# Novel Hesitant Triangular Fuzzy Portfolio with Parametric Entropy Based on Credibility Theory

Xue Deng, Jinyao Zhao

*Abstract*—**It is difficult for portfolio selections to adopt an appropriate way to describe information effectively. On the one hand, hesitant triangular fuzzy sets (HTFSs), composed of a limited number of membership degrees with the form of triangular fuzzy numbers, can transform uncertain variables into definite values more comprehensively than fuzzy sets with only one membership degree. On the other hand, we firstly parametrize the fuzzy entropy for triangular fuzzy numbers based on credibility theory. The rigorous mathematical derivation of the proposed entropy formula is proved in this paper, which is also extended to HTFS. Next, a novel bi-objective model is constructed from the perspectives of return maximization and risk minimization, which adopts credibility expectation and parametric entropy to describe returns and risks, respectively. Furthermore, the model solving process is designed for risk investors with different risk preferences. Finally, a case study is presented to inspect the feasibility of our proposed model. In the analysis of results, the reasons why optimal portfolio changes with investors' risk preferences are explained. We conclude that the parameter of our defined entropy has enriched the expression of preference information. Thus, the proposed model can be reasonably used in practical qualitative risk investment.**

*Index Terms*—**Portfolio selection, Hesitant triangular fuzzy sets, Parametric entropy, Credibility theory**

## I. Introduction

With the increasing complexity of practical decision-making problems and the uncertainty of decision makers' subjective ideas, the traditional decision-making method using accurate numbers can no longer be applied to all situations. Fuzzy theory proposed by Zadeh [1] was perceived as a major breakthrough, which can be used as a powerful quantitative tool in a fuzzy environment. The membership degree, as a concept describing the degree of an element belonging to a set, has various forms in different problem descriptions. The membership degrees of the elements in fuzzy sets are respectively designated as finite numbers in $[0,1]$, interval value, triangular fuzzy numbers and trapezoidal fuzzy numbers to describe investors' preferences from multiple angles. Intuitionistic fuzzy sets [2] contained both membership degree and non-membership degree, while hesitant fuzzy sets [3] described information with multiple membership degrees simultaneously. Furthermore, hesitant fuzzy sets were generalized to interval-valued hesitant fuzzy sets [4], dual hesitant fuzzy sets [5], and hesitant triangular fuzzy sets [6]. Fuzzy sets were extended to three-dimensions space by Type-2 fuzzy sets [7].

As experts combine fuzzy theory with Markowitz's portfolio theory [8], the appropriate form of decision processing instead of probability theory has become a research hotspot. After Katagiri and Ishii's pioneering work [9], scholars immersed themselves in the research of the fuzzy portfolio selection problem. Zhang et al. [10] used LR-power fuzzy number to describe the rate of return and proposed a fuzzy mean–semi-variance portfolio model. Pahade et al. [11] extended Markowitz's mean-variance portfolio model with credibility skewness of trapezoidal fuzzy variables. De et al. [12] presented a fuzzy programming approach considering risk, return and liquidity. Mansour and Cherif [13] took into account tradeoffs between investors' preferences regarding several incommensurable objectives in an imprecise environment.

In the research of hesitant fuzzy sets, risk measures such as distance measure [14], correlation coefficient [15], partial moments [16] and entropy [17] took up a large proportion. Beyond that, Deng and Li [18] defined hesitant semi-variance. With the advent of validated axiomatic definitions and concrete formulas, these risk measures were widely used in medical diagnosis [19], machine learning [20], hierarchical clustering [21], and most broadly, decision making [22]. Liu and Peng [23] proposed a new method for measuring fuzzy phenomena, called credibility theory. Models with credibility mean-variance [24]-[25], mean-absolute deviation [26] were classic. Gupta [27] researched multi-period multi-objective portfolio models using conditional value-at-risk in a credibility environment.

In this paper, a parametric entropy formula for hesitant triangular fuzzy sets is proposed inspired by credibility theory. Based on the guideline of risks measured by entropy, we construct a portfolio selection model with parametric hesitant triangular fuzzy entropy, which adopts credibility expectation to measure returns. Parameters are introduced to reduce information loss as much as possible and reflect investors' decision preferences more comprehensively. We present the specific portfolio selection processes in detail for investors with different risk preferences and demonstrate the model solving processes with a numerical example. The results obtained are reasonably analyzed and explained, which reveal the validity of our proposed model.

We organize this paper as follows. In Section II, we review the definitions of triangular fuzzy numbers, credibility theory and hesitant fuzzy sets. In Section III, a parametric entropy formula is presented and its

Manuscript received May 30, 2022; revised November 1, 2022. This research was supported by the National Social Science Foundation of China No. 21BTJ069, Teaching Reform Project of University Mathematics Teaching Research and Development Center of Colleges and Universities No. X2lxc9223049, Research Topic of Higher Science Education of Science Education Professional Committee of Chinese Association of Higher Education No. X2lxc9223047, Teaching Reform Projects of South China University of Technology No. X2lxc9217010, X2lxc 9213140.

Xue Deng is a professor of School of Mathematics, South China University of Technology, Guangzhou 510640, China (e-mail: dxue@scut.edu.cn).

Jinyao Zhao is a postgraduate student of School of Mathematics, South China University of Technology, Guangzhou 510640, China (corresponding author to provide e-mail: 675060348@qq.com).

mathematical derivation is deduced in detail. Parametric entropy plays an important role in the model proposed in Section IV, then the model solving process is described. In Section V, the efficiency of the proposed model is confirmed by solving a numerical example and discussing the results. Finally, we conclude the paper in Section VI.

## II. Preliminaries

In this section, we will review some concepts of the hesitant fuzzy sets and some extensions of fuzzy sets. In order to make these definitions more comprehensible, we provide some examples.

