Adaptive Control of Pure Feedback Stochastic Nonlinear Systems with Input Saturation and Partial State Constraints

Chenqi Zhai, Nannan Zhao, Xinyu Ouyang and Xianhong Chen

Abstract—**The adaptive fuzzy logic control problem for nonlinear systems with partial state constraints and input saturation is concerned in this paper. By using the implicit function theorem and mean value theorem, the pure feedback nonlinear system can be transformed. Barrier Lyapunov Function (BLF) is selected to prevent the state of some constraints from violating the constraints. The output tracking problem of this kind of system and the influence of input saturation are solved with Lyapunov's second method and backstepping method. The analysis of probabilistic stability is also carried out to ensure that all signals of the closed-loop system are bounded and the system output can track the given reference signal. Finally, the feasibility and effectiveness of the control scheme is verified by simulation.**

Index Terms—**Adaptive control; pure feedback system; partial state constraints; saturation input; fuzzy logic system (FLS)**

I. INTRODUCTIONED

PURE feedback systems are ubiquitous in real life, and many systems can be described by pure feedback systems. In recent years, the research on pure feedback nonlinear systems has become a hot spot and a series of achievements have been made [1]–[15]. Wang et al. [11] studied the pure feedback stochastic nonlinear system with input constraints, the control scheme which considered the influence of input saturation. However, the scheme didn't consider the state constraints of the system. Li et al. [14] proposed the adaptive control scheme for pure feedback stochastic nonlinear systems with dead time input and time-varying delay, but the scheme ignored the input saturation.

The solution of constraint problem in control system is also a very important work in industrial process control [16]. As a typical model in industry, strict feedback nonlinear system has been widely studied [16]–[25]. For the stochastic nonlinear system with full state constraints, two kinds of algorithms were described by using symmetric BLF and asymmetric BLF in [19]. Based on backstepping method, Tee et al. [20] used traditional BLF and symmetric BLF respectively, proposed the corresponding adaptive control strategy to settle the output constraint problem.

Partial state constraint control is only a part of the state in the control system that needs to meet certain specific constraints. In general, only partial states, not complete states, are constrained. For example, when the robot manipulator grabs the workpiece from the pipeline, it does not involve position constraints in the direction parallel to the pipeline, but it will impose strict motion constraints in other directions. In fact, output constraint control and full state constraint control can be regarded as special forms of partial state constraint control [26]. In addition, the existing research results can not solve the input saturation phenomenon when some states of the system must meet certain constraints. As a result, it is of great practical important to investigate the scheme under local state constraints.

In this context, this paper attempts to design an adaptive controller to achieve effective control of pure feedback nonlinear systems with partial state constraints and input saturation. The rest of this article is organized as follows. Section II gives the system description and basic knowledge. Section III develops the control scheme. Section IV shows the stability analysis. The simulation results are provided in Section V, and the conclusion is given in last Section.

Manuscript received July 5, 2022; revised Febuary 7, 2023.

Chenqi Zhai is a postgraduate student of School of Electronic and Information Engineering, University of Science and Technology Liaoning, Anshan, Liaoning, CO 114051, China. (e-mail: 1483020867@qq.com).

Nannan Zhao is a professor of School of Electronic and Information Engineering, University of Science and Technology Liaoning, Anshan, Liaoning, CO 114051, China. (Corresponding author, e-mail: 723306003@qq.com).

Xinyu Ouyang is a professor of School of Electronic and Information Engineering, University of Science and Technology Liaoning, Anshan, Liaoning, CO 114051, China.(Corresponding author, e-mail: 13392862@qq.com).

Xianhong Chen is a postgraduate student of School of Electronic and Information Engineering, University of Science and Technology Liaoning, Anshan, Liaoning, CO 114051, China. (e-mail: 1158992971@qq.com).

II. SYSTEM DESCRIPTIONS AND BASIC KNOWLEDGE

Considering the nonlinear system as follows:

$$dx = f(x,t)dt + h(x,t)dw, \forall x \in R^n \tag{1}$$

where x is the state of the system, $f : R^{i+1} \to R$ and $h : R^n \to R^r$ are local Lipschitz functions satisfying $f(0,t) = h(0,t) = 0$, $\forall t \geq 0$.

In order to facilitate subsequent analysis, the following definitions and lemmas are introduced.

Definition 1. *[17] For each given* $\mathrm{V}(x) \in C^2$*, associated with the differential (1), the differential operator* L *is defined as follows:*

$$LV = \frac{\partial v}{\partial x} f + \frac{1}{2} Tr\{h^T \frac{\partial^2 v}{\partial x^2} h\} \tag{2}$$

here $Tr(A)$ *is the trace of matrix* A.

Lemma 1. *[15] Assume that* $f(x,u) : R^n \times R \to R$ *is continuously differentiable,* $\forall (x,u) \in R^n \times R$ *, and there exists* d *such that* $\frac{\partial f(x,u)}{\partial u} > d > 0$, $\forall (x,u) \in R^n \times R$*. Then there exists* $u^* = u(x)$ *such that* $f(x,u^*) = 0$.

Lemma 2. *[27] For each constant k_b and each real number $|z| < k_b$, there always exists*

$$\log \frac{k_b^{2p}}{k_b^{2p} - z^{2p}} < \frac{z^{2p}}{k_b^{2p} - z^{2p}} \tag{3}$$

where $p > 0$.

Lemma 3. *[28] There is a C^2 function $v : R^n \to R_+$, $r > 0$, $\rho > 0$, k_∞-class functions $\overline{\alpha}_1$ and $\overline{\alpha}_2$ make $\overline{\alpha}_1(|x|) \le v(x) \le \overline{\alpha}_2(|x|)$ and for all $x \in R^n$, $t > t_0$, there are $L[V(x)] \le -rv(x) + \rho$. For each $x_0 \in R^n$, there exists*

$$E[V(x)] \le v(x_0)e^{-rt} + \frac{\rho}{r}, \forall t > t_0 \tag{4}$$

Lemma 4. *[29] For all $\varepsilon > 0$, $p > 1$, $q > 1$ and $(p-1)(q-1) = 1$, there exists*

$$xy \le \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q \tag{5}$$

Consider the following pure feedback nonlinear systems:

$$\begin{cases} dx_i = f_i(\overline{x}_i, x_{i+1})dt + \varphi_i^T(x)dw, 1 \le i \le n-1 \\ dx_n = f_n(\overline{x}_n, u)dt + \varphi_n^T(x)dw \\ y = x_1 \end{cases} \tag{6}$$

where $x = [x_1, x_2, ..., x_n]^T \in R^n$, $u \in R$ and $y \in R$ respectively represent the state variables, inputs and outputs of the system, $\overline{x}_i = [x_1, x_2, ..., x_i]^T \in R^i$, w is an r-dimensional independent standard Brownian motion, $f_i(\cdot)$ and $\varphi_i(\cdot)$ are unknown functions, u indicates the saturation input described by

$$u = sat(v) = \begin{cases} u_M, v \ge u_M \\ v, u_m \le v \le u_M \\ u_m, v \le u_m \end{cases} \tag{7}$$

where $sat(v)$ is the saturation function, v is the ideal control law, u_m is the minimum value of the known input u, u_M is the maximum value of the known input u. Method similar to [17], a piecewise smooth function is introduced to approach u as follows:

$$\begin{aligned} g(v) &= \begin{cases} u_{\max}\tanh(\frac{v}{u_{\max}}), v \ge 0 \\ u_{\min}\tanh(\frac{v}{u_{\min}}), v < 0 \end{cases} \\ &= \begin{cases} \frac{u_{\max}(e^{v/u_{\min}} - e^{-v/u_{\min}})}{e^{v/u_{\min}} + e^{-v/u_{\min}}}, v \ge 0 \\ \frac{u_{\min}(e^{v/u_{\max}} - e^{-v/u_{\max}})}{e^{v/u_{\max}} + e^{-v/u_{\max}}}, v < 0 \end{cases} \end{aligned} \tag{8}$$

