Minimum Cost Maximum Flow Problem in Continuous-Time Dynamic Networks

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Abstract—**In the application of networks, the capacity, storage cost, bandwidth and delay etc. can change over time. Therefore, the maximum flow problem in the dynamic network becomes an important research topic. In the dynamic network, each arc e is associated with two time-varying weight functions in a continuous time range: the cost function $w_e(t)$ of the unit flow on e and the capacity function $c_e(t)$ of e, where t is the departure time of flow on e. The objective in this work is to find an optimal scheme to send the maximum flow from a source to a sink with the minimum cost in a dynamic network. Due to the complexity of the problem in the network with time-varying cost and capacity functions, many researches consider the computing of the approximate solutions by using the technique of time discretization to transform the continuous-time dynamic network into classical static network. Then the degree of approximation is closely related to the degree of time discretization. In the changeable actual environment, some important changes of weight functions may be ignored due to time discretization. Therefore, in this paper, by using dynamic weight functions in different time periods, we consider the continuity of the transmission time and propose an efficient algorithm to find the maximum flow of minimum total cost with any departure time.**

Index Terms—**Minimum cost maximum flow problem, continuous-time dynamic capacity network, the residual network, the shortest dynamic path.**

I. Introduction

THE network flow is a problem domain that lies at the cusp between several fields of inquiry, including applied mathematics, computer science, engineering, and operations research. It is widely used in communication networks, transportation networks and many kinds of interconnection networks. Further more, many practical optimization problems need to be considered in the dynamic network environment, where the weight (e.g., cost, capacity, delay) function associated with each edge (or arc) will dynamically change over time and it is called the time-varying function. For example, the distance between any two nodes in the communication network will change over time; also in the transportation network, there are always a large number of vehicles in some special time period and may cause traffic jams. Therefore, the time-dependency becomes an indispensable element in the problems related to the maximum flow and so the dynamic flow problem needs to be studied in different models of continuous-time dynamic networks.

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Because of the complexity of the time-varying weight functions in the dynamic network, many research results obtained by using the method of time discretization. Then the classical method to solve the flow problem in time-varying network is to discretize a time horizon $[0, T]$ and reduce the dynamic problem to the static problem on a time-expanded network. In 1958, Ford et al. [7] put forward the concept of dynamic flow and obtained the classic Ford-Fulkerson maximum dynamic flow algorithm. Later Gale [10], Minieka[11] and other researchers [17], [16], [12] proposed discrete algorithms to solve the earliest arrival flow in continuous-time networks. In 2001, Cai et al. [5] solved the minimum cost flow problem and considered three time-varying arc weight functions: capacity, delay and cost. These functions depend on the departure time t of the flow on the arc where $t = 0, 1, 2, \ldots$. Fonoberova [8] considered the minimum cost multicommodity flow problem in dynamic networks with time-varying capacity and transmission time functions of arcs. Then Parpalea et al. [13] represented a generalisation of the maximum flow of minimum cost problem for the case of minimizing the travelling cost and time. The goal was to solve a series of maximum flow problems in different single objective functions. In 2019, Pyakurel [14] presented modified minimum cost flow algorithm that computed the maximum dynamic flow and the earliest arrival flow in strongly polynomial time when the time horizons of the weight functions were discrete.

Although the method of time discretization can simplify the optimization process, when the weight functions are sensitive to the changes of time in the process of transmission, the determination of the degree of time discretization will affect the approximation of the result and the complexity of the algorithm. Therefore, it has recently received much attention on the maximum flow problem with the time-varying weight functions in a continuous time horizon. There have been some phased results about this problem and the further research is necessary.

Different from using the auxiliary time-expanded network obtained by the time discretization method, in the continuous-time network, we need directly consider the weight functions in the time interval $[0, T]$ without changing the structure of the network. In 1982, Anderson et al. [1] examined the continuous-time dynamic flow problem with the constraint of storage capacity at each node in the absence of the traversal time. They proved the dynamic maximum flow minimum cut theorem by the continuous version of Ford-Fulkerson theory. Philpott [15] extended this result by adding the transmission time for each arc. Anderson and Philpott [2] developed a continuous-time version of the simplex method under the assumption that the cost function on each arc is piecewise linear, where the storage capacity and the cost of each node were considered. But the convergence of the algorithm and the optimality of the solution could not be guaranteed.

In recent years, there have been some new developments about the optimization problem of dynamic path in the continuous-time dynamic network. For example, Ding et al. [6] focused on the time-varying shortest path (TDSP) problem and found the optimal solution for the TDSP problem by using a continuous-time approach on First In First Out (FIFO) time-varying graphs. The time complexity of the main algorithm was $O((nlogn + m)\alpha(T))$ and the space complexity was $O((n + m)\alpha(T))$, where n and m were the numbers of the nodes and arcs, respectively, and $\alpha(T)$ was the number of time segments of the travel time function. Wang et al. [18] presented a fast path query algorithm of the TDSP problem by using the hierarchical graph partitioning. The complexity of the algorithm was $O(\log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T))$, where κ_f was the number of the subgraphs formed by graph partitioning.

In the aspect of maximum flow problem, there are few researches in continuous-time networks. We will investigate the minimum cost maximum flow problem in dynamic capacity networks with the time interval $[0, T]$. Each arc is assigned a time-varying capacity function and a time-varying cost function per unit flow on that arc. The flow starts from the source v_s at time t_s and any internal vertex may receive different parts of the flow at different times.

When the flow with value δ starts to enter the arc e at time t, we assume that the delay of the flow on e is $w_e(t)$ and the total cost of the flow on e is denoted by $g(t)$ in the time horizon $[t, t+w_e(t)]$. In the real network, especially the communication network and the transportation network, the longer the delay, the greater the cost value, that is, the delay $w_e(t)$ is proportional to the cost $g(t)$. Combining this with that $g(t) = \delta \cdot g_e(t)$, where $g_e(t)$ is the cost per unit flow on e, $w_e(t)$ is proportional to $g_e(t)$. Thus, in this paper, the delay function $w_e(t)$ is regarded as the cost function $g_e(t)$ per unit flow and we only consider the time-varying delay function $w_e(t)$.

In order to find the maximum flow that can be transmitted in the continuous-time network and to minimize the cost of the maximum flow, the main challenge in our research is that how to deal with the time-varying capacity function and the delay function of each arc in the residual capacity network in the process of increasing the value of flow. Since the flow will not wait at any node, it is necessary to ensure the time continuity of the flow passing through the adjacent arcs.

The organization of the paper is as follows. Section II states the basic concepts of dynamic network and some results that need to be used in this paper. Section III investigates the minimum cost maximum flow problem under the FIFO condition and presents an effective algorithm for any departure time of flow. Some examples will be given to illustrate the steps performed in the main algorithm in this section. In Section IV, we will prove the correctness of the algorithm and compute the time complexity. Section V concludes the paper.

II. Problem Formulation and Basic Concepts

A. The Dynamic Capacity Network and Dynamic Flow

The dynamic network is defined as follows.

