware & Soft Computing*, vo. 12, pp. 121-128, 2005.

[13] K. Hur, S. Y. Jang and H. W. Kang, "Intuitionistic fuzzy subgroupoids," *International Journal of Fuzzy Logic and Intelligent Systems*, vo. 3, no. 1, pp. 72-77, 2003.

[14] K. Hur, S. Y. Jang and P. K. Lim, "Intuitionistic fuzzy semigroups," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 8, no. 3, pp. 207-219, 2008.

[15] W. Khan, F. Yousafzai, W. Guo and M. Khan, "On (m,n)-ideals of left almost semigroups," *Journal of Semigroup Theory and Applications*, vol. 2014, Article ID 1, 2014.

[16] K. H. Kim and Y. B. Jun, "Intuitionistic fuzzy ideals of semigroups," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 4, pp. 443-449, 2002.

[17] D. N. Krgović, "On (m,n)-regular semigroups," *Publ. Inst. Math.*, vol. 18, no. 32, pp. 107-110, 1975.

[18] N. Kuroki, "On fuzzy ideals and fuzzy bi-ideals in semigroups," *Fuzzy Sets and Systems*, vol. 5, no. 2, pp. 203-215, 1981.

[19] N. Kuroki," On fuzzy semigroups," *Information Sciences*, vol. 53, no. 3, pp. 203-236, 1991.

[20] N. Kuroki, "Fuzzy generalized bi-ideals in semigroups," *Information Sciences*, vol. 66, no. 3, pp. 235-243, 1992.

[21] N. Kuroki, "Fuzzy semiprime quasi-ideals in semigroups," *Information Sciences*, vol. 75, no. 3, pp. 201-211, 1993.

[22] S. Lajos, "Generalize ideals in semigroups," *Acta Sci. Math.*, vol. 22, pp. 217-222, 1961.

[23] A. Mahboob, B. Davvaz and N. M. Khan, "Fuzzy (m,n)-ideals in semigroups," *Computational and Applied Mathematics*, vol. 38, no. 4, pp. 1-18, 2019.

[24] W. Nakkhasen, "Characterizing regular and intra-regular semigroups in terms of picture fuzzy bi-ideal," *International Journal of Innovative Computing, Information and Control*, vol. 17, no. 6, pp. 2115-2135, 2021.

[25] W. Nakkhasen, "Intuitionistic fuzzy ideals of ternary near-rings," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 20, no, 4, pp. 290-297, 2020.

[26] W. Nakkhasen and B. Pibaljommee, "Intuitionistic fuzzy k-Γ-hyperideals of Γ-semihyperrings," *International Journal of Mathematics and Computer Science*, vol. 13, no. 2, pp. 139–155, 2018.

[27] B. Pibaljommee and W. Nakkhasen, "Connections of (m,n)-bi-quasi hyperideals in semihyperrings," *Thai Journal of Mathematics*, Special Issue: Annual Meeting in Mathematics 2019, pp. 39-48, 2020.

[28] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512-517, 1971.

[29] M. Shabir and A. Khan, "Intuitionistic fuzzy interior ideals in ordered semigroups," *J. Appl. Math. & Informatics.*,vol. 27, pp. 1447-1457, 2009.

[30] P. Yiarayong, "Semigroup characterized by picture fuzzy sets,"" *International Journal of Innovative Computing, Information and Control*, vol. 16, no. 6, pp. 2121-2130, 2020.

[31] P. Yiarayong, "Applications of hesitant fuzzy sets to completely regular semigroups,"" *IAENG International Journal of Computer Science*, vol. 47, no. 1, pp. 98-109, 2020.

[32] P. Yiarayong, "Characterisations of semigroups by the properties of their picture fuzzy bi-ideals," *Journal of Control and Decision*, vol. 9, no. 1, pp. 111-116, 2022.

[33] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[34] J. Zhan and X. Ma, "Intuitionistic fuzzy ideals of near-rings," *Scientiae Mathematicae Japanicae Online*, vol. 2004, pp. 289-293, 2004.