Then the saturation function $sat(v)$ can be written as [30]

$$sat(v) = u = g(v) + \rho(v) \tag{9}$$

where $\rho(v) = sat(v) - g(v)$, its upper bound is

$$|\rho(v)| = |sat(v) - g(v)| \le Mu(1 - \tanh(1)) = D \tag{10}$$

By mean value theorem, there exists $\mu(0 < \mu < 1)$ to make the following equation hold

$$g(v) = g(v_0) + g_{vu}(v - v_0) \tag{11}$$

where $g_{vu} = \left.\frac{\partial g(v)}{\partial v}\right|_{v=v_\mu}$, $v_\mu = \mu v + (1-\mu)v_0$.

Selecting $v_0 = 0$, the following equation holds

$$g(v) = g_{vu}v \tag{12}$$

It is noteworthy that the thesis studies the solution of partial state constraints. All States are divided into constraint state $\overline{x}_t = [x_1, ..., x_t]^T$ and free state $\overline{x}_m = [x_{t+1}, ..., x_n]^T$. The constraint status needs to meet the following constraints:

$$|x_i| < k_i \tag{13}$$

where $k_i > 0$.

The control objective of the thesis is to design a controller for (6) so that:

1) All signals of the closed-loop system are bounded;
2) Output tracking error shall be as small as possible;
3) The constrained state satisfies the constraints.

In order to complete the control scheme, the assumptions are proposed.

Assumption 1. *The tracking signal $y_d(t)$ and its jth order derivative $y_d^{(i)}$ satisfy $|y_d| \le Y_0$ and $\left|y_d^{(i)}\right| \le Y_i$, here Y_0 and Y_i are unknown positive constants, $i = 1, ..., n$, respectively.*

Assumption 2. *The sign of smooth function $g_i(\overline{x}_i, x_{i+1})$, is known, and there exist b_m and b_M such that*

$$0 < b_m \le |g_i(\overline{x}_i, x_{i+1})| \le b_M < \infty \tag{14}$$

where $g_i(\overline{x}_i, x_{i+1}) = \frac{\partial f_i(\overline{x}_i, x_{i+1})}{\partial x_{i+1}}$, $i = 1, ..., n$.

Assumption 3. *g_m is an unknown constant and $0 < g_m < g_{vu} \le 1$.*

III. Controller Design

In this section, the controller will be designed for (6). The following coordinate transformation will be used later

$$\begin{aligned} z_1 &= x_1 - y_d \\ z_i &= x_i - \alpha_{i-1}, i = 2, ..., n \end{aligned} \tag{15}$$

here α_i is a virtual control signal. The detailed design steps of the controller are as follows.

Step 1: According to (6) and (15), we can get

$$dz_1 = [f_1(\overline{x}_1, x_2) - \dot{y}_d]dt + \varphi_1^T(x)dw \tag{16}$$

Define a Lyapunov function as follows:

$$V_1 = \frac{1}{4}\log\frac{k_{b1}^4}{k_{b1}^4 - z_1^4} + \frac{1}{2}b_m\tilde{\theta}_1^2 \tag{17}$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$. $\hat{\theta}_1$ is the estimation of θ_1 which is described as

$$\begin{aligned} \theta_i &= \max\{\frac{b_M^2\|w_i^*\|^2}{b_m}\}, i = 1, ..., n-1 \\ \theta_n &= \max\{\frac{g_m b_M^2\|w_n^*\|^2}{b_m}\} \end{aligned} \tag{18}$$

Then, by taking the differential operator L and (16) for (17), we can get:

$$\begin{aligned} LV_1 = &\frac{z_1^3}{k_{b1}^4 - z_1^4}[f_1(\overline{x}_1, x_2) - \dot{y}_d] \\ &+ \frac{z_1^3(3k_{b1}^4 + z_1^4)\varphi_1^T\varphi_1}{2(k_{b1}^4 - z_1^4)^2} - b_m\tilde{\theta}_1\dot{\hat{\theta}}_1 \end{aligned} \tag{19}$$

According to assumption 2, $\frac{\partial f_1(\overline{x}_1,x_2)}{\partial x_2} \geq b_m > 0$. Let's define

$$w_1 = -\dot{y}_d + k_1 z_1 + \frac{z_1^3}{2(k_{b1}^4 - z_1^4)} + \frac{\tau_1^{\frac{3}{2}}(3k_{b1}^4 + z_1^4)^{\frac{3}{2}}\|\varphi_1\|^3}{3\sqrt{2}(k_{b1}^4 - z_1^4)^2} + \frac{3z_1}{4(k_{b1}^4 - z_1^4)^{\frac{1}{3}}} \tag{20}$$

Because $\frac{\partial w_1}{\partial x_2} = 0$, there is

$$\frac{\partial[f_1(\overline{x}_1, x_2) + w_1]}{\partial x_2} \geq b_m > 0 \tag{21}$$

By lemma 1, for any x_1 and w_1, there exist $x_2 = \alpha_1^*(x_1, w_1)$, so that

$$f_1(\overline{x}_1, \alpha_1^*) + w_1 = 0 \tag{22}$$

According to the mean value theorem, there exists $\mu_1(0 < \mu_1 < 1)$ satisfied the following formula

$$f_1(\overline{x}_1, x_2) = f_1(\overline{x}_1, \alpha_1^*) + g_{\mu 1}(x_2 - \alpha_1^*) \tag{23}$$

where $g_{\mu 1} = g_1(\overline{x}_1, x_{\mu 1})$, $x_{\mu 1} = \mu_1 x_2 + (1 - \mu_1)\alpha_1^*$.

Substituting (22) and (23) into (19), we can get

$$\begin{aligned} LV_1 =& \frac{z_1^3}{k_{b1}^4 - z_1^4}[g_{\mu 1}(z_2 + \alpha_1 - \alpha_1^*) - k_1 z_1 - \frac{z_1^3}{2(k_{b1}^4 - z_1^4)} \\ &- \frac{\tau_1^{\frac{3}{2}}(3k_{b1}^4 + z_1^4)^{\frac{3}{2}}\|\varphi_1\|^3}{3\sqrt{2}(k_{b1}^4 - z_1^4)^2} - \frac{3z_1}{4(k_{b1}^4 - z_1^4)^{\frac{1}{3}}}] \\ &+ \frac{z_1^2(3k_{b1}^4 + z_1^4)\varphi_1^T\varphi_1}{2(k_{b1}^4 - z_1^4)^2} - b_m\tilde{\theta}_1\dot{\hat{\theta}}_1 \end{aligned} \tag{24}$$

According to assumption 2 and formula (5), there are

$$\frac{z_1^3 g_{\mu 1} z_2}{k_{b1}^4 - z_1^4} \leq \frac{3z_1^4}{4(k_{b1}^4 - z_1^4)^{\frac{4}{3}}} + \frac{1}{4}b_M^4 z_2^4 \tag{25}$$

Here α_1 can be designed as

$$\alpha_1 = -k_1 z_1 - \frac{z_1^3\hat{\theta}_1 S_1^T S_1}{2a_1^2(k_{b1}^4 - z_1^4)} \tag{26}$$

where $a_1 > 0$.