Definition 1 (**continuous-time dynamic network**). A continuous-time dynamic network is defined by $G = (V(G), E(G), W, C, T, v_s, v_d)$.

- $V(G)$ is the set of nodes in G and let $|V(G)| = n$.
- $E(G)$ is the set of arcs in G and let $|E| = m$. For any $v_x, v_y \in V(G)$, $(v_x, v_y) \in E(G)$ if there exists an arc e with tail v_x and head v_y (i.e., e is from v_x to v_y).
- T is the upper bound of the time horizon considered.
- $W = \{w_e(t), e \in E(G)\}$, where $w_e(t) : [0, T] \mapsto R^+$ denotes the time-varying continues delay function of arc e with the departure time t and R^+ is the set of positive real numbers.
- $C = \{c_e(t), e \in E(G)\}$, where $c_e(t) : [0, T] \mapsto N^+$ denotes the time-varying continues capacity function of arc e at time t and N^+ is the set of positive integers.
- Nodes v_s and v_d are source and sink, respectively. In addition, v_s (resp. v_d) is not the head (resp. tail) of any arc.

A walk [3] in a static directed graph G is a finite node sequence $P = \langle v_0, v_1, v_2, \ldots, v_k\rangle$ which is such that $(v_i, v_{i+1}) \in E(G)$ for $i = 0, 1, \ldots, k-1$. Note that P can pass through some node multiple times. If G is a dynamic network, then we need consider the time dimension. The states of a node v in different times are different and so the node v at time t and the node v at time t' are regarded as two different nodes in the dynamic network. Then, in the following, we give the definition of the dynamic path.

Definition 2 (**dynamic path**). *Given a continuous-time dynamic network $G = (V(G), E(G), W, C, T)$, a walk $P(t) = \langle v_0, v_1, v_2, \ldots, v_k\rangle$ in G is said to be a dynamic path with departure time $t \in [0, T]$ if the delay function of $P(t)$ can be recursively computed by:*

$$w_{P_{v_0 v_i}}(t) = w_{P_{v_0 v_{i-1}}}(t) + w_{(v_{i-1}, v_i)}(t + w_{P_{v_0 v_{i-1}}}(t)),$$

where $P_{v_0 v_i}(t)$ is the walk $\langle v_0, v_1, \ldots, v_{i-1}, v_i\rangle$ in $P(t)$ from v_0 to the node v_i.

Recall that v_s and v_d are the source and sink, respectively. The set of all dynamic $v_s v_d$-paths with departure time t is denoted by $\mathcal{P}_{v_s v_d}(t)$. A dynamic path $P^*(t)$ from the source v_s to some node u with departure time t is called a *dynamic shortest path* if and only if

$$P^*(t) = \operatorname{argmin}\{w_P(t) : P(t) \in \mathcal{P}_{v_s u}(t)\}.$$

For the problem of finding the dynamic shortest path in continuous-time dynamic network, Y. Wang et al. [18] devised efficient algorithms based on TD-G-tree to find the time-varying shortest path (TDSP) with starting time t and the time-interval shortest path (TIP) with optimal starting time in time interval $[0, T]$. We know that in a static network, the successive shortest path algorithm [4] is used to find a shortest path between the source and sink in the residual network. Then the flow can be increased along the path and a maximum flow with minimal cost can be obtained at last. Similarly, in time-varying graph, since the delay function $w_e(t)$ will be treated as the cost function per unit flow on e when the flow enters e at time t, the TDSP algorithm can be used to find the dynamic shortest path with minimal delay.

Since the flow on an arc will leave the arc after a period of time and different flows can enter the same arc at different times, the flow on the arc is dynamic. Based on the the dynamic flow defined in [9], we restate the definition of the flow which will be considered in this paper as follows.

Definition 3 (**v_sv_d-Dynamic flow** [9])**.** *A v_sv_d-dynamic flow (or simply a dynamic flow) f in a dynamic network G with time horizon $[0,T]$ is a real-valued function $f_e(t)$: $E(G)\times[0,T] \to R^+$ defined on any $e \in E(G)$ and $t \in [0,T]$ satisfying the following conditions:*

- *$f_e(t)$ is the value of flow on e at time t, where $0 \le t \le T$;*
- *For each $v \in V(G) \setminus \{v_s, v_d\}$, if the value of flow received by a vertex v at time t is denoted by $f_v^-(t)$ and the value of flow leaving v at time t is denoted by $f_v^+(t)$, then $f_v^-(t) = f_v^+(t)$.*

Definition 4 (**Dynamic feasible flow**)**.** *For any arc $e = (v_x, v_y)$, if the dynamic flow f can arrive v_x at time t_x and leave v_y at time t_y. Then f is feasible if it satisfies the capacity constraint: $0 \le f_e(t) \le c_e(t)$ when $t \in [t_x, t_y]$.*

In this paper, we suppose that the departure time of the dynamic feasible flow f at vertex v_s is t_s. Then we give following definition.

Definition 5 (**v_sv_d-Dynamic feasible flow with departure time** t_s)**.** *A dynamic feasible flow f that departs from v_s at time t_s and arrives at v_d in the time horizon $[0,T]$ is called a v_sv_d-dynamic feasible flow with departure time t_s.*

According to the conditions in Definitions 1-5, it can be seen that all the dynamic feasible flow with departure time t_s can be received by the destination v_d and so

$$f_{v_s}^+(t_s) = \int_{t_s}^{T} f_{v_d}^-(t)dt.$$

Therefore, the value of the dynamic feasible flow f with departure time t_s is $Val(f) = f_{v_s}^+(t_s)$.

By the above definitions, for any dynamic feasible flow f, the cost of the flow $\phi(f)$ can be computed by following equation:

$$\phi(f) = \sum_{e\in E(G)} \int_0^T w_e(t) f_e^-(t)dt,$$

where $f_e^-(t)$ denotes the value of flow entering e at time t. We conclude this section by formulating the minimum cost maximum flow problem as follows.

Problem 1 (**Minimum cost maximum flow problem in continuous-time dynamic networks**)**.** *Given a continuous-time dynamic network $G = (V(G), E(G), W, C, T, v_s, v_d)$, find a v_sv_d-dynamic feasible flow $f(t)$ in G with the given departure time t_s at v_s such that the value of $Val(f)$ is maximum and the cost $\phi(f)$ is minimum.*

B. The Dynamic Cut

For any vertex v_x, if there exists a dynamic v_sv_x-path $P(t_s)$ such that the v_sv_d-dynamic feasible flow f can start from v_s at time t_s and arrive v_x at time t_x passing through $P(t_s)$, then the time t_x is called a *reachable time at* v_x and the pair (v_x, t_x) is called a *reachable node-time pair*.

We can observe that when the departure time of f is t_s which has been given, it is enough to consider the reachable times of other vertices to compute the value of flow. Then, in the continuous-time dynamic network, the dynamic cut is defined as follows.

