Cross-diffusion-driven Instability and Pattern Formation in a Nonlinear Predator-prey System

Wenbin Yang, Yimamu Maimaiti

***Abstract*—In this paper, a spatial predator-prey model with an alternative food sources and cross-diffusion is studied. We show that the self-diffusion can not induce a Turing Instability theoretically, but the cross-diffusion can. Moreover, the one-dimensional morphological spatial pattern is characterized, and the effects of parameter μ or ρ on pattern formation are discussed numerically.**

***Index Terms*—reaction-diffusion equations, predator-prey model, pattern dynamics, cross-diffusion.**

I. Introduction

MANY of the most interesting dynamics in biology are related to the interactions between species, and the spatio-temporal dynamics of a predator-prey system has been investigated by many researchers [1]–[4]. For the predator population without any alternative source, the general Lotka-Volterra predator-prey system can be described as follows:

$$\begin{cases} \frac{du}{dt} = \gamma u(1-\frac{u}{K}) - \alpha uv, \\ \frac{dv}{dt} = \rho uv - \beta v, \end{cases} \tag{1}$$

where u and v are the biomass of the prey and predator at any time t, respectively; γ is the intrinsic growth rate, k is the environmental carrying capacity of the prey species, α is the predation rate, $\rho(<1)$ is called the conversion rate of prey to predator biomass, and β is the natural death rate of the predator.

Ghosh and Kar [5] have considered a predator-prey ODE model having some alternative source to predator, as follows:

$$\begin{cases} \frac{du}{dt} = \gamma u(1-\frac{u}{K}) - \alpha uv, \\ \frac{dv}{dt} = \mu v(1-\frac{u}{K}) + \rho uv - \beta v, \end{cases} \tag{2}$$

where μ is the maximum growth rate due to alternative source for food. *The term $(1-u/K)$ adds a density-dependent effect to the focal prey, and it is observed that as the focal prey population u increases, the predator uses less amount of alternative source and the consumption of alternative source tends to zero when u approaches K.* [5] showed that alternative source of food to the predator hurts the growth of the prey species.

Recently, much attention has been focused on the Turing instability of the predator-prey model by taking into account the effect of cross-diffusion [6], [7]. Cross-diffusion, the phenomenon in which a gradient in the concentration of one species induces a flux of another chemical species, has generally been neglected in the study of reaction-diffusion systems [8]–[14]. Now, we consider the Lotka-Volterra predator-prey model (2) with cross-diffusion and an alternative source of food for the predator:

$$\begin{cases} \frac{\partial u}{\partial t} = d_{11}\frac{\partial^2 u}{\partial x^2} + \gamma u(1-\frac{u}{K}) - \alpha uv, \\ \frac{\partial v}{\partial t} = d_{21}\frac{\partial^2 u}{\partial x^2} + d_{22}\frac{\partial^2 v}{\partial x^2} + \mu v(1-\frac{u}{K}) + \rho uv - \beta v, \end{cases} \tag{3}$$

where $x \in (0,l)$, d_{11} and d_{22} are diffusion coefficients of the prey and predator population, d_{21} is cross-diffusion coefficient of the predator population, and $l > 0$ is a positive constant.

The paper is organized as follows. In section II, we will derive the sufficient conditions of the asymptotic stability and Turing Instability of our proposed models. The one-dimensional morphological spatial pattern will be characterized, and the effects of parameter μ or ρ on pattern formation will be discussed in section III.

II. Mathematical analysis

In this section, we will consider the asymptotic stability and Turing instability of our proposed models. First, it is obvious that system (1) has the following nonnegative constant solutions:

1) $e_1 = (0,0)$;
2) $e_2 = (K,0)$;
3) $e_2 = (u_*, v_*) = (\frac{\beta}{\rho}, \frac{\gamma(\rho K-\beta)}{\alpha\beta k})$, if $\rho K > \beta$.

Accordingly, system (2), and thus system (3) has the following nonnegative constant solutions:

1) $U_1 = (0,0)$;
2) $U_2 = (K,0)$;
3) $U_3 = (0,c)$, where c is an arbitrary positive constant, if $\mu \neq \beta$;
4) $U_4 = (u^*, v^*)$, where
$$u^* \equiv u(\mu) = K(\frac{\beta-\mu}{\rho K-\mu}),$$
$$v^* \equiv v(\mu) = (\frac{\gamma}{\alpha})(\frac{\rho K-\beta}{\rho K-\mu}),$$
if either $\mu > \beta > \rho K$ or $\mu < \beta < \rho K$ holds.