Substituting (26) into (24), we can get

$$\frac{z_1^3 g_{\mu 1}\alpha_1}{k_{b1}^4 - z_1^4} \leq -\frac{z_1^4 b_m k_1}{k_{b1}^4 - z_1^4} - \frac{b_m z_1^6\hat{\theta}_1 S_1^T S_1}{2a_1^2(k_{b1}^4 - z_1^4)^2} \tag{27}$$

The fuzzy logic system is used to approximate α_1^* as follows

$$\alpha_1^* = \omega_1^{*T} S_1(z_1) + \varepsilon_1(z_1) \tag{28}$$

where ω_1^* is the unknown optimal parameter and $\varepsilon_1(z_1)$ is the minimum fuzzy approximation error. Suppose $|\varepsilon_1(z_1)| \leq \varepsilon_1^*$ and $\varepsilon_1^* > 0$.

Substituting α_1^* into (24) and using Young's inequality, there has:

$$\begin{aligned} -\frac{z_1^3 g_{\mu 1}\alpha_1^*}{k_{b1}^4 - z_1^4} \leq& \frac{z_1^6 b_m\theta_1 S_1^T S_1}{2a_1^2(k_{b1}^4 - z_1^4)^2} \\ &+ \frac{1}{2}a_1^2 + \frac{1}{2}b_M^2\varepsilon_1^{*2} + \frac{z_1^6}{2(k_{b1}^4 - z_1^4)^2} \end{aligned} \tag{29}$$

Combining (5), there has

$$\begin{aligned} \frac{z_1^2(3k_{b1}^4 + z_1^4)\varphi_1^T\varphi_1}{2(k_{b1}^4 - z_1^4)^2} \leq& \frac{z_1^3\tau_1^{\frac{3}{2}}(3k_{b1}^4 + z_1^4)^{\frac{3}{2}}\|\varphi_1\|^3}{3\sqrt{2}(k_{b1}^4 - z_1^4)^3} \\ &+ \frac{1}{3\tau_1^3} \end{aligned} \tag{30}$$

where τ_1 is the positive design parameter.

Substituting (25), (27) and (29) into (30), we can obtain

$$\begin{aligned} LV_1 \leq& -\frac{c_1 z_1^4}{k_{b1}^4 - z_1^4} + \frac{1}{2}a_1^2 + \frac{1}{2}b_M^2\varepsilon_1^{*2} + \frac{1}{3\tau_1^3} + \frac{1}{4}b_M^4 z_2^4 \\ &+ b_m\tilde{\theta}_1[\frac{z_1^6 S_1^T S_1}{2a_1^2(k_{b1}^4 - z_1^4)^2} - \dot{\hat{\theta}}_1] \end{aligned} \tag{31}$$

where $c_1 = k_1(1 + b_m)$.

$\dot{\hat{\theta}}_1$ can be designed as follows:

$$\dot{\hat{\theta}}_1 = \frac{z_1^6 S_1^T S_1}{2a_1^2(k_{b1}^4 - z_1^4)^2} - \hat{\theta}_1 \tag{32}$$

Substituting (32) into (31) and using (5) , one has:

$$\begin{aligned} LV_1 \leq& -\frac{c_1 z_1^4}{k_{b1}^4 - z_1^4} + \frac{1}{2}a_1^2 + \frac{1}{2}b_M^2\varepsilon_1^{*2} + \frac{1}{3\tau_1^3} \\ &+ \frac{1}{4}b_M^4 z_2^4 + \frac{1}{2}b_m\theta_1^2 - \frac{1}{2}b_m\tilde{\theta}_1^2 \end{aligned} \tag{33}$$

Step i: According to $z_i = x_i - \alpha_{i-1}, i = 2, ..., t$ and (6), it leads to

$$\begin{aligned} dz_i =& [f_i(\bar{x}_i, x_{i+1}) - L\alpha_{i-1}]dt \\ &+ (\varphi_i(x) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{j-1}}{\partial x_j}\varphi_j(x))^T dw \end{aligned} \tag{34}$$

where $L\alpha_{i-1} = \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}f_j(\bar{x}_j, x_{j+1}) + \sum_{j=1}^{i-1}\frac{\partial\alpha_i - 1}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^T(x)\varphi_q(x)$.

Let's choose the following Lyapunov function

$$V_i = V_{i-1} + \frac{1}{4}\log\frac{k_{bi}^4}{k_{bi}^4 - z_i^4} + \frac{1}{2}b_m\tilde{\theta}_i^2 \tag{35}$$

Taking the time derivative of (35) yields

$$\begin{aligned} LV_i =& LV_{i-1} + \frac{z_i^3}{k_{bi}^4 - z_i^4}[f_i(\bar{x}_i, x_i) - L\alpha_{i-1}] \\ &+ \frac{z_i^2(3k_{bi}^4 + z_i^4)\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right\|^2}{2(k_{bi}^4 - z_i^4)^2} \\ &- b_m\tilde{\theta}_i\dot{\hat{\theta}}_i \end{aligned} \tag{36}$$

According to assumption 2, $\frac{\partial f_i(\bar{x}_i, x_{i+1})}{\partial x_{i+1}} \geq b_m > 0$. Define

$$\begin{aligned} w_i =& -L\alpha_{i-1} + k_i z_i + \frac{z_i^3}{2(k_{bi}^4 - z_i^4)^2} \\ &+ \frac{\tau_i^{\frac{3}{2}}(3k_{bi}^4 + z_i^4)^{\frac{3}{2}}\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right\|^3}{3\sqrt{2}(k_{bi}^4 - z_i^4)^2} \\ &+ \frac{3z_i}{4(k_{bi}^4 - z_i^4)^{\frac{1}{3}}} + \frac{1}{4}b_M^4 z_i(k_{bi}^4 - z_i^4) \end{aligned} \tag{37}$$

Because $\frac{\partial w_i}{\partial x_{i+1}} = 0$, there has

$$\frac{\partial[f_i(\bar{x}_i, x_{i+1}) + w_i]}{\partial x_{i+1}} \geq b_m > 0 \tag{38}$$

According to Lemma 1, for any x_i and w_i, there exists $x_{i+1} = \alpha_i^*(x_i, w_i)$, so that

$$f_i(\bar{x}_i, \alpha_i^*) + w_i = 0 \tag{39}$$

Similar to step 1, there exists $\mu_i (0 < \mu_i < 1)$ satisfied

$$f_i(\bar{x}_i, x_{i+1}) = f_i(\bar{x}_i, \alpha_i^*) + g_{\mu i}(x_{i+1} - \alpha_i^*) \tag{40}$$

where $g_{\mu i} = g_i(\bar{x}_i, x_{\mu i})$, $x_{\mu i} = \mu_i x_{i+1} + (1 - \mu_i)\alpha_i^*$.