Definition 6. *For any $\tilde{S} \subset V$ such that $v_s \in \tilde{S}$ and $v_d \notin \tilde{S}$, given a set of reachable node-time pairs $S = \{(v_x, t_x) : v_x \in \tilde{S}, t_x \in [0,T]\}$, let*

$$\begin{aligned}\bar{S} =&\{(v_y, t_y) : v_y \notin \tilde{S} \text{ and there exists } (v_x, v_y) \in E(G)\\ &\text{such that } (v_x, t_x) \in S \text{ and } t_y = t_x + w_{(v_x,v_y)}(t_x)\}.\end{aligned}$$

Then

$$\begin{aligned}K =&(S, \bar{S})\\ =&\{((v_x, t_x), (v_y, t_y)) : (v_x, t_x) \in S, (v_y, t_y) \in \bar{S},\\ &(v_x, v_y) \in E(G), t_y = t_x + w_{(v_x,v_y)}(t_x)\}\end{aligned}$$

is called a dynamic cut.

The minimum value of the function $c_e(t)$ when $t \in [a, b]$ is denoted by $c_e^*[a, b]$. Then we have the following definition.

Definition 7. *The capacity function of a dynamic cut K is defined as:*

$$Cap(K) = \sum_{((v_x,t_x),(v_y,t_y))\in K} c^*_{(v_x,v_y)}[t_x, t_y].$$

For any $((v_x, t_x), (v_y, t_y)) \in K$, we have known that the value of the dynamic feasible flow which can enter (v_x, v_y) at time t_x and pass through v_y until the time t_y is denoted by $f^-_{(v_x,v_y)}(t_x)$. Therefore,

$$f^-_{(v_x,v_y)}(t_x) \le c^*_{(v_x,v_y)}[t_x, t_y].$$

Combining this with Definition 7, following result holds.

Theorem 8. *Given a dynamic feasible flow f in a dynamic network G, for any dynamic cut $K = (S, \bar{S})$,*

$$\begin{aligned}&\sum_{((v_x,t_x),(v_y,t_y))\in K} f^-_{(v_x,v_y)}(t_x)\\ \le&\sum_{((v_x,t_x),(v_y,t_y))\in K} c^*_{(v_x,v_y)}[t_x, t_y] = Cap(K).\end{aligned}$$

Recall that, for any $v_x \in \tilde{S}$, $(v_x, t_x) \in S$ if and only if t_x is an reachable time at v_x. Then

$$Val(f) = f_{v_s}^+(t_s) = \sum_{((v_x,t_x),(v_y,t_y))\in K} f^-_{(v_x,v_y)}(t_x).$$

Therefore, by Theorem 8, we have a corollary as follows.

Corollary 9. *Given a dynamic feasible flow f in a dynamic network, for any dynamic cut $K = (S, \bar{S})$, $Val(f) \le Cap(K)$.*

In following sections, the main algorithms will be given to obtain the v_sv_d-dynamic feasible flow f (abbreviated as flow f if there is no ambiguity) with departure time t_s which is such that $Val(f) = Cap(K)$ and then prove that f is the flow as required.

III. Algorithm Scheme

The main algorithms will be given under the FIFO condition that $t + w_{(v_x,v_y)}(t) \le t' + w_{(v_x,v_y)}(t')$ for any times $t \le t'$ and arc (v_x, v_y).

According to the definition of Problem 1, the minimum cost maximum flow problem in continuous-time dynamic network requires finding the flow f with maximum $Val(f)$

and minimum cost under the condition that each arc has a dynamic capacity constraint. Recall that the delay function $w_e(t)$ is regarded as the cost function per unit flow on e in this paper. The core idea of the main algorithm is to increase flow iteratively. In each iteration, we find a dynamic v_sv_d-unsaturated path $P(t_s)$ with the minimum dynamic cost (i.e. delay) in the updated residual dynamic network. Then, by sending an additional flow along $P(t_s)$, a new flow f' with greater value and small cost is obtained and we can update the residual dynamic network for next iteration. The flow as required in the dynamic network is obtained until the algorithm can not find the time-varying unsaturated path.

In the process of solving Problem 1, for any feasible flow f, in order to send an additional flow and increase $Val(f)$, the residual capacity of each arc should be determined. Since the capacity function is time-varying, the residual capacity is changing over time as well. Then, in the residual dynamic network of each iteration, the unsaturated path $P(t_s)$ is a time dependent path. That is the delay and the additional flow along path $P(t_s)$ need to be computed in different time segments of $[0,T]$ because that the residual capacity of each arc in $P(t_s)$ may be 0 in some periods.

Now, according to the steps of the increasing $Val(f)$, in following sections III-A and III-B, we will give some concepts which are used in the algorithms. Meanwhile, the main steps of algorithms will be introduced and the main notations used are listed in Table I.

A. The initial flow, dynamic residual capacity network and the calculation of the increment of flow

Given a dynamic network $G=(V(G),E(G),W,C,T,v_s,v_d)$, in this section, we consider the initial step firstly. The flow $f(t_s)$ with value 0 is used as the initial flow. Then $\phi(f)=0$.

Furthermore, the initial state of dynamic residual capacity network $G'=(V(G),E(G'),W',C',T,v_s,v_d)$ can be defined as follows.

- Let $E(G')=E(G)\cup E(G)^-$, where $E(G)^-=\{(v_x,v_y):(v_y,v_x)\in E(G)\}$. To avoid confusion, each arc of $E(G)$ is called a *positive arc* and each arc of $E(G)^-$ is called a *reverse arc* (see Fig. 1).
- $W'=\{w'_{(v_x,v_y)}(t):(v_x,v_y)\in E(G')\}$, where the delay function $w'_{(v_x,v_y)}(t)$ of (v_x,v_y) for any $t\in[0,T]$ is defined by:

$$w'_{(v_x,v_y)}(t)=\begin{cases} w_{(v_x,v_y)}(t), & (v_x,v_y)\in E(G);\\ \infty, & (v_x,v_y)\in E(G)^-.\end{cases}$$

- $C'=\{c'_{(v_x,v_y)}(t):(v_x,v_y)\in E(G')\}$ is a set of residual capacity functions, where the residual capacity $c'_{(v_x,v_y)}(t)$ of (v_x,v_y) for any $t\in[0,T]$ is defined by:

$$c'_{(v_x,v_y)}(t)=\begin{cases} c_{(v_x,v_y)}(t), & (v_x,v_y)\in E(G);\\ 0, & (v_x,v_y)\in E(G)^-.\end{cases}$$

Choose a dynamic v_sv_d-path $P(t_s)$ in G'. Let the time that a flow in G' passing through v_x along $P(t_s)$ be t_x. Then, for any $(v_x,v_y)\in E(P(t_s))$, $t_y=t_x+w_{(v_x,v_y)}(t_x)$ and the *transmission time period* on (v_x,v_y) is $T_{(v_x,v_y)}=[t_x,t_y]$.