### *A. Triangular Fuzzy Numbers*

**Definition 1** [1] A triangular fuzzy number $a$ can be represented by a triplet $\left(\gamma^L,\gamma^M,\gamma^R\right)$, while $\gamma^L$ is the lower bound and $\gamma^R$ is the upper bound. The membership degree function $\mu_a(x)$ is defined as

$$\mu_a(x)=\begin{cases}0 , & x\le\gamma^L\\ \dfrac{x-\gamma^L}{\gamma^M-\gamma^L}, & \gamma^L\le x\le\gamma^M\\ \dfrac{\gamma^R-x}{\gamma^R-\gamma^M}, & \gamma^M\le x\le\gamma^R\\ 0 , & x\ge\gamma^R\end{cases} \tag{1}$$

### *B. Credibility Theory*

**Definition 2** [23] Let $\xi$ be a fuzzy variable with membership degree function $\mu(x)$, $r$ is a real number. Then the possibility measure of fuzzy event $\xi\ge r$ is

$$Pos\{\xi\ge r\}=\sup_{x\ge r}\mu(x). \tag{2}$$

The necessity measure is

$$Nec\{\xi\ge r\}=1-Pos\{\xi<r\}=1-\sup_{x<r}\mu(x). \tag{3}$$

Liu [23] defined the credibility measure as the average of possibility measure and necessity measure:

$$\begin{aligned}Cr\{\xi\ge r\}&=\frac{1}{2}\left(Pos\{\xi\ge r\}+Nec\{\xi\ge r\}\right)\\&=\frac{1}{2}\left(\sup_{x\ge r}\mu(x)+1-\sup_{x<r}\mu(x)\right).\end{aligned} \tag{4}$$

Based on the credibility measure, expectation and entropy of fuzzy variables were proposed.

**Definition 3** [23] The credibility expectation of fuzzy variable $\xi$ is

$$E[\xi]=\int_0^{+\infty}Cr\{\xi\ge r\}dr-\int_{-\infty}^0 Cr\{\xi\le r\}dr. \tag{5}$$

Assume that at least one of the above two integrals is finite.

**Definition 4** [23]The entropy of a continuous fuzzy variable $\xi$ is

$$H[\xi]=\int_{-\infty}^{\infty}S\left(Cr\{\xi=r\}\right)dr. \tag{6}$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$ is a continuously differentiable function, and is monotonically increasing over $[0,0.5]$, monotonically decreasing over $[0.5,1]$.

After a simple calculation, it is clear that for triangular fuzzy number $a=\left(\gamma^L,\gamma^M,\gamma^R\right)$, the credibility expectation $E\left[\overline{\xi}\right]$ and entropy $H\left[\overline{\xi}\right]$ can be expressed as

$$E\left[\overline{\xi}\right]=\frac{\left(\gamma^L+2\gamma^M+\gamma^R\right)}{4}, \tag{7}$$

$$H\left[\overline{\xi}\right]=\frac{\left(\gamma^R-\gamma^L\right)}{2}. \tag{8}$$

Entropy is the measure of fuzzy degree. Formula (8) enlightens us that triangular fuzzy numbers' fuzzy degree is entirely determined by the distance between the upper bound $\gamma^R$ and the lower bound $\gamma^L$.

### *C. Hesitant Fuzzy Set and Hesitant Triangular Fuzzy Set*

**Definition 5** [3] Let $X$ be a fixed set, a hesitant fuzzy set (HFS) $A$ on $X$ is described as: $A=\left\{\left\langle x,h_A(x)\right\rangle\middle| x\in X\right\}$, where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x\in X$ in the HFS $A$. $h_A(x)$ is called a hesitant fuzzy element (HFE).

When the membership degrees of HFSs turned into triangular fuzzy numbers, hesitant triangular fuzzy sets were proposed.

**Definition 6** [28] Let $X$ be a fixed set, a hesitant triangular fuzzy set (HTFS) on $X$ is expressed by $E=\left\{\left\langle x,h_E(x)\right\rangle\middle| x\in X\right\}$. $h_E(x)$ is a set of some possible triangular fuzzy values in $[0,1]$, representing the possible membership degrees of the element $x\in X$ in the HTFS $E$. For convenience, we call $h_E(x)=\left(\gamma^L,\gamma^M,\gamma^R\right)$ a hesitant triangular fuzzy element (HTFE).

For HTFE $h_E(x)=\left(\gamma^L,\gamma^M,\gamma^R\right)$, the larger the value of $\gamma^R-\gamma^L$, the fuzzier the element.

**Example 1** Let $X=\{x_1,x_2,x_3\}$ be the universe of discourses. Assume that $h_A(x_1)=\{(0.1,0.2,0.4),(0.6,0.7,0.8)\}$, $h_A(x_2)=\{(0.05,0.1,0.3),(0.3,0.5,0.6),(0.85,0.9,0.98)\}$, $h_A(x_3)=\{(0.1,0.2,0.5)\}$ are HTFEs on $x_i\in X(i=1,2,3)$ to set $A$. Then

$$\begin{aligned}A=\{&\langle x_1,\{(0.1,0.2,0.4),(0.6,0.7,0.8)\}\rangle,\\&\langle x_2,\{(0.05,0.1,0.3),(0.3,0.5,0.6),(0.85,0.9,0.98)\}\rangle,\\&\langle x_3,\{(0.1,0.2,0.5)\}\rangle\}\end{aligned}$$

is a HTFS.

**Definition 7** [28] Let $a=\left(\gamma^L,\gamma^M,\gamma^R\right)$, $a_1=\left(\gamma_1^L,\gamma_1^M,\gamma_1^R\right)$, $a_2=\left(\gamma_2^L,\gamma_2^M,\gamma_2^R\right)$ be HTFEs, several operations on HTFEs can be represented as follows in (9.1)-(9.4).

**Remark:** According to Definition 7, we can obtain Formula (10), which will be used frequently.

(i) $a^{\lambda}=\bigcup_{\gamma\in a}\left\{\left(\left(\gamma^{L}\right)^{\lambda},\left(\gamma^{M}\right)^{\lambda},\left(\gamma^{R}\right)^{\lambda}\right)\right\}$; (9.1)

(ii) $\lambda a=\bigcup_{\gamma\in a}\left\{\left(1-\left(1-\gamma^{L}\right)^{\lambda},1-\left(1-\gamma^{M}\right)^{\lambda},1-\left(1-\gamma^{R}\right)^{\lambda}\right)\right\}$; (9.2)

(iii) $a_1\oplus a_2=\bigcup_{\gamma_1\in a_1,\gamma_2\in a_2}\left\{\left(\gamma_1{}^{L}+\gamma_2{}^{L}-\gamma_1{}^{L}\gamma_2{}^{L},\gamma_1{}^{M}+\gamma_2{}^{M}-\gamma_1{}^{M}\gamma_2{}^{M},\gamma_1{}^{R}+\gamma_2{}^{R}-\gamma_1{}^{R}\gamma_2{}^{R}\right)\right\}$; (9.3)