Substituting (39) and (40) into (36), one has:

$$\begin{aligned} LV_i =& LV_{i-1} + \frac{z_i^3}{k_{bi}^4 - z_i^4}[g_{\mu i}(z_{i+1} + \alpha_i - \alpha_i^*) - k_i z_i \\ &- \frac{z_i^3}{2(k_{bi}^4 - z_i^4)} - \frac{\tau_i^{\frac{3}{2}}(3k_{bi}^4 + z_i^4)^{\frac{3}{2}}\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j\right\|^3}{3\sqrt{2}(k_{bi}^4 - z_i^4)^2} \\ &- \frac{3z_i}{4(k_{bi}^4 - z_i^4)^{\frac{1}{3}}} - \frac{1}{4}b_M^4 z_i (k_{bi}^4 - z_i^4)] \\ &+ \frac{z_i^2(3k_{bi}^4 + z_i^4)\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j\right\|^2}{2(k_{bi}^4 - z_i^4)^2} - b_m\tilde{\theta}_i\dot{\hat{\theta}}_i \end{aligned} \tag{41}$$

Applying Assumption 2 and (5) to $\frac{z_i^3 g_{\mu i} z_{i+1}}{k_{bi}^4 - z_i^4}$, one has:

$$\frac{z_i^3 g_{\mu i} z_{i+1}}{k_{bi}^4 - z_i^4} \leq \frac{3z_i^4}{4(k_{bi}^4 - z_i^4)^{\frac{4}{3}}} + \frac{1}{4}b_M^4 z_{i+1}^4 \tag{42}$$

α_i can be designed as

$$\alpha_i = -k_i z_i - \frac{z_i^3 \hat{\theta}_i S_i^T S_i}{2a_i^2(k_{bi}^4 - z_i^4)} \tag{43}$$

where $a_i > 0$.

Substituting (43) into (41), one has:

$$\frac{z_i^3 g_{\mu i}\alpha_i}{k_{bi}^4 - z_i^4} \leq -\frac{b_m z_i^6 \hat{\theta}_2 S_i^T S_i}{2a_i^2(k_{bi}^4 - z_i^4)^2} - \frac{z_i^4 b_m k_i}{k_{bi}^4 - z_i^4} \tag{44}$$

Using fuzzy logic system to approach α_i^*, there are

$$\alpha_i^* = \omega_i^{*T} S_i(z_i) + \varepsilon_i(z_i) \tag{45}$$

where ω_i^* is the unknown optimal parameter and $\varepsilon_i(z_i)$ is the minimum fuzzy approximation error. Suppose $|\varepsilon_i(z_i)| \leq \varepsilon_i^*$ and $\varepsilon_i^* > 0$.

Substituting α_i^* into (41) and using (5), one has:

$$\begin{aligned} -\frac{z_i^3 g_{\mu i}\alpha_i^*}{k_{bi}^4 - z_i^4} \leq& \frac{b_m z_i^6 \theta_2 S_i^T S_i}{2a_i^2(k_{bi}^4 - z_i^4)^2} + \frac{1}{2}a_i^2 \\ &+ \frac{1}{2}b_M^2\varepsilon_i^{*2} + \frac{z_i^6}{2(k_{bi}^4 - z_i^4)^2} \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\frac{z_i^2(3k_{bi}^4 + z_i^4)\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j\right\|^3}{2(k_{bi}^4 - z_i^4)^2} \leq \\ &\frac{1}{3\tau_i^3} + \frac{z_i^3\tau_i^{\frac{3}{2}}(3k_{bi}^4 + z_i^4)^{\frac{3}{2}}\left\|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j\right\|^3}{3\sqrt{2}(k_{bi}^4 - z_i^4)^2} \end{aligned} \tag{47}$$

where τ_i is the positive design parameter.

Substituting (42), (44), (46) and (47) into (41), one has:

$$\begin{aligned} LV_i \leq& LV_{i-1} - \frac{c_i z_i^4}{k_{bi}^4 - z_i^4} + \frac{1}{2}a_i^2 + \frac{1}{2}b_M^2\varepsilon_i^{*2} \\ &+ \frac{1}{3\tau_i^3} + \frac{1}{4}b_M^4 z_{i+1}^4 + b_m\tilde{\theta}_i[\frac{z_i^6 S_i^T Si}{2a_i^2(k_{bi}^4 - z_i^4)^2} - \dot{\hat{\theta}}_i] \\ &- \frac{1}{4}b_M^4 z_i^4 \end{aligned} \tag{48}$$

where $c_i = k_i(1 + b_m)$.

$\dot{\hat{\theta}}_i$ can be constructed as follows:

$$\dot{\hat{\theta}}_i = \frac{z_i^6 S_i^T S_i}{2a_i^2(k_{bi}^4 - z_i^4)^2} - \hat{\theta}_i \tag{49}$$

Substituting (49) into (48), we can get

$$\begin{aligned} LV_i \leq& -\sum_{j=1}^{i}\frac{c_j z_j^4}{k_{bj}^4 - z_j^4} + \frac{1}{2}\sum_{j=1}^{i}a_j^2 + \frac{1}{2}b_M^2\sum_{j=1}^{i}\varepsilon_j^{*2} + \sum_{j=1}^{i}\frac{1}{3\tau_j^3} \\ &+ \frac{1}{4}b_M^4 z_{i+1}^4 + \frac{1}{2}b_m\sum_{j=1}^{i}\theta_j^2 - \frac{1}{2}b_m\sum_{j=1}^{i}\tilde{\theta}_j^2 \end{aligned} \tag{50}$$

Remark 1. *This thesis studies the solution of partial state constraints. The full states are divided into constraint state $\overline{x}_t = [x_1, ..., x_t]^T$ and free state $\overline{x}_m = [x_{t+1}, ..., x_n]^T$. Step 1 - Step t constructs the virtual controller for the constrained state $\overline{x}_t$, and Step t + 1 - Step n constructs the controller of the free state $\overline{x}_m$.*

Step m: According to $z_m = x_m - \alpha_{m-1}, m = t+1, ..., n-1$ and (6), we can get

$$\begin{aligned} dz_m =& [f_m(\bar{x}_m, x_{m+1}) - L\alpha_{m-1}]dt \\ &+ (\varphi_m(x) - \sum_{j=1}^{m-1}\frac{\partial \alpha_{m-1}}{\partial x_j}\varphi_j(x))^T dw \end{aligned} \tag{51}$$

where $L\alpha_{m-1} = \sum_{j=1}^{m-1}\frac{\partial \alpha_{m-1}}{\partial x_j}f_j(\bar{x}_j, x_{j+1}) + \sum_{j=1}^{m-1}\frac{\partial \alpha_{m-1}}{\partial \hat{\theta}_j}\dot{\hat{\theta}}_j + \sum_{j=0}^{m-1}\frac{\partial \alpha_{m-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{1}{2}\sum_{p,q=1}^{m-1}\frac{\partial^2 \alpha_{m-1}}{\partial x_p \partial x_q}\varphi_p^T(x)\varphi_q(x)$.