Noting that the arc capacity of the dynamic network changes with time and the maximum amount that the flow on (v_x,v_y) can increase at time t is $c'_{(v_x,v_y)}(t)$, in order to

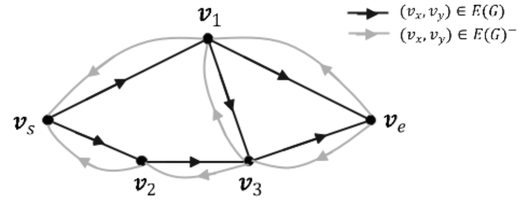


Fig. 1: $E(G')$

ensure that the value of flow on (v_x,v_y) cannot exceed the dynamic capacity in the time period $[t_x,t_y]$, it is necessary to obtain the function

$$c^*_{(v_x,v_y)}=\min_{t\in T_{(v_x,v_y)}} c'_{(v_x,v_y)}(t)$$

which is the minimum value of the residual capacity function in the time interval $[t_x,t_y]$. Then, for the dynamic v_sv_d-path $P(t_s)$ in G',

$$\delta(P(t_s))=\min_{(v_x,v_y)\in E(P(t_s))} c^*_{(v_x,v_y)}$$

is the maximum value that the flow f can be increased along $P(t_s)$. In addition, $P(t_s)$ is called an *f-unsaturated path*. Since the initial flow f is such that $Val(f)=0$, after increasing the flow value on each arc in $P(t_s)$, we obtain a new flow f' with greater value $Val(f')=0+\delta(P(t_s))$. Recall that the delay function is regarded as the cost function per unit flow. Hence, in order to reduce the cost of flow, we find the f-unsaturated path $P(t_s)$ with minimal delay $w_P(t_s)$ (i.e., the shortest dynamic path from v_s to v_d in G' with departure time t_s) and the cost of the flow f' is

$$\phi(f')=\phi(f)+\sum_{(v_x,v_y)\in E(P(t_s))} w_{(v_x,v_y)}(t_x)\cdot\delta(P(t_s))$$

In addition, it can be seen that we can repeat above processes of constructing dynamic residual capacity network and finding the shortest unsaturated path. Then the value of flow is increased by Algorithm 1.

Algorithm 1 Increase the value of flow

Input: an residual capacity network $G'=(V(G),E(G'),\ W',C',T,v_s,v_d)$, an initial flow f with departure time t_s
Output: a flow f' with greater value
find the f-unsaturated path $P(t_s)$ with minimal $w_P(t_s)$ in G' by using TDSP Algorithm.
if the $P(t_s)$ exists **then**
  for each $(v_x,v_y)\in E(P(t_s))$ **do**
    $c^*_{(v_x,v_y)}=\min_{t\in T_{(v_x,v_y)}} c'_{(v_x,v_y)}(t)$
  end for
  $\delta(P(t_s))=\min_{(v_x,v_y)\in E(P(t_s))} c^*_{(v_x,v_y)}$
  the flow value of each arc in $P(t_s)$ is increased by $\delta(P(t_s))$ and obtain new flow f';
  $Val(f')=Val(f)+\delta(P(t_s))$;
  return $P(t_s), f'$ and $Val(f)$;
end if
return $P(t_s)==NULL$ and $f'=f$;

TABLE I: Main notations

$Val(f)$	The flow value of f in the network
$w'_{(v_x,v_y)}(t)$	The delay function of the arc (v_x, v_y) in the residual network G'
$c'_{(v_x,v_y)}(t)$	The capacity function of the arc (v_x, v_y) in the residual network G'
$T_{(v_x,v_y)}$	The time period $[t_x, t_y]$ during which the flow passes through arc (v_x, v_y)
$c^*_{(v_x,v_y)}[t_1, t_2]$	The minimum value of the function $c'_{(v_x,v_y)}(t)$ in the time period $[t_1, t_2]$
$P(t_s)$	The f-unsaturated path with a departure time t_s
$w_P(t_s)$	The delay of path $P(t_s)$
$\delta(P(t_s)$	The maximum increase in flow f along $P(t_s)$
$\phi(f')$	The cost of transporting flow f' over the network G'
$E(P(t_s))$	The edge set of path $P(t_s)$
$f'_e(t)$	The flow f' on the arc e at time t
$f'^{-}_{(v_x,v_y)}(t)$	The flow f' entering arc (v_x, v_y) at time t

B. Update the dynamic residual capacity network

Based on the dynamic residual capacity network G' and f-unsaturated path $P(t_s)$, we further construct new residual capacity network $\hat{G}$ and obtain a flow $\hat{f}$ with greater value than f' by using Algorithm 1.

According to the definition of f', since the value of the flow on each arc e is $\delta(P(t_s))$, the residual capacity of e is reduced in the corresponding transmission time period. Then, in next step, there are two choices as follows.

- If the flow on e continues to increase, then the residual capacity needs to be calculated and seemed as the new capacity of e.
- If the flow on e will be reduced, then it is reduced by at most $f'_e(t)$ at time t. Thus $f'_e(t)$ can be used as the capacity of the reverse arc of e.

Therefore, we need to update the dynamic residual capacity network and obtain $\hat{G} = (V(G), E(G'), \hat{W}, \hat{C}, T, v_s, v_d)$. Since the flow on the arc outside of $P(t_s)$ is unchanged, in order to define $\hat{G}$, it is enough to distinguish the following two cases to update the dynamic delay function and capacity function of any arc (v_x, v_y) in $P(t_s)$.

Case 1. (v_x, v_y) is a positive arc.

Firstly, we update the residual capacity function $c'_{(v_x,v_y)}(t)$ and the delay function $w'_{(v_x,v_y)}(t)$.

According to the dynamic f-unsaturated path $P(t_s)$ with minimal delay which has been found in Section III-A, the flow on (v_x, v_y) increases by $\delta(P(t_s))$ in the time period $T_{(v_x,v_y)} = [t_x, t_y]$. We can observe that $\delta(P(t_s))$ is the maximum flow increased along path $P(t_s)$ and then f' is obtained. Now, the residual capacity of (v_x, v_y) is $c'_{(v_x,v_y)}(t) - \delta(P(t_s))$ in the time period $[t_x, t_y]$. Then, the residual capacity function is updated to

$$\hat{c}_{(v_x,v_y)}(t) = \begin{cases} c'_{(v_x,v_y)}(t) - \delta(P(t_s)), & t \in [t_x, t_y]; \\ c'_{(v_x,v_y)}(t), & \text{otherwise.} \end{cases} \quad (1)$$

Recall that $\hat{c}_{(v_x,v_y)}(t)$ is a time-varying continuous function. If $\hat{c}_{(v_x,v_y)}(t) = 0$ in some pairwise disjoint time segments $[a_1, b_1], [a_2, b_2] \ldots, [a_k, b_k]$ in $[t_x, t_y]$, then, these time segments are called *saturated time periods* and we can not continue to increase the flow on (v_x, v_y) in these time periods. For any $[a_i, b_i]$ where $i \in \{1, 2, \ldots, k\}$, let

$$t_i = \arg\min_{t \in [0,T]} \{a_i \leq t + w_{(v_x,v_y)}(t) \leq b_i\}. \quad (2)$$

Then we have following lemma.