(iv) $a_1\otimes a_2=\bigcup_{\gamma_1\in a_1,\gamma_2\in a_2}\left\{\left(\gamma_1{}^{L}\gamma_2{}^{L},\gamma_1{}^{M}\gamma_2{}^{M},\gamma_1{}^{R}\gamma_2{}^{R}\right)\right\}$. (9.4)

$$\oplus_{i=1}^{n}\omega_i\bar{a}_i=\bigcup_{\gamma_i\in\bar{a}_1,\gamma_2\in\bar{a}_2,\dots,\gamma_n\in\bar{a}_n}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i{}^{L}\right)^{\omega_i},1-\prod_{i=1}^{n}\left(1-\gamma_i{}^{M}\right)^{\omega_i},1-\prod_{i=1}^{n}\left(1-\gamma_i{}^{R}\right)^{\omega_i}\right)\right\}. \quad (10)$$

$$\begin{aligned}\oplus_{i=1}^{3}\omega_i h_i&=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\gamma_3\in h_3}\left\{\left(1-\prod_{i=1}^{3}\left(1-\gamma_i{}^{L}\right)^{\omega_i},1-\prod_{i=1}^{3}\left(1-\gamma_i{}^{M}\right)^{\omega_i},1-\prod_{i=1}^{3}\left(1-\gamma_i{}^{R}\right)^{\omega_i}\right)\right\}\\
&=\{\left(1-0.9^{\omega_1}\times0.95^{\omega_2}\times0.9^{\omega_3},1-0.8^{\omega_1}\times0.9^{\omega_2}\times0.8^{\omega_3},1-0.6^{\omega_1}\times0.7^{\omega_2}\times0.5^{\omega_3}\right),\\
&\left(1-0.9^{\omega_1}\times0.7^{\omega_2}\times0.9^{\omega_3},1-0.8^{\omega_1}\times0.5^{\omega_2}\times0.8^{\omega_3},1-0.6^{\omega_1}\times0.4^{\omega_2}\times0.5^{\omega_3}\right),\\
&\left(1-0.9^{\omega_1}\times0.15^{\omega_2}\times0.9^{\omega_3},1-0.8^{\omega_1}\times0.1^{\omega_2}\times0.8^{\omega_3},1-0.6^{\omega_1}\times0.02^{\omega_2}\times0.5^{\omega_3}\right),\\
&\left(1-0.4^{\omega_1}\times0.95^{\omega_2}\times0.9^{\omega_3},1-0.3^{\omega_1}\times0.9^{\omega_2}\times0.8^{\omega_3},1-0.2^{\omega_1}\times0.7^{\omega_2}\times0.5^{\omega_3}\right),\\
&\left(1-0.4^{\omega_1}\times0.7^{\omega_2}\times0.9^{\omega_3},1-0.3^{\omega_1}\times0.5^{\omega_2}\times0.8^{\omega_3},1-0.2^{\omega_1}\times0.4^{\omega_2}\times0.5^{\omega_3}\right),\\
&\left(1-0.4^{\omega_1}\times0.15^{\omega_2}\times0.9^{\omega_3},1-0.3^{\omega_1}\times0.1^{\omega_2}\times0.8^{\omega_3},1-0.2^{\omega_1}\times0.02^{\omega_2}\times0.5^{\omega_3}\right)\}.\end{aligned} \quad (11)$$

**Example 2** For three HTFEs $h_A(x_1)$, $h_A(x_2)$ and $h_A(x_3)$ in Example 1, based on Formula (10), we get the expression for the weighted sum (11).

## III. Parametric Entropy of Htfs and Its Proof

In this section, a new entropy measure of HTFS with parameter is defined based on credibility theory. We give the concrete mathematical expression of parametric entropy and perform a rigorous derivation of the mathematical formula.

**Definition 8** Let $X=\{x_1,x_2...,x_n\}$ be a non-empty finite set and $T=\{\langle x_i,t_T(x_i)\rangle\mid x_i\in X\}$ be a HTFS on $X$ with $t_T(x_i)=\left\{\left(\gamma_i^{L_1},\gamma_i^{M_1},\gamma_i^{R_1}\right),...,\left(\gamma_i^{L_{l_i}},\gamma_i^{M_{l_i}},\gamma_i^{R_{l_i}}\right)\right\}$, where $l_i$ is the number of elements in $t_T(x_i)$. Then $H_\alpha[\xi]$ is the parametric fuzzy entropy of HTFS $A$:

$$H_\alpha[\xi]=\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{1}{l_i}\sum_{\lambda=1}^{l_i}S\left(Cr\{\xi_\lambda=r\}\right)dr \quad (12)$$

Where $S(m)=\frac{1}{1-\alpha}\left(m^{\alpha}\ln m^{\alpha}+(1-m)^{\alpha}\ln(1-m)^{\alpha}\right)$, $\alpha>1$. Given the function $S(m)$ is monotonically increasing over $[0,0.5]$, monotonically decreasing over $[0.5,1]$, fuzzy entropy is maximal if and only if $Cr\{\xi=r\}=\frac{1}{2}$.

**Theorem 1** Let $X=\{x_1,x_2...,x_n\}$ be a non-empty finite set and $T=\{\langle x_i,t_T(x_i)\rangle\mid x_i\in X\}$ be a HTFS on $X$ with $t_T(x_i)=\left\{\left(\gamma_i^{L_1},\gamma_i^{M_1},\gamma_i^{R_1}\right),...,\left(\gamma_i^{L_{l_i}},\gamma_i^{M_{l_i}},\gamma_i^{R_{l_i}}\right)\right\}$, and $l_i$ is the number of elements in $t_T(x_i)$. Then $H_\alpha[\xi]$ is the parametric fuzzy entropy of HTFS T with $\alpha>1$:

$$H_\alpha[T]=-\frac{1}{n}\sum_{i=1}^{n}\frac{4}{(\alpha+1)^2}\frac{\alpha}{1-\alpha}\frac{1}{l_i}\sum_{\lambda=1}^{l_i}\left(\gamma_i^{R_\lambda}-\gamma_i^{L_\lambda}\right). \quad (13)$$

Formula (13) illustrates that parametric entropy of HTFS is determined by the parameter's value and the HTFEs' distance between upper bound and lower bound, which is consistent with Formula (8).