Construct the Lyapunov function as follows

$$V_m = V_{m-1} + \frac{1}{4}z_m^4 + \frac{1}{2}b_m\tilde{\theta}_m^2 \tag{52}$$

Thus we have

$$\begin{aligned} LV_m =& LV_{m-1} + z_m^3[f_m(\bar{x}_m, x_{m+1}) - L\alpha_{m-1}] \\ &+ \frac{3z_m^2\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial \alpha_{m-1}}{\partial x_j}\varphi_j\right\|^2}{2} - b_m\tilde{\theta}_m\dot{\hat{\theta}}_m \end{aligned} \tag{53}$$

According to Assumption 2, $\frac{\partial f_m(\bar{x}_m,x_{m+1})}{\partial x_{m+1}} \geq b_m > 0$. Let

$$\begin{aligned} w_m =& - L\alpha_{m-1} + k_m z_m + \frac{1}{2}z_m^3 \\ &+ \sqrt{\frac{3}{2}}\tau_m^{\frac{3}{2}}\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial\alpha_{m-1}}{\partial x_j}\varphi_j\right\|^3 \\ &+ \frac{3}{4}z_m + \frac{1}{4}b_M^4 z_m \end{aligned} \tag{54}$$

Because $\frac{\partial w_m}{\partial x_{m+1}} = 0$, there is

$$\frac{\partial[f_m(\bar{x}_m, x_{m+1}) + w_m]}{\partial x_{m+1}} \geq b_m > 0 \tag{55}$$

By Lemma 1, for any x_m and w_m, there exists an ideal smooth control input $x_{m+1} = \alpha_m^*(x_m, w_m)$, so that

$$f_m(\bar{x}_m, {\alpha_m}^*) + w_m = 0 \tag{56}$$

Similar to step 1, there exists $\mu_m (0 < \mu_m < 1)$ satisfied

$$f_m(\bar{x}_m, x_{m+1}) = f_m(\bar{x}_m, {\alpha_m}^*) + g_{\mu m}(x_{m+1} - \alpha_m^*) \tag{57}$$

where $g_{\mu m} = g_m(\bar{x}_m, x_{\mu m})$, $x_{\mu m} = \mu_m x_{m+1} + (1-\mu_m)\alpha_m^*$.

Substituting (56) and (57) into (53), we then have

$$\begin{aligned} LV_m =& LV_{m-1} + z_m^3[g_{\mu m}(z_{m+1} + \alpha_m - \alpha_m^*) - k_m z_m \\ &- \frac{1}{2}z_m^3 - \sqrt{\frac{3}{2}}\tau_m^{\frac{3}{2}}\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial\alpha_{m-1}}{\partial x_j}\varphi_j\right\|^3 \\ &- \frac{3}{4}z_m - \frac{1}{4}b_M^4 z_m] + \frac{3z_m^2\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial\alpha_{m-1}}{\partial x_j}\varphi_j\right\|^2}{2} \\ &- b_m\tilde{\theta}_m\dot{\hat{\theta}}_m \end{aligned} \tag{58}$$

Applying Assumption 2 and (5) to $z_m^3 g_{\mu m} z_{m+1}$, one has:

$$z_m^3 g_{\mu m} z_{m+1} \leq \frac{3}{4}z_m^4 + \frac{1}{4}b_M^4 z_{m+1}^4 \tag{59}$$

α_m can be designed as

$$\alpha_m = -k_m z_m - \frac{z_m^3\hat{\theta}_m S_m^T S_m}{2a_m^2} \tag{60}$$

where a_m is the positive design constant.

Substituting (60) into (58), one has:

$$z_m^3 g_{\mu m}\alpha_m \leq -k_m b_m z_m^4 - \frac{b_m z_m^6\hat{\theta}_m S_m^T S_m}{2a_m^2} \tag{61}$$

Using fuzzy logic system to approach α_m^*, there are

$$\alpha_m^* = \omega_m^{*T}S_m(z_m) + \varepsilon_m(z_m) \tag{62}$$

where ω_m^{*T} is the unknown optimal parameter and ε_m is the minimum fuzzy approximation error. Suppose $|\varepsilon_m| \leq \varepsilon_m^*$ and $\varepsilon_m^* > 0$.

Substituting α_m^* into (58), then

$$\begin{aligned} -z_m^3 g_{\mu m}\alpha_m^* \leq& \frac{b_m z_m^6\theta_m S_m^T S_m}{2a_m^2} + \frac{1}{2}a_m^2 \\ &+ \frac{1}{2}b_M^2\varepsilon_m^{*2} + \frac{1}{2}z_m^6 \end{aligned} \tag{63}$$

By using (5), one has

$$\begin{aligned} &\frac{3z_m^2\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial\alpha_{m-1}}{\partial x_j}\varphi_j\right\|^2}{2} \leq \\ &\frac{1}{3\tau_m^3} + \sqrt{\frac{3}{2}}z_m^3\tau_m^{\frac{3}{2}}\left\|\varphi_m - \sum_{j=1}^{m-1}\frac{\partial\alpha_{m-1}}{\partial x_j}\varphi_j\right\|^3 \end{aligned} \tag{64}$$

where τ_m is the positive design parameter.

Substituting (59), (61), (63) and (64) into (58), one has:

$$\begin{aligned} LV_m \leq& LV_{m-1} - c_m z_m^4 + \frac{1}{2}a_m^2 + \frac{1}{2}b_M^2\varepsilon_m^{*2} + \frac{1}{3\tau_m^3} \\ &+ \frac{1}{4}b_M^4 z_{m+1}^4 + b_m\tilde{\theta}_m[\frac{z_m^6 S_m^T S_m}{2a_m^2} - \dot{\hat{\theta}}_m] \\ &- \frac{1}{4}b_M^4 z_m^4 \end{aligned} \tag{65}$$

where $c_m = k_m(1 + b_m)$.

$\dot{\hat{\theta}}_m$ can be constructed as follows

$$\dot{\hat{\theta}}_m = \frac{z_m^6 S_m^T S_m}{2a_m^2} - \hat{\theta}_m \tag{66}$$

Substituting (66) into (65) and use (5), we can obtain that

$$\begin{aligned} LV_m \leq& -\sum_{j=1}^{t}\frac{c_j z_j^4}{k_{bj}^4 - z_j^4} - \sum_{j=t+1}^{m}c_j z_j^4 + \frac{1}{2}\sum_{j=1}^{m}a_j^2 \\ &+ \frac{1}{2}b_M^2\sum_{j=1}^{m}\varepsilon_j^{*2} + \sum_{j=1}^{m}\frac{1}{3\tau_j^3} + \frac{1}{4}b_M^4 z_{m+1}^4 \\ &+ \frac{1}{2}b_m\sum_{j=1}^{m}\theta_j^2 - \frac{1}{2}b_m\sum_{j=1}^{m}\tilde{\theta}_j^2 \end{aligned} \tag{67}$$

Step n: By $z_n = x_n - \alpha_{n-1}$, we can get

$$\begin{aligned} dz_n =& [f_n(\bar{x}_n, u) - L\alpha_{n-1}]dt \\ &+ (\varphi_n(x) - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j(x))^T dw \end{aligned} \tag{68}$$

where $L\alpha_{n-1} = \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}f_j(\bar{x}_j, x_{j+1}) + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j + \sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T(x)\varphi_q(x)$.

Construct the Lyapunov function as follows:

$$V_n = V_{n-1} + \frac{1}{4}z_n^4 + \frac{1}{2}g_m b_m\tilde{\theta}_n^2 \tag{69}$$

Taking the time derivative of (69) leads to

$$\begin{aligned} LV_n =& LV_{n-1} + z_n^3[f_n(\bar{x}_n, u) - L\alpha_{n-1}] \\ &+ \frac{3z_n^2\left\|\varphi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j\right\|^2}{2} \\ &- g_m b_m\tilde{\theta}_n\dot{\hat{\theta}}_n \end{aligned} \tag{70}$$

According to Assumption 2, $\frac{\partial f_n(\bar{x}_n,u)}{\partial u} \geq b_m > 0$.