Theorem 1. *i) For system (1):*

- *e_1 is a unstable saddle point.*
- *If $\beta > \rho K$ ($\beta < \rho K$), then e_2 is a asymptotically stable node (a unstable saddle point).*
- *Assume the $\rho K > \beta$. If $\frac{\beta^2\gamma}{\rho^2 K} > 4(\rho K - \beta)$, then e_3 is a asymptotically stable node; if $\frac{\beta^2\gamma}{\rho^2 K} < 4(\rho K - \beta)$, then e_3 is a asymptotically stable spiral point.*

ii) For system (2):

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- *If $\mu > \beta$ ($\mu < \beta$), then U_1 is a unstable node (a unstable saddle point).*
- *If $\beta > \rho K$ ($\beta < \rho K$), then U_2 is a asymptotically stable node (a unstable saddle point).*
- *If $\gamma < \alpha c$ and $\mu < \beta$ ($\gamma > \alpha c$ and $\mu > \beta$), then U_3 is a asymptotically stable node (unstable); if $(\gamma - \alpha c)(\mu - \beta) < 0$, then U_3 is a unstable saddle point.*
- *If $\mu > \beta > \rho K$, then U_4 is unstable. Therefore, variation of the growth rate μ due to alternative source never stabilizes the system (2). If $\mu < \beta < \rho K$, U_4 is stable.*

iii) For system (3):

- *If $\mu < \beta < \rho K$ and $d_{21} = 0$, then U_4 is still stable.*
- *If $\mu < \beta < \rho K$, then a necessary condition for the emergence of cross-diffusion instability is $d_{21} \geq \frac{(2a_{11}-1)-\sqrt{1+8(2Det(J)+\sqrt{Det(J)})}}{2a_{12}}$.*

The results show that, self-diffusion can not induce Turing instability, but cross-diffusion may lead to Turing instability.

Proof: We only consider cases ii) and iii), since a similar proof of case i) can be made in more straightforward way.

ii) For system (2), the Jacobian matrix at some a equilibrium is

$$J(U) = \begin{pmatrix} \gamma(1-\frac{2u}{K}) - \alpha v & -\alpha u \\ v(\rho - \frac{\mu}{K}) & \mu(1-\frac{u}{K}) + \rho u - \beta \end{pmatrix}. \quad (4)$$

At the equilibria U_1, U_2 and U_3, the corresponding Jacobian matrix can be respectively, calculated as

$$J(U_1) = \begin{pmatrix} \gamma & 0 \\ 0 & \mu - \beta \end{pmatrix}, \quad J(U_2) = \begin{pmatrix} -\gamma & -\alpha K \\ 0 & \rho K - \beta \end{pmatrix},$$

$$J(U_3) = \begin{pmatrix} \gamma - \alpha c & 0 \\ c(\rho - \frac{\mu}{K}) & \mu - \beta \end{pmatrix}.$$

By the linear stability theory, we obtain that

- If $\mu > \beta$, then U_1 is a unstable node; if $\mu < \beta$, then U_1 is a unstable saddle point.
- If $\beta > \rho K$, then U_2 is a asymptotically stable node; if $\beta < \rho K$, then U_2 is a unstable saddle point.
- If $\gamma < \alpha c$ and $\mu < \beta$, then U_3 is a asymptotically stable node; if $\gamma > \alpha c$ and $\mu > \beta$, then U_3 is unstable; If $(\gamma - \alpha c)(\mu - \beta) < 0$, then U_3 is a unstable saddle point.

At the equilibrium U_4, the Jacobian matrix is

$$J \equiv J(U_4) = \begin{pmatrix} -\frac{\gamma u^*}{K} & -\alpha u^* \\ v^*(\rho - \frac{\mu}{K}) & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

since $\gamma(1-\frac{2u}{K}) - \alpha\rho = [\gamma(1-\frac{u}{K}) - \alpha\rho] - \frac{\gamma u}{K} = -\frac{\gamma u}{K}$ and $\mu(1-\frac{u}{K}) + \rho u - \beta = 0$ at the equilibrium U_4.

If $\rho K > \mu$, then

$$a_{11} < 0, \quad a_{12} < 0, \quad a_{21} > 0, \quad a_{22} = 0.$$

Now the first principle diagonal minor is $-\frac{\gamma u}{K} < 0$, the second principle diagonal minor is

$$Det\begin{pmatrix} -\frac{\gamma u^*}{K} & -\alpha u^* \\ v^*(\rho - \frac{\mu}{K}) & 0 \end{pmatrix} = \alpha u^* v^* (\rho - \frac{\mu}{K}).$$

The second principle diagonal minor would be positive if and only if $\rho K > \mu$. Hence both the eigenvalues of the Jacobian matrix have negative real part. Now we can demand that the system is stable if $\rho K > \mu$, and unstable if $\rho K < \mu$.