**Proof:** Before we calculate the parametric fuzzy entropy of triangular fuzzy number $t=\left(\gamma^{L},\gamma^{M},\gamma^{R}\right)$ in Formula (14)-(16), we state that:

let $z=\frac{1}{2}\frac{r-\gamma^{L}}{\gamma^{M}-\gamma^{L}}$, then $dz=\frac{1}{2}\frac{1}{\gamma^{M}-\gamma^{L}}dr$.

Likewise, let $s=\frac{1}{2}\frac{\gamma^{M}-r}{\gamma^{M}-\gamma^{L}}$, then $ds=-\frac{1}{2}\frac{1}{\gamma^{M}-\gamma^{L}}dr$.

Constructed in the same way, and we can get

$$P=Q=-2\frac{\alpha}{(\alpha+1)^2}\left(\gamma^{R}-\gamma^{M}\right). \quad (17)$$

To sum up, the parametric fuzzy entropy of triangular fuzzy number $t=\left(\gamma^{L},\gamma^{M},\gamma^{R}\right)$ is

$$H_\alpha[t]=-\frac{4}{(\alpha+1)^2}\frac{\alpha}{1-\alpha}\left(\gamma^{R}-\gamma^{L}\right) \quad (18)$$

Simple fuzzy element's entropy Formula (18) should be extended to fuzzy sets', which are widely used in application. Then, the sum of HTFEs' entropy is averaged, as shown in Formula (19).

$$H_\alpha[T]=-\frac{1}{n}\sum_{i=1}^{n}\frac{4}{(\alpha+1)^2}\frac{\alpha}{1-\alpha}\frac{1}{l_i}\sum_{\lambda=1}^{l_i}\left(\gamma_i^{R_\lambda}-\gamma_i^{L_\lambda}\right) \quad (19)$$

It is easy to know that the parametric entropy $H_\alpha[T]$ decreases monotonically with the increase of parameter $\alpha$. Furthermore, as a measure of fuzzy degree, the smaller entropy is, of course, the better. In terms of decision makers' information expression facing practical application, a larger value assigned to parameter $\alpha$ means the stock is more trusted.

$$
\begin{aligned}
H_\alpha[t] &= \int_{\gamma^L}^{\gamma^M} S\left(Cr\{t=r\}\right)dr + \int_{\gamma^M}^{\gamma^R} S\left(Cr\{t=r\}\right)dr \\
&= \left\{\int_{\gamma^L}^{\gamma^M} S\left(\frac{1}{2}\frac{r-\gamma^L}{\gamma^M-\gamma^L}\right)dr + \int_{\gamma^M}^{\gamma^R} S\left(\frac{1}{2}\frac{\gamma^R-r}{\gamma^R-\gamma^M}\right)dr\right\} \\
&= \frac{1}{(1-\alpha)}\left\{\int_{\gamma^L}^{\gamma^M}\left(\frac{1}{2}\frac{r-\gamma^L}{\gamma^M-\gamma^L}\right)^\alpha \ln\left(\frac{1}{2}\frac{r-\gamma^L}{\gamma^M-\gamma^L}\right)^\alpha dr + \left(\frac{1}{2}\frac{\gamma^M-r}{\gamma^M-\gamma^L}\right)^\alpha \ln\left(\frac{1}{2}\frac{\gamma^M-r}{\gamma^M-\gamma^L}\right)^\alpha dr\right. \\
&\quad \left. + \int_{\gamma^M}^{\gamma^R}\left(\frac{1}{2}\frac{\gamma^R-r}{\gamma^R-\gamma^M}\right)^\alpha \ln\left(\frac{1}{2}\frac{\gamma^R-r}{\gamma^R-\gamma^M}\right)^\alpha dr + \left(\frac{1}{2}\frac{r-\gamma^M}{\gamma^R-\gamma^M}\right)^\alpha \ln\left(\frac{1}{2}\frac{r-\gamma^M}{\gamma^R-\gamma^M}\right)^\alpha dr\right\} \\
&= \frac{1}{(1-\alpha)}\left(M+N+P+Q\right).
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
M &= 2(\gamma^M-\gamma^L)\int_0^1 z^\alpha \ln z^\alpha dz = 2(\gamma^M-\gamma^L)\left(\ln z^\alpha \bullet \frac{z^{\alpha+1}}{\alpha+1}\Big|_0^1 - \int_0^1 \frac{\alpha}{\alpha+1} z^\alpha dz\right) \\
&= 2(\gamma^M-\gamma^L)\left(\ln z^\alpha \bullet \frac{z^{\alpha+1}}{\alpha+1}\Big|_0^1 - \frac{\alpha}{(\alpha+1)^2} z^{\alpha+1}\Big|_0^1\right) = -2(\gamma^M-\gamma^L)\frac{\alpha}{(\alpha+1)^2}.
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
N &= 2(\gamma^M-\gamma^L)\int_0^1 s^\alpha \ln s^\alpha ds = 2(\gamma^M-\gamma^L)\left(\ln s^\alpha \bullet \frac{s^{\alpha+1}}{\alpha+1}\Big|_0^1 - \int_0^1 \frac{\alpha}{\alpha+1} s^\alpha dz\right) \\
&= 2(\gamma^M-\gamma^L)\left(\ln s^\alpha \bullet \frac{s^{\alpha+1}}{\alpha+1}\Big|_0^1 - \frac{\alpha}{(\alpha+1)^2} s^{\alpha+1}\Big|_0^1\right) = -2(\gamma^M-\gamma^L)\frac{\alpha}{(\alpha+1)^2}.
\end{aligned} \tag{16}
$$

## IV. Fuzzy Portfolio Model with Parametric Entropy under Hesitant Triangular Fuzzy Environment

In this section, risks are measured by entropy, returns are measured by credibility expectation function. We construct the PE-CE Model with two objective functions and list its solving process.