Define

$$w_n = - L\alpha_{n-1} + k_n z_n + \frac{1}{2}z_n^3 + \sqrt{\frac{3}{2}}\tau_n^{\frac{3}{2}} \left\| \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\varphi_j \right\|^3 + \frac{1}{2}b_M^2 z_n^3 + \frac{1}{4}b_M^4 z_n \quad (71)$$

Because $\frac{\partial w_n}{\partial u} = 0$, then

$$\frac{\partial[f_n(\bar{x}_n, u) + w_n]}{\partial u} \geq b_m > 0 \quad (72)$$

By Lemma 1, for any x_n and w_n, there exists $u = \alpha_n^*(x_n, w_n)$, so that

$$f_n(\bar{x}_n, \alpha_n^*) + w_n = 0 \quad (73)$$

Similar to step 1, there exists $\mu_n(0 < \mu_n < 1)$ satisfied

$$f_n(\bar{x}_n, u) = f_n(\bar{x}_n, \alpha_n^*) + g_{\mu n}(u - \alpha_n^*) \quad (74)$$

where $g_{\mu n} = g_n(\bar{x}_n, x_{\mu n})$, $x_{\mu n} = \mu_n u + (1 - \mu_n)\alpha_n^*$.

Substituting (73) and (74) into (70), one has:

$$LV_n = LV_{n-1} + z_n^3[g_{\mu n}(u - \alpha_m^*) - k_n z_n - \frac{1}{2}z_n^3 - \sqrt{\frac{3}{2}}\tau_n^{\frac{3}{2}} \left\| \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\varphi_j \right\|^3 - \frac{1}{2}b_M^2 z_n^3 - \frac{1}{4}b_M^4 z_n] + \frac{3z_n^2\left\| \varphi_n - \sum_{j=1}^{n} \frac{\partial \alpha_n}{\partial x_j}\varphi_j \right\|^2}{2} - g_m b_m \tilde{\theta}_n \dot{\theta}_n \quad (75)$$

Substituting (9) and (12) into (75), we have

$$z_n^3 g_{\mu n} u = z_n^3 g_{\mu n}\rho(v) + z_n^3 g_{\mu n} g_{vu} v \quad (76)$$

Using (5), one has

$$z_n^3 g_{\mu n}\rho(v) \leq \frac{1}{2}b_M^2 z_n^6 + \frac{1}{2}D^2 \quad (77)$$

The ideal control signal v can be designed as

$$v = \alpha_n = -k_n z_n - \frac{z_n^3 \hat{\theta}_n S_n^T S_n}{2a_n^2} \quad (78)$$

where a_n is the positive design constant.

Substituting (78) into (76), one has:

$$z_n^3 g_{\mu n} g_{vu} v \leq -k_n g_m b_m z_n^4 - \frac{g_m b_m z_n^6 \hat{\theta}_n S_n^T S_n}{2a_n^2} \quad (79)$$

Using a fuzzy logic system to approach α_n^*, we have

$$\alpha_n^* = \omega_n^{*T} S_n(z_n) + \varepsilon_n(z_n) \quad (80)$$

where ω_n^{*T} is the unknown optimal parameter and ε_n is the minimum fuzzy approximation error. Suppose $|\varepsilon_n| \leq \varepsilon_n^*$ and $\varepsilon_n^* > 0$.

Substituting α_n^* into (75) yields

$$-z_n^3 g_{\mu n}\alpha_n^* \leq \frac{g_m b_m z_n^6 \theta_n S_n^T S_n}{2a_n^2} + \frac{1}{2}a_n^2 + \frac{1}{2}b_M^2 \varepsilon_n^{*2} + \frac{1}{2}z_n^6 \quad (81)$$

Using (5), one has

$$\frac{3z_n^2\left\| \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\varphi_j \right\|^2}{2} \leq \frac{1}{3\tau_n^3} + \sqrt{\frac{3}{2}} z_n^3 \tau_n^{\frac{3}{2}} \left\| \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\varphi_j \right\|^3 \quad (82)$$

where τ_n is the positive design parameter.

Substituting (77), (79), (81) and (82) into (75), one has:

$$LV_n \leq LV_{n-1} - c_n z_n^4 + \frac{1}{2}a_n^2 + \frac{1}{2}b_M^2 \varepsilon_n^{*2} + \frac{1}{3\tau_n^3} + g_m b_m \tilde{\theta}_n[\frac{z_n^6 S_n^T S_n}{2a_n^2} - \dot{\hat{\theta}}_n] - \frac{1}{4}b_M^4 z_n^4 + \frac{1}{2}D^2 \quad (83)$$

where $c_n = k_n(g_m b_m + 1)$.

$\dot{\hat{\theta}}_n$ can be constructed as follows:

$$\dot{\hat{\theta}}_n = \frac{z_n^6 S_n^T S_n}{2a_n^2} - \hat{\theta}_n \quad (84)$$

Substituting (84) into (83) yields

$$LV_n \leq -\sum_{j=1}^{c} \frac{c_j z_j^4}{k_{bj}^4 - z_j^4} - \sum_{j=c+1}^{n} c_j z_j^4 + \frac{1}{2}\sum_{j=1}^{n} a_j^2 + \frac{1}{2}b_M^2 \sum_{j=1}^{n} \varepsilon_j^{*2} + \sum_{j=1}^{n} \frac{1}{3\tau_j^3} + \frac{1}{2}b_m \sum_{j=1}^{n-1} \theta_j^2 - \frac{1}{2}b_m \sum_{j=1}^{n-1} \tilde{\theta}_j^2 + \frac{1}{2}D^2 + \frac{1}{2}g_m b_m \theta_n^2 - \frac{1}{2}g_m b_m \tilde{\theta}_n^2 \quad (85)$$

According to Lemma 2 :

$$-\frac{c_i z_i^4}{k_{bi}^4 - z_i^4} \leq -c_i \log \frac{k_{bi}^4}{k_{bi}^4 - z_i^4} \quad (86)$$

Substituting (86) into (85), we have

$$LV_n \leq -\sum_{j=1}^{c} c_i \log \frac{k_{bi}^4}{k_{bi}^4 - z_i^4} - \sum_{j=c+1}^{n} c_j z_j^4 + \frac{1}{2}\sum_{j=1}^{n} a_j^2 + \frac{1}{2}b_M^2 \sum_{j=1}^{n} \varepsilon_j^{*2} + \sum_{j=1}^{n} \frac{1}{3\tau_j^3} + \frac{1}{2}b_m \sum_{j=1}^{n-1} \theta_j^2 - \frac{1}{2}b_m \sum_{j=1}^{n-1} \tilde{\theta}_j^2 + \frac{1}{2}D^2 + \frac{1}{2}g_m b_m \theta_n^2 - \frac{1}{2}g_m b_m \tilde{\theta}_n^2 \quad (87)$$

The following definitions are given:

$$\psi = \min\{4c_i, g_m, i = 1, ..., n\} \quad (88)$$

$$\lambda = \frac{1}{2}\sum_{j=1}^{n} a_j^2 + \frac{1}{2}b_M^2 \sum_{j=1}^{n} \varepsilon_j^{*2} + \sum_{j=1}^{n} \frac{1}{3\tau_j^3} + \frac{1}{2}b_m \sum_{j=1}^{n-1} \theta_j^2 + \frac{1}{2}D^2 + \frac{1}{2}g_m b_m \theta_n^2 \quad (89)$$

Combining (69), (87), (88) and (89), it leads to

$$LV \leq -\psi V + \lambda, t \geq 0 \quad (90)$$

IV. STABILITY ANALYSIS

Theorem 1. *For the system (6), based on Assumptions 1-3, the virtual controller is designed as shown in (26), (43), and (60), the controller is designed as shown in (78), and the adaptive rate is shown in (32), (49), (66), and (84).*

Then the designed adaptive controller can guarantee that:

1) All signals in the closed-loop system are bounded;

2) The constrained state is within the constraint boundary;

3) The output signal of the system can effectively track the desired signal.