Lemma 10. *Any two time periods $[t, t + w_{(v_x,v_y)}(t)]$ and $[a_i, b_i]$ are intersecting if and only if $t \in [t_i, b_i]$.*

Proof: For the necessity, by the FIFO condition and the choice of t_i, if $t < t_i$, then $t + w_{(v_x,v_y)}(t) < a_i$ and so $[t, t+w_{(v_x,v_y)}(t)]$ and $[a_i, b_i]$ are disjoint, a contradiction. Hence $t \geq t_i$. Since $t \leq b_i$ is clearly, we have that $t \in [t_i, b_i]$.

It remains to establish its sufficiency. Noting that $t \leq b_i$, then we prove that $t + w_{(v_x,v_y)}(t) \geq a_i$. Suppose that $t + w_{(v_x,v_y)}(t) < a_i$. By the FIFO condition, it can be seen that $t < t_i$. This is a contradiction. Therefore, the result holds. ∎

Let

$$\mathcal{T} = [t_1, b_1] \cup [t_2, b_2] \cup \ldots \cup [t_k, b_k]. \quad (3)$$

Recall that $[a_i, b_i]$ is a saturated time period for $i = 1, 2, \ldots, k$. By Lemma 10 and Eq. 3, we observe that it is enough to update the delay function to $\hat{w}_{(v_x,v_y)}(t) = \infty$ when $t \in \mathcal{T}$ and any shortest unsaturated path in $\hat{G}$ will not pass through the arc (v_x, v_y) in the saturated time periods.

Therefore, the delay function of the positive arc (v_x, v_y) of path P is updated to:

$$\hat{w}_{(v_x,v_y)}(t) = \begin{cases} \infty, & t \in \mathcal{T}; \\ w'_{(v_x,v_y)}(t), & \text{otherwise.} \end{cases} \quad (4)$$

Now, an example will be given to illustrate this process.

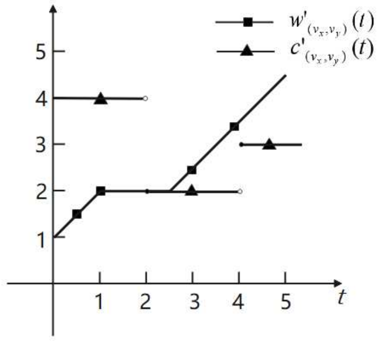
Example 1. Suppose that (v_x, v_y) is a positive arc in $P(t_s)$. According to the Section III-A, see Fig. 2(a), let

$$w'_{(v_x,v_y)}(t) = w_{(v_x,v_y)} = \begin{cases} t+1, & t \in [0, 1); \\ 2, & t \in [1, 2.5); \\ t - 0.5, & t \in [2.5, T]; \end{cases} \quad (5)$$

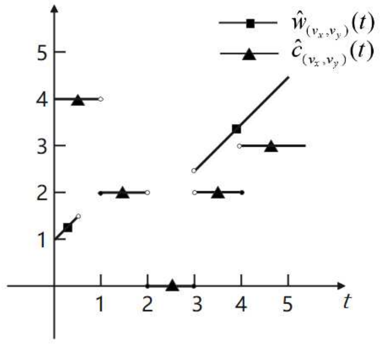
and

$$c'_{(v_x,v_y)}(t) = c_{(v_x,v_y)} = \begin{cases} 4, & t \in [0, 2); \\ 2, & t \in [2, 4); \\ 3, & t \in [4, T]. \end{cases} \quad (6)$$

Suppose that $\delta(P(t_s)) = 2$ and the transmission time period of (v_x, v_y) is $[t_x, t_y]$. Let $t_x = 1$, then $t_y = 1 + w_{(v_x,v_y)}(1) = 3$. According to the Eq. 1, see Fig. 2(b), the



(a) $c'_{(v_x,v_y)}(t)$, $w'_{(v_x,v_y)}(t)$



(b) $\hat{c}_{(v_x,v_y)}(t)$, $\hat{w}_{(v_x,v_y)}(t)$

Fig. 2: The update of the weight functions in the the residual graph

residual capacity function of the arc is updated to:

$$\hat{c}_{(v_x,v_y)}(t) = \begin{cases} 4, & t \in [0,1); \\ 2, & t \in [1,2); \\ 0, & t \in [2,3]; \\ 2, & t \in (3,4]; \\ 3, & t \in (4,T]. \end{cases}$$

Note that $c'_{(v_x,v_y)}(t) = 0$ for $t \in [2,3]$. Then $[a_1, b_1] = [2,3]$. By Eq. 2, we have that $t_1 = 0.5$. Therefore, $\mathcal{T} = [0.5, 3]$ and so

$$\hat{w}_{(v_x,v_y)}(t) = \begin{cases} t+1, & t \in [0, 0.5); \\ \infty, & t \in [0.5, 3]; \\ t - 0.5, & t \in (3,T]. \end{cases}$$

Secondly, we consider the function update of the reverse arc (v_y, v_x). Recall that the residual capacity of (v_y, v_x) is the value of flow which can be reduced on (v_x, v_y). Then the flow on (v_y, v_x) is called the *back-flow* which starts from v_y at time t_y and arrives v_x at time t_x. It implies that the transmission time period on (v_y, v_x) is $T_{(v_y,v_x)} = [t_x, t_y]$.

Since the flow value of (v_x, v_y) increases by $\delta(P(t_s))$ in the time period $[t_x, t_y]$, we have that the residual capacity of (v_y, v_x) increases by $\delta(P(t_s))$ as well. Therefore, let

$$\hat{c}_{(v_y,v_x)}(t) = \begin{cases} c'_{(v_y,v_x)}(t) + \delta(P(t_s)), & t \in [t_x, t_y]; \\ c'_{(v_y,v_x)}(t), & \text{otherwise.} \end{cases}$$

Because the residual capacity of (v_y, v_x) is increased, the f'-unsaturated path to be found in next step can pass through (v_y, v_x) in the time period $[t_x, t_y]$. Therefore, the delay function of this arc needs to be updated. As the back-flow is from v_y to v_x and the corresponding time period is from time t_y back to time t_x, that is $\hat{w}_{(v_y,v_x)}(t) = t_x - t_y$ when $t = t_y$.

Then the delay function of the reverse arc (v_y, v_x) is updated to:

$$\hat{w}_{(v_y,v_x)}(t) = \begin{cases} t_x - t_y, & t = t_y; \\ w'_{(v_y,v_x)}(t), & \text{otherwise.} \end{cases}$$

For example, in Example 1, as $\delta(P(t_s)) = 2$ and $[t_x, t_y] = [1, 3]$, we have that

$$\hat{c}_{(v_y,v_x)}(t) = \begin{cases} 2, & t \in [1,3]; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

and

$$\hat{w}_{(v_y,v_x)}(t) = \begin{cases} -2, & t = 3; \\ \infty, & \text{otherwise.} \end{cases} \tag{8}$$

Case 2. (v_x, v_y) is a reverse arc.