### A. PE-CE Model Construction

Assume that an investor chooses from $n$ stocks $\{x_1, x_2, ..., x_n\}$ by $m$ criteria $\{C_1, C_2, ..., C_m\}$. Hesitant triangular fuzzy matrix $H = [h_{ij}]_{n\times m}$ is composed of hesitant triangular fuzzy set $h_{ij}(i=1,2,...,n; j=1,2,...,m)$, which can be transformed into a collective column vector $\overline{H} = [\overline{h}_i]_{m\times 1}$.

In order to obtain the optimal investment ratios and the optimal portfolio, we construct the following portfolio model (20) (PE-CE Model) based on parametric hesitant triangular fuzzy entropy proposed in Formula (13): $E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$ is the credibility expectation function of HTFE $h$, $H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$ is the parametric hesitant triangular fuzzy entropy of HTFE $h$. $\overline{h}_i$ is the aggregated HTFE based on $\overline{h}_i = \oplus_{j=1}^n h_{ij}$, $h_{ij}$ is the hesitant triangular fuzzy information of the alternative $x_i$ with respect to the criterion $C_j$. $W = \{w_1, w_2, \ldots, w_n\}$ denotes the optimal investment ratios of this fund on these stocks, $l_i$ and $u_i$ denote the upper limit and lower limit of *i-th* stock, $\# h$ denotes the length of HTFE $h$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

### B. PE-CE Model Solving Process

Before starting, we should normalize credibility expectation function $E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$ and fuzzy entropy function $H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$. These two objective functions are weighted to construct satisfaction function $A(x)$, then the PE-CE Model turns into Formula (21).

Where $k_1 + k_2 = 1$, $\phi_1\left(\oplus_{i=1}^n w_i \overline{h}_i\right)$ and $\phi_2\left(\oplus_{i=1}^n w_i \overline{h}_i\right)$ are fuzzy entropy function $H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$ and credibility expectation function $E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]$ after normalization, respectively:

$$
\phi_1\left(\oplus_{i=1}^n w_i \overline{h}_i\right) = \frac{H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\max} - H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]}{H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\max} - H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\min}}. \tag{22}
$$

$$
\phi_2\left(\oplus_{i=1}^n w_i \overline{h}_i\right) = \frac{E\left[\oplus_{i=1}^n w_i \overline{h}_i\right] - E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\min}}{E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\max} - E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\min}}. \tag{23}
$$

$H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\max}$ and $H_\alpha\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\min}$ in Formula (22) represents the maximum and minimum values obtained under the same constraint with the PE-CE Model when entropy function is the only objective function, accordingly, $E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\max}$ and $E\left[\oplus_{i=1}^n w_i \overline{h}_i\right]_{\min}$ in Formula (23) represent the maximum and minimum values obtained when credibility expectation function is the only objective function.

$$\max E\left[\oplus_{i=1}^{n} w_i \bar{h}_i\right]$$
$$\min H_\alpha\left[\oplus_{i=1}^{n} w_i \bar{h}_i\right]$$
$$\text{s.t.}\begin{cases}\oplus_{i=1}^{n} w_i \bar{h}_i = \bigcup_{t_1\in\bar{h}_1, t_2\in\bar{h}_2,\ldots,t_n\in\bar{h}_n}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i^{L}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\gamma_i^{M}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\gamma_i^{R}\right)^{w_i}\right)\right\}\\ H_\alpha\left[\oplus_{i=1}^{n} w_i \bar{h}_i\right] = -\frac{1}{n}\sum_{i=1}^{n}\left(\frac{4}{(\alpha+1)^2}\frac{\alpha}{1-\alpha}\frac{1}{\#h}\sum_{t_i\in\bar{h}_i}\left(-\prod_{i=1}^{n}\left(1-\gamma_i^{R}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\gamma_i^{L}\right)^{w_i}\right)\right)\\ E\left[\oplus_{i=1}^{n} w_i \bar{h}_i\right] = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\#h}\sum_{t_i\in\bar{h}_i}\frac{\gamma_i^{L}+2\gamma_i^{M}+\gamma_i^{R}}{4}\right)\\ \sum_{i=1}^{n} w_i \le 1, l_i \le w_i \le u_i, i=1,2,\ldots n.\end{cases} \tag{20}$$

$$\max A\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) = k_1\phi_1\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) + k_2\phi_2\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right)$$
$$\text{s.t.}\begin{cases}\oplus_{i=1}^{n} w_i \bar{h}_i = \bigcup_{t_1\in\bar{h}_1, t_2\in\bar{h}_2,\ldots,t_n\in\bar{h}_n}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i^{L}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\gamma_i^{M}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\gamma_i^{R}\right)^{w_i}\right)\right\}\\ \sum_{i=1}^{n} w_i \le 1, l_i \le w_i \le u_i, i=1,2,\ldots,n.\end{cases} \tag{21}$$

**Step1:** Transform the hesitant fuzzy matrix $H=\left[h_{ij}\right]_{n\times m}$ ( $i=1,2,\ldots,n$ ; $j=1,2,\ldots,m$ ) into a collective column vector $\bar{H}=\left[\bar{h}_i\right]_{n\times 1}$ by aggregating all the values on one line.

**Step2:** Construct the PE-CE Model, which takes parametric entropy and credibility expectation as two objective functions. The return should be as high as possible and the risk as low as possible.

**Step3:** Construct satisfaction function, thus the dual objective programming problem is transformed into a single objective programming problem. The parameters' value $k_1$ and $k_2$ are set according to the investor's risk preferences.

Case 1: If there is a risk seeker, set $k_1 < k_2$ ;

Case 2: If there is a risk neutral, set $k_1 = k_2$ ;

Case 3: If there is a risk averter, set $k_1 > k_2$ .

**Step4:** Solve the portfolio model, then obtain the optimal investment ratios $w_i\left(i=1,2,\ldots,n\right)$.

Thus, we get optimal portfolios with different risk preferences.

## V. Case Study

In this section, we demonstrate the validity of the model proposed in Section IV through empirical analysis and analyze the data results.

### *A. Example and Calculations*

In this subsection, we assume that the investor constructs the hesitant triangular fuzzy matrix $H=\left[h_{ij}\right]_{3\times 3}$ of $x_i\left(i=1,2,3\right)$ with respect to the criteria $C_j\left(j=1,2,3\right)$, which is shown in Table I.

### *B. Solution Process for PE-CE Model in Case Study*

We demonstrate the portfolio selection process for risk investors under hesitant triangular fuzzy environment with parametric entropy and credibility expectation measuring risks and returns, respectively. The numerical calculation process is also given.