Proof. *Based on (90), one has:*

$$\frac{d(E[v_n])}{dt} = E[LV_n] \le -\psi E[v_n] + \lambda \tag{91}$$

Defining $E[v_n] = l$ *and* $\psi > \frac{\lambda}{l}$, *can get* $\frac{d(E[v_n])}{dt} \le 0$. *For all* $t \ge 0$, *when* $E[v_n(0)] \le l$, *it can be obtained from Lemma 3 as follows*

$$0 \le E[v_n(t)] \le v_n(0)e^{-\psi t} + \frac{\lambda}{\psi}, \forall t \ge 0 \tag{92}$$

From (92), it can be seen that $\log \frac{k_{bi}^4}{k_{bi}^4 - z_i^4}$ *and* $\tilde{\theta}$ *are bounded, so* z_i *is bounded. Because* θ_i *is a constant,* $\hat{\theta}_i$ *is also bounded. According to the definition of* α_i, *it can be concluded that* α_i *is also bounded and* $|\alpha_i| \le \tilde{\alpha}_i$,$\tilde{\alpha}_i$ *are positive constants. In addition, because* $x_1 = z_1 + y_d$, $x_i = z_i + \alpha_{i-1}, i = 2, ..., n$, *and* y_d *is bounded, it can be concluded that* $x_i, i = 1, ..., n$ *is bounded. According to the above analysis, all signals are bounded.*

Using $x_1 = z_1 + y_d, |y_d| \le Y_0$, *we have* $|x_1| = |z_1 + y_d| \le |z_1| + |y_d| < k_{b1} + Y_0$. *Set the parameter* $k_{b1} = k_1 - Y_0$, *there is* $|x_1| < k_1$. *From* $x_2 = z_2 + \alpha_1$, *it is concluded that* $|x_2| \le |z_2| + |\alpha_1| < k_{b2} + \tilde{\alpha}_1$. *If choose* $k_{b2} = k_2 - \tilde{\alpha}_1$, *there exists* $|x_2| < k_2$. *Recursively, we can get* $|x_i| < k_i, i = 3, ..., c$, *so the system state will not be violated.*

From (92), there are $\log \frac{k_{b1}^4}{k_{b1}^4 - z_1^4} \le 4v_n(0)e^{-\psi t} + 4\frac{\lambda}{\psi}$, $\frac{k_{b1}^4}{k_{b1}^4 - z_1^4} \le e^{4v_n(0)e^{-\psi t} + 4\frac{\lambda}{\psi}}$ *can be obtained through transformation, and we can further get* $|z_1| \le k_{b1}\sqrt[4]{1 - e^{-4v_n(0) - 4\frac{\lambda}{\psi}}}$.

V. SIMULATION EXAMPLE

The following simulation examples will illustrate the application of the provided control scheme. Consider the nonlinear system as follows

$$\begin{cases} dx_1 = [(1 + x_1^2)x_2 + 0.5x_2^3]dt + x_1^2 \sin(x_2)dw \\ dx_2 = (x_1^2 x_2 + \frac{1}{5}u^3 + u)dt + [1 + \sin(x_1^2)]x_2 dw \\ y = x_1 \end{cases} \tag{93}$$

The tracking signal is selected as follows

$$y_d = \sin(t) + 0.5\sin(0.5t) \tag{94}$$

The system has the nonlinear property of input saturation, in which the saturated input model is

$$u = sat(v) = \begin{cases} 5, v \ge 5 \\ v, -10 \le v \le 5 \\ -10, v \le -10 \end{cases} \tag{95}$$

The initial conditions are $[x_1(0), x_2(0)]^T = [0.3, 0.2]^T, \hat{\theta}_1(0) = \hat{\theta}_2(0) = 0, a_1 = 0.02, a_2 = 50, k_{b1} = 1, k_1 = 15, k_2 = 2$, and the simulation results are shown in Figure 1-4.

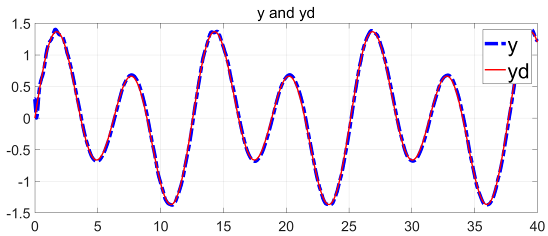

Fig. 1: System output y and reference signal y_d

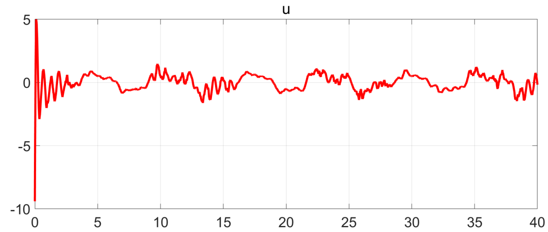

Fig. 2: Control signal u

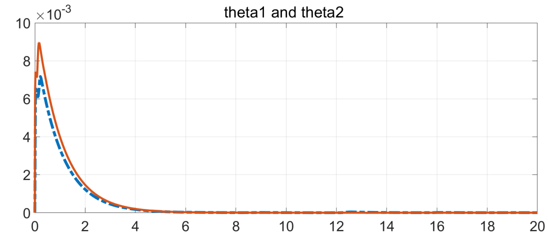

Fig. 3: Adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$

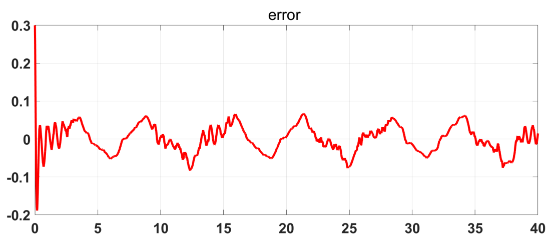

Fig. 4: Tracking error

Figure 1 shows that under the action of the designed controller, the output signal can well track the given expected signal. The trajectory of the control signal is described in Figure 2. The trajectory of adaptive parameters are shown in Figure 3, it shows that $\hat{\theta}_1$ and $\hat{\theta}_2$ are ultimately bounded. The trajectory of the tracking error is shown in Figure 4, it fluctuates between 0.1 and -0.1. From the simulation results, the designed controller can ensure that the closed-loop system has good tracking performance.

VI. CONCLUSION

The problem of adaptive tracking control for pure feedback nonlinear systems with input saturation and partial state constraints is studied. In order to facilitate the research, the implicit function theorem and mean value theorem are used to transform the pure feedback system. Then, the FLS is used to approach the unknown functions in the system, and the controller of the system is designed using the backstepping technique. Here, the BLF can make the constrained local s-tate converge to the constraint boundary. Through simulation, the provided method can make the system control achieve the desired effect.