In this case, the flow on (v_x, v_y) is the back-flow and so $t_x > t_y$. Then the corresponding transmission time period along path $P(t_s)$ is $T_{(v_x,v_y)} = [t_y, t_x]$.

Firstly, we update the residual capacity function $c'_{(v_x,v_y)}(t)$ and the delay function $w'_{(v_x,v_y)}(t)$.

Similar to the Eq. 1 in Case 1, since the value of flow on (v_x, v_y) increases by $\delta(P(t_s))$, the residual capacity function is updated to:

$$\hat{c}_{(v_x,v_y)}(t) = \begin{cases} c'_{(v_x,v_y)}(t) - \delta(P(t_s)), & t \in [t_y, t_x]; \\ c'_{(v_x,v_y)}(t), & \text{otherwise.} \end{cases} \tag{9}$$

Since the residual capacity function of a reverse arc is obtained by increasing or decreasing the flow along the unsaturated paths, it can be seen that $\hat{c}_{(v_x,v_y)}(t)$ is a constant function in the time period $[t_y, t_x]$.

If $\hat{c}_{(v_x,v_y)}(t) > 0$, then this arc can continue to be used to increase back-flow and so the delay function does not change in the time period $[t_y, t_x]$.

If $\hat{c}_{(v_x,v_y)}(t) = 0$ with $t \in [t_y, t_x]$, then, in the residual capacity network $\hat{G}$, this arc can not in any shortest dynamic path $\hat{P}(t_s)$ which will be used as a unsaturated path to increase flow in next iteration. Therefore, the delay function of the reverse arc (v_x, v_y) in $P(t_s)$ is updated to:

$$\hat{w}_{(v_x,v_y)}(t) = \begin{cases} \infty, & t = t_y; \\ w'_{(v_x,v_y)}(t), & \text{otherwise.} \end{cases} \tag{10}$$

By the equations 7-10, we have following result.

Observation 11. *For each reverse arc (v_x, v_y), the time horizon $[0, T]$ can be divided into some time periods and the residual capacity function is a constant in each time period. In addition, the delay function is a finite constant only at a finite number of time points.*

Secondly, we consider the functions of (v_y, v_x) which is a positive arc.

As the back-flow on (v_x, v_y) increases by $\delta(P(t_s))$, we have that the residual capacity function of (v_y, v_x) is increased by $\delta(P(t_s))$ in the corresponding time period. Then, let

$$\hat{c}_{(v_y,v_x)}(t) = \begin{cases} c'_{(v_y,v_x)}(t) + \delta(P(t_s)), & t \in [t_y, t_x]; \\ c'_{(v_y,v_x)}(t), & \text{otherwise.} \end{cases}$$

By the above equation, it is easy to see that if there exists a time period in $[t_y, t_x]$ such that $c'_{(v_y,v_x)}(t) = 0$, then this time period is called a saturated time period. When $\hat{c}_{(v_y,v_x)}(t) >$

0, the positive arc (v_y, v_x) can be used in $\hat{G}$ to increase flow. Therefore, the delay function needs to be updated in the saturated time period.

Without loss of generality, suppose that $c'_{(vy,vx)}(t) = 0$ in the pairwise disjoint time periods $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$. For any $[a_i, b_i]$ where $i \in \{1, 2, \ldots, k\}$, let

$$t_i = \arg\min_{t\in[0,T]} \{a_i \le t + w_{(v_y,v_x)}(t) \le b_i\}.$$

Then, by Lemma 10, we have that any two time periods $[t, t + w_{(v_y,v_x)}(t)]$ and $[a_i, b_i]$ are disjoint if and only if $t \in [t_i, b_i]$. Hence, a time period $\mathcal{T}$ is obtained by Eq. 3 and we can observe that $f'^{-}_{(v_y,v_x)}(t)$ can be continue increased in next iteration when $t \in \mathcal{T}$. Therefore, the delay function of the positive arc (v_y, v_x) is updated to:

$$\hat{w}_{(v_y,v_x)}(t) = \begin{cases} w_{(v_y,v_x)}(t), & t \in \mathcal{T}; \\ w'_{(v_y,v_x)}(t), & \text{otherwise.} \end{cases}$$

According to the two cases as above, the sets $\hat{W} = \{\hat{w}_e(t) : e \in E(G')\}$ and $\hat{C} = \{\hat{c}_e(t) : e \in E(G')\}$ are obtained and dynamic residual capacity network is updated to $\hat{G} = (V(G), E(G'), \hat{W}, \hat{C}, T, v_s, v_d)$. Therefore, the Algorithm 2 of updating the dynamic residual capacity network is given. Then, similar to the $P(t_s)$ in G', if there exists a shortest dynamic path $\hat{P}(t_s)$ from v_s to v_d in $\hat{G}$ with departure time t_s, then $\hat{P}(t_s)$ is called a f'-unsaturated path. Furthermore, this path can be used to increase the flow f' and update the functions $\hat{c}_e(t)$ and $\hat{w}_e(t)$ for each $e \in E(G')$. Repeat this process, the dynamic feasible flow as required in Problem 1 is obtained until there is no unsaturated path.

Then an example related to the reverse arc is given as follows.

Example 2. Let (v_x, v_y) be a reverse arc. By Observation 11, we suppose that

$$\hat{c}_{(v_x,v_y)}(t) = \begin{cases} 0, & t \in [0, 1); \\ 4, & t \in [1, 3]; \\ 3, & t \in (3, 4]; \\ 0, & t \in (4, T]. \end{cases}$$

and

$$\hat{w}_{(v_x,v_y)}(t) = \begin{cases} -2, & t = 3; \\ -1, & t = 4; \\ \infty, & \text{otherwise.} \end{cases}$$

Assume that there exists an unsaturated path $\hat{P}(t_s)$ containing (v_x, v_y) in $\hat{G}$ and $T_{(v_x,v_y)} = [1, 3]$. If $\delta(\hat{P}(t_s)) = 4$, then $Val(f')$ increases by 4 in the time period $[1, 3]$. Eq. 9 implies that the residual capacity function is updated to

$$\hat{c}'_{(v_x,v_y)}(t) = \begin{cases} 0, & t \in [0, 3]; \\ 3, & t \in (3, 4]; \\ 0, & t \in (4, T]. \end{cases}$$

Moreover, by Eq. 10, the delay function is updated to

$$\hat{w}'_{(v_x,v_y)}(t) = \begin{cases} -1, & t = 4; \\ \infty, & \text{otherwise.} \end{cases}$$

In Example 2, we note that, after updating the residual network, the delay function $\hat{w}'_{(v_x,v_y)}(t)$ of the reverse arc $(v_x, v_y) \in E(P(t_s)) \cap E(G)^-$ may be the negative value.