**Step1:** Transform the hesitant triangular fuzzy matrix $H=\left[h_{ij}\right]_{3\times 3}$ into a collective column vector, then we get $\bar{H}=\left[\bar{h}_1,\bar{h}_2,\bar{h}_3\right]_{3\times 1}$. HTFEs are presented in Formula (24)-(26).

**Step2:** Construct the PE-CE Model by Formula (27), which takes parametric entropy and credibility expectation as two objective functions.

During the solution process, we set $\alpha = 2$ .

**Step3:** Construct satisfaction function by Formula (28), thus the PE-CE Model (27) is transformed into a single objective programming problem. To normalize two objective functions: credibility expectation and entropy, we are going to find their maximum and minimum values:

$H_\alpha\left[\oplus_{i=1}^{3} w_i \bar{h}_i\right]_{\max} = 0.3991$ , $H_\alpha\left[\oplus_{i=1}^{3} w_i \bar{h}_i\right]_{\min} = 0.1980$ ,

$E\left[\oplus_{i=1}^{3} w_i \bar{h}_i\right]_{\max} = 0.8913$ , $E\left[\oplus_{i=1}^{3} w_i \bar{h}_i\right]_{\min} = 0.7956$ ,

which are assigned to Formula (22) and Formula (23):

$$\phi_1\left(\oplus_{i=1}^{3} w_i \bar{h}_i\right) = \frac{0.3991 - H_\alpha\left[\oplus_{i=1}^{3} w_i \bar{h}_i\right]}{0.3991-0.1980} \tag{29}$$

$$\phi_2\left(\oplus_{i=1}^{3} w_i \bar{h}_i\right) = \frac{E\left[\oplus_{i=1}^{n} w_i \bar{h}_i\right]-0.7956}{0.8913-0.7956} \tag{30}$$

The parameters' value $k_1$ and $k_2$ are set according to the investors' risk preference.

Case 1: If there is a risk seeker, set $k_1 < k_2$ ;

Case 2: If there is a risk neutral, set $k_1 = k_2$ ;

Case 3: If there is a risk averter, set $k_1 > k_2$ .

**Step4:** Solve the portfolio model, then obtain the optimal investment ratios $w_i\left(i=1,2,3\right)$, which is shown in Table II.

Thus, we get optimal portfolios with different risk preferences.

Table I The hesitant triangular fuzzy matrix $H=\left[h_{ij}\right]_{3\times 3}$

| Stocks | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $x_1$ | $\{(0.1,0.3,0.5)\}$ | $\{(0.3,0.7,0.8),(0.1,0.4,0.8),(0.5,0.8,0.9)\}$ | $\{(0.2,0.3,0.5)\}$ |
| $x_2$ | $\{(0.6,0.7,0.8)\}$ | $\{(0.2,0.4,0.5)\}$ | $\{(0.3,0.4,0.7),(0.3,0.4,0.8)\}$ |
| $x_3$ | $\{(0.1,0.3,0.5)\}$ | $\{(0.5,0.8,0.9),(0.6,0.9,1.0)\}$ | $\{(0.2,0.3,0.4),(0.4,0.5,0.6)\}$ |

$$\overline{h_1}=\{(0.496,0.853,0.950),(0.352,0.706,0.950),(0.640,0.902,0.975)\}; \tag{24}$$

$$\overline{h_2}=\{(0.776,0.892,0.970),(0.776,0.892,0.980)\}; \tag{25}$$

$$\overline{h_3}=\{(0.640,0.902,0.970),(0.784,0.965,1.000),(0.730,0.930,0.980),(0.712,0.951,1.000)\}. \tag{26}$$

$$\begin{aligned}
&\min H_\alpha\left[\oplus_{i=1}^3 w_i\overline{h_i}\right]\\
&\max E\left[\oplus_{i=1}^3 w_i\overline{h_i}\right]\\
&\text{s.t.}\begin{cases}
\oplus_{i=1}^3 w_i\overline{h_i}=\bigcup_{t_1\in\overline{h}_1,t_2\in\overline{h}_2,t_3\in\overline{h}_3}\left\{\left(1-\prod_{i=1}^n\left(1-\gamma_i^L\right)^{w_i},1-\prod_{i=1}^n\left(1-\gamma_i^M\right)^{w_i},1-\prod_{i=1}^n\left(1-\gamma_i^R\right)^{w_i}\right)\right\}\\
H_\alpha\left[\oplus_{i=1}^3 w_i\overline{h_i}\right]=-\frac{1}{3}\sum_{i=1}^3\left(\frac{4}{(\alpha+1)^2}\frac{\alpha}{1-\alpha}\frac{1}{\#\overline{h}_i}\sum_{t_i\in\overline{h}_i}\left(-\prod_{i=1}^n\left(1-\gamma_i^R\right)^{w_i}+\prod_{i=1}^n\left(1-\gamma_i^L\right)^{w_i}\right)\right)\\
E\left[\oplus_{i=1}^3 w_i\overline{h_i}\right]=\frac{1}{3}\sum_{i=1}^3\left(\frac{1}{\#\overline{h}_i}\sum_{t_i\in\overline{h}_i}\frac{\gamma_i^L+2\gamma_i^M+\gamma_i^R}{4}\right)\\
\sum_{i=1}^3 w_i\le 1, 0.1\le w_i\le 1, i=1,2,3.
\end{cases}
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\max A\left(\oplus_{i=1}^3 w_i\overline{h_i}\right)=k_1\phi_1\left(\oplus_{i=1}^3 w_i\overline{h_i}\right)+k_2\phi_2\left(\oplus_{i=1}^3 w_i\overline{h_i}\right)\\
&\text{s.t.}\begin{cases}
\oplus_{i=1}^3 w_i\overline{h_i}=\bigcup_{t_1\in\overline{h}_1,t_2\in\overline{h}_2,t_3\in\overline{h}_3}\left\{\left(1-\prod_{i=1}^3\left(1-\gamma_i^L\right)^{w_i},1-\prod_{i=1}^3\left(1-\gamma_i^M\right)^{w_i},1-\prod_{i=1}^3\left(1-\gamma_i^R\right)^{w_i}\right)\right\}\\
\sum_{i=1}^3 w_i\le 1, 0.1\le w_i\le 1, i=1,2,3\\
k_1+k_2=1.
\end{cases}
\end{aligned} \tag{28}$$