REFERENCES

[1] J. Na, J. Yang, S. Wang, G. Gao, and C. Yang, "Unknown dynamics estimator-based output-feedback control for nonlinear pure-feedback systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 6, pp. 3832–3843, 2019.

[2] F. Jia, J. Lu, and Y. Li, "Output regulation problem of a class of pure-feedback nonlinear systems via adaptive neural control," *Journal of the Franklin Institute*, vol. 358, no. 11, pp. 5659–5675, 2021.

[3] N. Wang, Z. Fu, F. Tao, S. Song, and T. Wang, "A simplified adaptive tracking control for nonlinear pure-feedback systems with input delay and full-state constraints," *International Journal of Adaptive Control and Signal Processing*, vol. 35, no. 12, pp. 2521–2536, 2021.

[4] A.-M. Zou, Z.-G. Hou, and M. Tan, "Adaptive control of a class of nonlinear pure-feedback systems using fuzzy backstepping approach," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 4, pp. 886–897, 2008.

[5] S. Sui, S. Tong, and Y. Li, "Adaptive fuzzy backstepping output feedback tracking control of mimo stochastic pure-feedback nonlinear systems with input saturation," *Fuzzy Sets and Systems*, vol. 254, pp. 26–46, 2014.

[6] S. Tong, Y. Li, and Y. Liu, "Adaptive fuzzy output feedback decentralized control of pure-feedback nonlinear large-scale systems," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 5, pp. 930–954, 2014.

[7] H. Wang, B. Chen, and C. Lin, "Adaptive fuzzy control for pure-feedback stochastic nonlinear systems with unknown dead-zone input," *International Journal of Systems Science*, vol. 45, no. 12, pp. 2552–2564, 2014.

[8] H. Su and W. Zhang, "Adaptive fuzzy control for pure-feedback stochastic nonlinear systems with unknown dead zone outputs," *International Journal of Systems Science*, vol. 49, no. 14, pp. 2981–2995, 2018.

[9] S. Tong, Y. Xu, and Y. Li, "Adaptive fuzzy decentralised control for stochastic nonlinear large-scale systems in pure-feedback form," *International Journal of Systems Science*, vol. 46, no. 8, pp. 1510–1524, 2015.

[10] H. Wang, X. Liu, and K. Liu, "Adaptive fuzzy tracking control for a class of pure-feedback stochastic nonlinear systems with non-lower triangular structure," *Fuzzy Sets and Systems*, vol. 302, pp. 101–120, 2016.

[11] H. Wang, B. Chen, X. Liu, K. Liu, and C. Lin, "Robust adaptive fuzzy tracking control for pure-feedback stochastic nonlinear systems with input constraints," *IEEE Transactions on Cybernetics*, vol. 43, no. 6, pp. 2093–2104, 2013.

[12] T. Wang, M. Ma, J. Qiu, and H. Gao, "Event-triggered adaptive fuzzy tracking control for pure-feedback stochastic nonlinear systems with multiple constraints," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 6, pp. 1496–1506, 2020.

[13] T. Yoshimura, "Adaptive fuzzy dynamic surface control for uncertain nonlinear systems in pure-feedback form with input and state constraints using noisy measurements," *International Journal of Systems Science*, vol. 50, no. 1, pp. 104–115, 2019.

[14] Z. Li, T. Li, G. Feng, R. Zhao, and Q. Shan, "Neural network-based adaptive control for pure-feedback stochastic nonlinear systems with time-varying delays and dead-zone input," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 50, no. 12, pp. 5317–5329, 2018.

[15] S. S. Ge and C. Wang, "Adaptive nn control of uncertain nonlinear pure-feedback systems," *Automatica*, vol. 38, no. 4, pp. 671–682, 2002.

[16] Y. Li, J. Zhang, W. Liu, and S. Tong, "Observer-based adaptive optimized control for stochastic nonlinear systems with input and state constraints," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 33, no. 12, pp. 7791-7805, 2021.

[17] Y. Ma, N. Zhao, X. Ouyang, H. Xu, and Y. Zhou, "Adaptive fuzzy prescribed performance control for strict-feedback stochastic nonlinear system with input constraint." *Engineering Letters*, vol. 29, no. 2, pp. 650–657, 2021.

[18] Z. Song, W. Fang, X. Liu, and A. Lu, "Adaptive fuzzy control for a class of mimo nonlinear systems with bounded control inputs." *Engineering Letters*, vol. 28, no. 3, pp. 820–826, 2020.

[19] Y.-J. Liu, S. Lu, S. Tong, X. Chen, C. P. Chen, and D.-J. Li, "Adaptive control-based barrier lyapunov functions for a class of stochastic nonlinear systems with full state constraints," *Automatica*, vol. 87, pp. 83–93, 2018.

[20] K. P. Tee, S. S. Ge, and E. H. Tay, "Barrier lyapunov functions for the control of output-constrained nonlinear systems," *Automatica*, vol. 45, no. 4, pp. 918–927, 2009.

[21] G. Zong, Y. Wang, H. R. Karimi, and K. Shi, "Observer-based adaptive neural tracking control for a class of nonlinear systems with prescribed performance and input dead-zone constraints," *Neural Networks*, vol. 147, pp. 126–135, 2022.

[22] J. Wu, W. Sun, S.-F. Su, and J. Xia, "Neural-based adaptive control for nonlinear systems with quantized input and the output constraint," *Applied Mathematics and Computation*, vol. 413, pp. 126637, 2022.

[23] M. M. Zirkohi, "Adaptive backstepping control design for mems gyroscope based on function approximation techniques with input saturation and output constraints," *Computers & Electrical Engineering*, vol. 97, pp. 107547, 2022.

[24] D. Ye, K. Wang, H. Yang, and X. Zhao, "Integral barrier lyapunov function-based adaptive fuzzy output feedback control for nonlinear delayed systems with time-varying full-state constraints," *International Journal of Adaptive Control and Signal Processing*, vol. 34, no. 11, pp. 1677–1696, 2020.

[25] K. Wang, X. Liu, and Y. Jing, "Adaptive finite-time command filtered controller design for nonlinear systems with output constraints and input nonlinearities," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 33, no. 11, pp. 6893–6904, 2022.

[26] K. P. Tee and S. S. Ge, "Control of nonlinear systems with partial state constraints using a barrier lyapunov function," *International Journal of Control*, vol. 84, no. 12, pp. 2008–2023, 2011.

[27] Y.-Q. Han, "Adaptive control of a class of stochastic nonlinear systems with full state constraints and input saturation using multi-dimensional taylor network," *Asian Journal of Control*, vol. 24, no. 4, pp. 1609–1621, 2022.

[28] W. Chen, L. Jiao, J. Li, and R. Li, "Adaptive nn backstepping output-feedback control for stochastic nonlinear strict-feedback systems with time-varying delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 40, no. 3, pp. 939–950, 2009.

[29] H. Deng and M. Krstic, "Output-feedback stochastic nonlinear stabilization," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 328–333, 1999.

[30] X. Tian and Z. Yang, "Adaptive stabilization of fractional-order nonlinear system considering input saturation phenomenon," *Engineering Letters*, vol. 27, no. 4, pp. 876–881, 2019.