Algorithm 2 Update the residual network

Input: an residual capacity network $G' = (V(G), E(G'), W', C', T, v_s, v_d)$, an unsaturated path $P(t_s)$;
Output: an updated residual capacity network;
for each $(v_x, v_y) \in E(P(t_s)) \cap E(G)$ **do**
  $c'_{(v_x,v_y)}(t) = c'_{(v_x,v_y)}(t) - \delta(P(t_s))$ when $t \in [t_x, t_y]$;
  $c'_{(v_y,v_x)}(t) = c'_{(v_y,v_x)}(t) + \delta(P(t_s))$ when $t \in [t_x, t_y]$;
  $w'_{(v_y,v_x)}(t) = t_x - t_y$ when $t = t_y$;
  if $c'_{(v_x,v_y)}(t) = 0$ in some time period of $[t_x, t_y]$ **then**
    Find the saturated time periods $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$;
    for each $i = 1, 2, \ldots, k$ **do**
      $t_i = \arg\min_{t\in[0,T]}\{a_i \le t + w_{(v_x,v_y)}(t) \le b_i\}$;
    end for
    $\mathcal{T} = [t_1, b_1] \cup [t_2, b_2] \cup \ldots \cup [t_k, b_k]$;
    $w'_{(v_x,v_y)}(t) = \infty$ when $t \in \mathcal{T}$;
  end if
end for
for each $(v_x, v_y) \in E(P(t_s)) \cap E(G)^-$ **do**
  $c'_{(v_x,v_y)}(t) = c'_{(v_x,v_y)}(t) - \delta(P(t_s)), t \in [t_y, t_x]$
  if $c'_{(v_x,v_y)}(t) = 0$ in $[t_y, t_x]$ **then**
    $w'_{(v_x,v_y)}(t) = \infty$ when $t = t_x$
  end if
  if $c'_{(v_y,v_x)}(t) = 0$ in some time periods of $[t_y, t_x]$ **then**
    Find the saturated time periods $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$;
    for each $i = 1, 2, \ldots, k$ **do**
      $t_i = \arg\min_{t\in[0,T]}\{a_i \le t + w_{(v_y,v_x)}(t) \le b_i\}$;
    end for
    $\mathcal{T} = [t_1, b_1] \cup [t_2, b_2] \cup \ldots \cup [t_k, b_k]$;
    $w'_{(v_y,v_x)}(t) = w_{(v_y,v_x)}$ when $t \in \mathcal{T}$;
  end if
  $c'_{(v_y,v_x)}(t) = c'_{(v_y,v_x)}(t) + \delta(P(t_s)), t \in [t_y, t_x]$
end for

Then we need to consider the dynamic cycle which is defined as follows.

Definition 12 (Dynamic feasible cycle). *For any dynamic path $P(t) = \langle u_1, u_2, \ldots, u_k\rangle$ in the dynamic residual capacity network G', suppose that the flow f' enters $P(t)$ from u_1 at time t and leaves the path from u_k. Then $P(t)$ is called a dynamic feasible cycle if it satisfies following conditions:*

(1) $u_1 = u_k$;
(2) $f'^{-}_{u_i}(t) = f'^{+}_{u_i}(t)$ *for any* $t \in [0, T]$.

In above definition, if the flow leaves the path from u_k at time t', then $t' = t + w'_P(t)$, where $w'_P(t)$ is the delay of $P(t)$ in G'. Therefore, the conditions (1) and (2) imply that $t = t'$ and so $t = t + w'_P(t)$. That is $w'_P(t) = 0$. Hence, we have following observation.

Observation 13. *For any dynamic feasible cycle $P(t)$ in the dynamic residual capacity network G', the total delay function of $P(t)$ in G' is 0.*

By Observation 13, the dynamic residual capacity network has no negative feasible cycle. In a static network without negative cycles, Floyd algorithm is known to be able to solve the shortest path problem with negative arcs. Thus, by Observation 13, in Line 3 of Algorithm 1, for any time-varying residual network, the shortest unsaturated path $P(t_s)$ can be obtained by the shortest path algorithm TDSP [18] which uses the TD-Floyd Algorithm.

C. Dynamic minimum cost maximum flow algorithm

The main problem of this paper is to calculate the minimum cost and maximum flow in continuous-time dynamic networks. Therefore, similar to the Successive Shortest Path Algorithm [4], for any f which is not the maximum flow, Algorithm 1 is given to calculate the shortest unsaturated path $P(t_s)$ and increase the $Val(f)$ along the path. Then the f is updated to a new flow f'. In Algorithm 2, by the value of f' on each arc in $P(t_s)$, the residual network is updated. Thus, the required maximum flow will be obtained in Algorithm 3 by calling Algorithms 1 and 2 iteratively and the steps are as follows:

- Firstly, in the initial residual network G' obtained in Section A, according to the Algorithm 1, a shortest unsaturated path $P(t_s)$ with any departure time $t_s \in [0, T]$ is calculated from the source v_s to the sink v_d. In addition, the maximum increasable flow value $\delta(P(t_s))$ and the corresponding dynamic network flow f' are obtained;
- Compute the cost $\phi(f')$ by using the $\delta(P(t_s))$ and f';
- Update the capacity function $c'_{(v_x,v_y)}(t)$ and delay function $w'_{(v_x,v_y)}(t)$ in the residual network G';
- Return to Step 1. If the unsaturated path exists, then we can repeat this process. Otherwise, the algorithm output maximum flow f' and minimum cost $\phi(f')$.

In this way, we get the maximum network flow f' with the minimum cost $\phi(f')$ for any departure time $t_s \in [0, T]$.

Algorithm 3 Find the minimum cost maximum flow

1: **Input**: an residual capacity network $G'(V(G), E(G'), W', C', T, v_s, v_d)$, an initial flow f with departure time t_s and value 0;
2: **Output**: the minimum cost maximum flow f';
3: run Algorithm 1 to increase the value of initial flow;
4: **if** $P(t_s)$ is not NULL **then**
5:
$$\phi(f') = \phi(f') + \sum_{(v_x,v_y)\in E(P(t_s))} w_{(v_x,v_y)}(t_x) \cdot \delta(P(t_s))$$
6: run Algorithm 2 to update the residual network;
7: return step 3;
8: **else**
9: return f',$\phi(f')$;
10: **end if**

IV. Correctness and complexity

In this section, the correctness of the Algorithm 3 will be proved. First, we consider the following theorem.

Theorem 14. *In Algorithm 3, the flow f' returned by Line 9 is a maximum flow if and only if the updated residual network G' in Line 6 has no unsaturated path $P(t_s)$.*

Proof: Suppose first that f' is a maximum flow. If an unsaturated path $P(t_s)$ can be found in the residual network G', then we can further increase the flow by Algorithm 1, a contradiction. Thus, the necessity holds.

Suppose next that the unsaturated path $P(t_s)$ is not found in the residual network G'. Let $K' = (S, \bar{S})$ be a dynamic cut when the departure time is t_s in G' such that $(v_x, t_x) \in S$ if and only if there is an unsaturated path in G' departing at v_s at time t_s and arriving at v_x at time t_x.