Table II Optimal portfolio for risk investors under different values of $k_i\,(i=1,2)$ when $\alpha=2$ in PE-CE Model

| Indexes | $k_1=0.1$ $k_2=0.9$ | $k_1=0.2$ $k_2=0.8$ | $k_1=0.3$ $k_2=0.7$ | $k_1=0.4$ $k_2=0.6$ | $k_1=0.5$ $k_2=0.5$ | $k_1=0.6$ $k_2=0.4$ | $k_1=0.7$ $k_2=0.3$ | $k_1=0.8$ $k_2=0.2$ | $k_1=0.9$ $k_2=0.1$ |
|---|---|---|---|---|---|---|---|---|---|
| $w_1$ | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 |
| $w_2$ | **0.2475** | **0.3981** | **0.5582** | **0.7319** | **0.8000** | **0.8000** | **0.8000** | **0.8000** | **0.8000** |
| $w_3$ | **0.6525** | **0.5019** | **0.3418** | **0.1681** | 0.1000 | 0.1000 | 0.1000 | 0.1000 | 0.1000 |
| $A(\cdot)$ | 0.9789 | 0.9638 | 0.9555 | 0.9551 | 0.9620 | 0.9696 | 0.9772 | 0.9848 | 0.9924 |

The most striking data are highlighted in Table II. From Table II, we can see that the ability for risk tolerance of investors is reflected by the changes of optimal portfolios' investment proportions. Risk seekers focus more on $x_3$, while risk neutrals and risk averters prefer $x_2$. With the increase of $k_1$, satisfaction degree decreases and then increases. The satisfaction degree falls to a low point of 0.9551 when $k_1=0.4$ and $k_2=0.6$. In this case, investors with the clearest risk attitude obtain the most satisfaction degree.

Table III The parametric entropy and credibility expectation values of $\overline{h_i}$

| Aggregated HTFE | Parametric entropy | Credibility expectation |
|---|---|---|
| $\overline{h_1}$ | 0.4623 | 0.7904 |
| $\overline{h_2}$ | **0.1990** | 0.8838 |
| $\overline{h_3}$ | 0.2710 | **0.8945** |

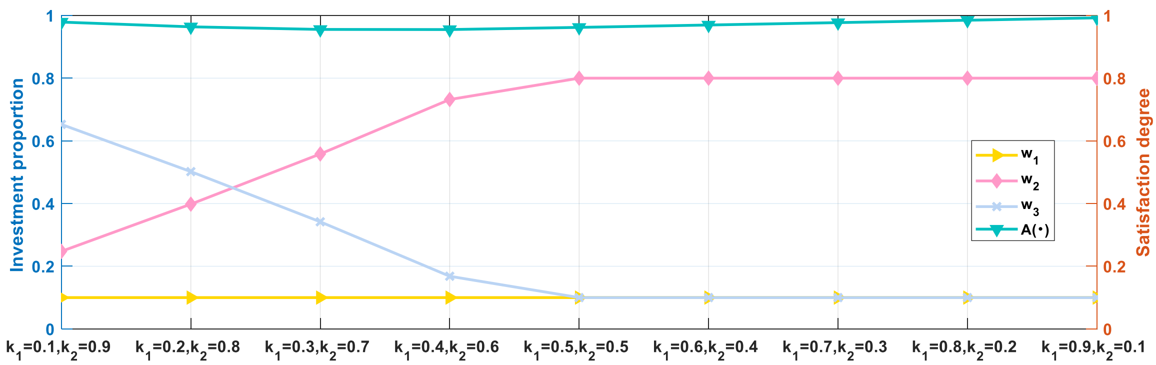


Figure 1 Optimal portfolio for risk investors under different values of $k_i\,(i=1,2)$ when $\alpha=2$ in PE-CE Model

### C. Discussions of Results

To visualize data, we graph the information presented in Table II by Figure 1. Table III provides the results obtained from the preliminary calculation of each stock's parametric entropy and credibility expectation.

By analyzing the results of existing data, we can conclude that:

(1) As shown in Table III, the order of entropy value is $H_\alpha\left[\overline{h_2}\right] < H_\alpha\left[\overline{h_3}\right] < H_\alpha\left[\overline{h_1}\right]$, the order of credibility expectation value is $E\left[\overline{h_3}\right] > E\left[\overline{h_2}\right] > E\left[\overline{h_1}\right]$. Overall, these results indicate that $x_2$ has minimal risk and $x_3$ has maximal return. From the perspective of minimizing risks, focusing on $x_2$ is the best choice. On the other hand, from the point of view of maximizing returns, $x_3$ has more significant advantages.

(2) When $k_1 < k_2$, the optimal investment proportion $w_2$ decreases with the rise of $k_1$ for risk seekers. This result may be explained by the fact that investors seek higher returns with acceptable risks. As we have discussed above, $x_3$ brings highest return. Thus, assets of risk seekers with more risk-bearing capacity ( $k_1=0.1$, $k_2=0.9$ and $k_1=0.2$, $k_2=0.8$ ) are skewed towards $x_3$. Nevertheless, investors take lower risks but show little difference between returns when focusing on $x_2$. So, risk seekers with less risk-bearing capacity ($k_1=0.3$, $k_2=0.7$ and $k_1=0.4$, $k_2=0.6$) prefer $x_2$.

(3) When $k_1 = k_2$ and $k_1 > k_2$, investors tend to concentrate on $x_2$. $x_2$ is a high-quality choice because of its absolute advantage in risk and nearly the same return as the largest one for risk neutrals and risk averters.

(4) Investors' satisfaction degree decreases and then increases. A possible explanation for this might be that clear risk attitude leads to higher satisfaction. The more risk investors focus on risk or return, the more satisfaction degree they get. For risk seekers, investors with lowest risk tolerance level ($k_1=0.4$, $k_2=0.6$) should make a few concessions after trade-off between risk and return. At this point, portfolio can neither achieve the maximum return nor the minimum risk.

In conclusion, the present results are significant in at least three major respects. Firstly, our proposed PE-CE Model (20) with parametric entropy measuring risk and credibility expectation measuring return is reasonable. Next, the changes of optimal portfolio can be interpreted with caution based on numerical analysis. Most important of all, the parameter introduced to entropy has important implications for developing risk investors' preference information.

## VI. Conclusions

In this paper, we pioneer the application of hesitant triangular fuzzy sets to portfolio selection model. Inspired by credibility theory, we introduce the parameter to entropy, which can be used to measure risks, and make the description of decision information more comprehensive. The satisfaction function constructed by transforming the dual objective model into a single objective model, shows the optimal portfolio changing with the weight of the objective function in the case study. The optimal portfolio changes are explained reasonably according to the specific situation of each stock. Thus, it is a strong proof of the validity of the PE-CE Model. The findings of this investigation complement those of earlier studies.

Despite these promising results, questions remain. Further research should be undertaken to investigate more objective functions measuring risks and returns which can be added to the PE-CE Model. Also, considerably more work will need to be done to determine the parameter's value according to the investors' risk preferences. It is also an alternative direction to establish other more effective risk measures in such a complex portfolio environment.

## References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no.3, pp. 338-353, 1965.

[2] K. T. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 37-45, 1989.

[3] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no.6, pp. 529-539, 2010.

[4] N. Chen, Z. Xu and M. Xia, "Interval-valued hesitant preference relations and their applications to group decision making," *Knowledge-Based Systems*, vol. 37, pp. 528-540, 2013.

[5] B. Zhu, Z. Xu and M. Xia, "Dual Hesitant Fuzzy Sets," *Journal of Applied Mathematics*, vol. 2012, pp. 1-13, 2012.

[6] D. Yu, “Triangular hesitant fuzzy set and its application to teaching quality evaluation,” *Journal of Information and Computational Science*, vol. 10, no.7, pp. 1925-1934, 2013.

[7] H. Prade and D. Dubois, *Fundamentals of Fuzzy Sets.* 2012.

[8] G. Tintner and H. M. Markowitz, “Portfolio Selection: Efficient Diversification of Investments,” *The Journal of Finance*, vol. 15, no.3, pp. 447-447, 1960.

[9] H. Katagiri and H. Ishii, “Fuzzy portfolio selection problem,” *IEEE International Conference on Systems, Man, and Cybernetics*, vol. 3, pp.973-978, 1999.

[10] Y. Zhang, W. Liu and X. Yang, “An automatic trading system for fuzzy portfolio optimization problem with sell orders,” *Expert Systems with Applications*, vol. 187, pp. 115822, 2022.

[11] J. K. Pahade and M. Jha, “Credibilistic variance and skewness of trapezoidal fuzzy variable and mean–variance–skewness model for portfolio selection,” *Results in Applied Mathematics*, vol. 11, pp. 100159, 2021.

[12] M. De, B. K. Mangaraj and K. B. Das, “A fuzzy goal programming model in portfolio selection under competitive-cum-compensatory decision strategies,” *Applied Soft Computing*, vol. 73, pp. 635-646, 2018.

[13] N. Mansour, M. S. Cherif and W. Abdelfattah, “Multi-objective imprecise programming for financial portfolio selection with fuzzy returns,” *Expert Systems with Applications*, vol. 138, pp. 112810, 2019.

[14] Z. Xu and M. Xia, “Distance and similarity measures for hesitant fuzzy sets,” *Information Sciences*, vol. 181, no.11, pp. 2128-2138, 2011.

[15] Z. Xu and M. Xia, “On distance and correlation measures of hesitant fuzzy information,” *International Journal of Intelligent Systems*, vol. 26, no.5, pp. 410-425, 2011.

[16] X. Deng and J. Chen, “The lower partial moments risk measure in a novel fuzzy framework based on possibility density function,” *Computers & Industrial Engineering*, vol. 169, pp. 108309, 2022.

[17] X. Deng, T. Lin, and C. Chen, “Comparison and research on diversified portfolios with several entropy measures based on different psychological states,” *Entropy*, vol. 22, no.10, 2020.

[18] X. Deng, W. Li and Y. Liu, “Hesitant fuzzy portfolio selection model with score and novel hesitant semi-variance,” *Computers & Industrial Engineering*, vol. 164, pp. 107879, 2022.

[19] X. Liu *et al*, “Novel correlation coefficient between hesitant fuzzy sets with application to medical diagnosis,” *Expert Systems with Applications*, vol. 183, pp. 115393, 2021.

[20] F. Karaaslan and Ş. Özlü, “Correlation coefficients of dual type-2 hesitant fuzzy sets and their applications in clustering analysis,” *International Journal of Intelligent Systems*, vol. 35, no.7, pp. 1200-1229, 2020.

[21] K. Rezaei and H. Rezaei, “New distance and similarity measures for hesitant fuzzy sets and their application in hierarchical clustering,” *Journal of Intelligent & Fuzzy Systems*, vol. 39, no.3, pp. 4349-4360, 2020.

[22] B. Farhadinia and Z. Xu, “Distance and Aggregation-Based Methodologies for Hesitant Fuzzy Decision Making,” *Cognitive Computation*, vol. 9, no.1, pp. 81-94, 2016.

[23] B. Liu and J. Peng, “Uncertainty theory tutorial,” Tsinghua University Press, Beijing, 2005.

[24] X. Zhang, W. Zhang and R. Cai, "Portfolio adjusting optimization under credibility measures," *Journal of Computational and Applied Mathematics*, vol. 234, no.5, pp. 1458-1465, 2010.

[25] W. Zhang, X. Zhang and Y. Chen, “Portfolio adjusting optimization with added assets and transaction costs based on credibility measures,” *Insurance, Mathematics & Economics*, vol. 49, no.3, pp. 353-360, 2011.

[26] E. Vercher and J. D. Bermúdez, “Portfolio optimization using a credibility mean-absolute semi-deviation model,” *Expert Systems with Applications*, vol. 42, no.20, pp. 7121-7131, 2015.

[27] P. Gupta, M. K. Mehlawat and A. Z. Khan, “Multi-period portfolio optimization using coherent fuzzy numbers in a credibilistic environment,” *Expert Systems with Applications*, vol. 167, pp. 114135, 2021.

[28] X. Zhao, R. Lin and G. Wei, “Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making,” *Expert Systems with Applications*, vol. 41, no.4, pp. 1086-1094, 2014.