For any $((v_x, t_x), (v_y, t_y)) \in K'$, by the definition of the dynamic cut, we have that $(v_x, v_y) \in E(G)$ or $(v_x, v_y) \in E(G)^-$.

Assume that $(v_x, v_y) \in E(G)$. If the residual capacity function $c'_{(v_x,v_y)}(t) > 0$ for each $t \in [t_x, t_y]$, then, there is an unsaturated path from v_s to v_y passing through (v_x, x_y). This is a contradiction to $(v_y, t_y) \in \bar{S}$. Then $c'_{(v_x,v_y)}(t) = 0$ in some time period of $[t_x, t_y]$. That is $f'_{(v_x,v_y)}(t) = c^*_{(v_x,v_y)}[t_x, t_y]$.

Suppose that $(v_x, v_y) \in E(G)^-$. If $f'_{(v_y,v_x)}(t) > 0$ for each $t \in [t_x, t_y]$. According to the Line 29 in Algorithm 2, it can be seen that $c'_{(v_x,v_y)}(t)$ can continue to be increased in the time period $[t_x, t_y]$. Therefore, there exists an unsaturated path from v_s to v_y containing (v_x, x_y), a contradiction. Hence, $f'_{(v_y,v_x)}(t) = 0$ in the time period $[t_x, t_y]$.

Let $K = K' \cap \{((v_x, t_x), (v_y, t_y)) : (v_x, v_y) \in E(G)\}$. Then K is a dynamic cut in G. It can be seen that $Val(f') = \sum_{((v_x,t_x),(v_y,t_y))\in K} c^*_{(v_x,v_y)}[t_x, t_y] = Cap(K)$. By Corollary 9, f' is a maximum flow. ■

Theorem 15. *Let f' and G' be the updated flow and residual network in some iteration of Algorithm 3, respectively. Then f' is a minimum cost flow with value $Val(f')$.*

Proof: Suppose that f' is improved from the flow f by increasing the flow of each arc in the shortest f-unsaturated path $P(t_s)$. Each arc in G' is assigned with a time-varying delay function $w'_e(t)$. By contradiction. Assume that there exists another dynamic feasible flow $f''(t)$ in G' such that $Val(f'') = Val(f')$ and $\phi(f'') < \phi(f')$. Let $g_e(t) = f''_e(t) - f_e(t)$ for $t \in [0, T]$. Then g is a dynamic feasible flow of G'.

Since $P(t_s)$ is the path in G' with minimal delay $w'_P(t_s)$, we have that the delay of the flow g from v_s to v_d is not less than $w'_P(t_s)$. As $\phi(f'') < \phi(f')$ and $Val(f' - f) = Val(f'' - f) = Val(g)$, g has some dynamic feasible cycles $C_1, C_2, \cdots, C_l$ in the residual network G' such that $w'_{C_i}(t) < 0$ for some $i \in \{1, 2, \ldots, l\}$. This is a contradiction to Observation 13. Then the result holds. ■

By the proof of above theorem, the flow f' with departure time t_s returned by Algorithm 3 is a flow in G. Combining this with Theorems 14 and 15, f' is the minimum cost and maximum flow. In Example 1, note that the time interval $[0, T]$ is divided into some time periods and the weight functions of each arc will be different in any two time periods. Therefore, the unsaturated path passing through e should be considered in each time period. The number of the time periods is denoted by $\alpha(T)$. Then we can compute the time complexity of Algorithm 3 as follows.

Theorem 16. *The time complexity of Algorithm 3 is*

$O(mc_{max}\alpha(T)(m\alpha^2(T) + \log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T)))$, *where* c_{max} *is the upper bound of* $c_e(t)$ *for each* $e \in E(G)$ *and* $t \in [0, T]$.

Proof: Since Algorithms 1 and 2 are two sub-procedures of Algorithm 3, we consider the two algorithms firstly.

In Algorithm 1, the algorithm TDSP in line 3 needs $O(\log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T))$ time. Note that $P(t_s)$ has at most m arcs and $\alpha(T)$ time periods. Then the **for** loop in line 5 needs $O(m\alpha(T))$ time and so the complexity of Algorithm 1 is $O(m\alpha(T) + \log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T))$.

In Algorithm 2, for each arc (v_x, v_y) in $E(P(t_s)) \cap E(G)$, since there are $O(\alpha(T))$ time periods in each time-varying weight function of (v_x, v_y), lines 4-6 need $O(\alpha(T))$ time. Note that the number of saturated time periods in line 8 is $O(\alpha(T))$. For each saturated time period $[a_i, b_i]$, if there exists a time t such that $t + w'_{(v_x,v_y)}(t) \in [a_i, b_i]$, then the flow which will be increased on (v_x, v_y) can not start at node v_x at time t. As there are $O(\alpha(T))$ time periods of t should be considered in line 10, the time complexity of the **if** loop in lines 7-14 is $O(\alpha(T)^2)$. Therefore, the **for** loop in line 3 needs $O(\alpha(T)^2)$ time. Similarly, the time complexity of the **for** loop in line 16 is $O(\alpha(T)^2)$ as well. Note that $P(t_s)$ has at most m arcs. Therefore, Algorithm 2 needs $O(m\alpha(T)^2)$ time.

The main process of Algorithm 3 is to continuously increase the value of dynamic flow of each time period in the residual capacity network G'. Since the capacity of each arc at any time is an integer, the value of the flow with departure time t_s can be increased by at least 1 in each iteration by a shortest unsaturated path. By Corollary 9, the value of the flow is at most $|N(v_s)|c_{max} = O(mc_{max})$, where $N(v_s) = \{v_x : (v_s, v_x) \in E(G')\}$. There are $\alpha(T)$ time periods of t_s in T. Then the number of the iterations is at most $mc_{max}\alpha(T)$ and the total running time is $O(mc_{max}\alpha(T)(m\alpha(T)^2 + \log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T)))$. ■

V. Conclusion

In this paper, we studied the minimum cost maximum flow problem for any departure time $t_s \in [0, T]$ in the continuous-time dynamic capacity network $G(V(G), E(G), W, C, T, v_s, v_d)$ under the FIFO condition. We proposed a minimum cost maximum dynamic flow algorithm based on the shortest dynamic path algorithm and the update algorithm of the residual dynamic network to find the optimal flow with time complexity $O(mc_{max}\alpha(T)(m\alpha(T)^2 + \log_2^2 \kappa_f \cdot n \cdot \log_2^2 \alpha(T)))$. According to the time variability of the delay function and capacity function, we mainly considered the process of updating the dynamic residual network. Since the flow will not wait at each node, the most important is to deal with the continuity of the transmission process when we update the time-varying capacity function and delay function of each positive arc and reverse arc in the dynamic residual network. By the scheme of updating, some problems related to network flow can also be considered similarly in the dynamic networks.

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