iii) Let us define

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where $d_{12} = 0$ and

$$M_k \equiv J - k^2 D = \begin{pmatrix} -k^2 d_{11} + a_{11} & -k^2 d_{12} + a_{12} \\ -k^2 d_{21} + a_{21} & -k^2 d_{22} + a_{22} \end{pmatrix},$$

where $k = 0, 1, 2, \cdots$. Then

$$\lambda I - M_k = \begin{pmatrix} \lambda + k^2 d_{11} - a_{11} & k^2 d_{12} - a_{12} \\ k^2 d_{21} - a_{21} & \lambda + k^2 d_{22} - a_{22} \end{pmatrix},$$

where $k = 0, 1, 2, \cdots$. By some calculations, we obtain

$$Trace(M_k) = Trace(J) - k^2(d_{11} + d_{22}),$$
$$Det(M_k) = k^4(d_{11}d_{22} - d_{12}d_{21}) + k^2(-d_{22}a_{11} + d_{21}a_{12} + d_{12}a_{21} - d_{11}a_{22}) + Det(J),$$

and

$$Det(\lambda I - M_k) = \lambda^2 + [k^2(d_{11} + d_{22}) - (a_{11} + a_{22})]\lambda + (k^2 d_{11} - a_{11})(k^2 d_{22} - a_{22}) - (k^2 d_{21} - a_{21})(k^2 d_{12} - a_{12}).$$

Since $Trace(J) < 0$, $Trace(M_k) < 0$ is always true since $d_{11} > 0, d_{22} > 0$. Hence if M_k has an eigenvalue with positive real part, then it must be a real value one and the other eigenvalue must be a negative real one. A necessary condition is

$$H \equiv -d_{22}a_{11} + d_{21}a_{12} + d_{12}a_{21} - d_{11}a_{22} < 0,$$

if $Det(D) > 0$ and $Det(J) > 0$.

Notice that $Det(M_k)$ achieves its minimum

$$\min_{k \in \mathbb{R}^+} Det(M_k) = -\frac{H^2}{4(d_{11}d_{22} - d_{12}d_{21})} + Det(J)$$

at the critical value $k_* > 0$ where

$$k_*^2 = -\frac{H}{2(d_{11}d_{22} - d_{12}d_{21})}.$$

Since

$$d_{12} = 0, \quad a_{11} < 0, \quad a_{12} < 0, \quad a_{21} > 0, \quad a_{22} = 0,$$

the necessary condition for cross-diffusion driven instability of (u_*, v_*) is given by

$$\min_{k \in \mathbb{R}^+} Det(M_k) = -\frac{H^2}{4(d_{11}d_{22} - d_{12}d_{21})} + Det(J) < 0, \quad (5)$$

and

$$k_*^2 = -\frac{H}{2(d_{11}d_{22} - d_{12}d_{21})} > 0. \quad (6)$$

By simple calculations,

$$(5) \Rightarrow -d_{22}a_{11} + d_{21}a_{12} < 0$$
$$\Rightarrow d_{22}\frac{\gamma u_*}{k} - d_{21}\alpha u_* < 0 \Rightarrow d_{21} > \frac{\gamma d_{22}}{k\alpha},$$

$$
\begin{aligned}
(6) &\Rightarrow -\frac{(-d_{22}a_{11}+d_{21}a_{12})^2}{4d_{11}d_{22}} + Det(J) < 0 \\
&\Rightarrow -\frac{(d_{22}\frac{\gamma u_*}{k} - d_{21}\alpha u_*)^2}{4d_{11}d_{22}} + \alpha u_* v_*(\rho - \frac{\mu}{k}) < 0 \\
&\Rightarrow \alpha v_*(\rho - \frac{\mu}{k}) > \frac{(d_{22}\frac{\gamma}{k} - d_{21}\alpha)^2 u_*}{4d_{11}d_{22}} \\
&\Rightarrow 4\alpha k d_{11}d_{22}\frac{(\rho k - \mu)v_*}{u_*} > (\gamma d_{22} - \alpha k d_{21})^2 \\
&\Rightarrow \sqrt{4\alpha k d_{11}d_{22}\frac{(\rho k - \mu)v_*}{u_*}} > \alpha k d_{21} - \gamma d_{22} \\
&\Rightarrow d_{21} < \sqrt{4d_{11}d_{22}\frac{(\rho k - \mu)v_*}{\alpha k u_*}} + \frac{\gamma d_{22}}{\alpha k}.
\end{aligned}
$$

To sum up, we have

$$
\frac{\gamma d_{22}}{\alpha k} < d_{21} < \sqrt{4d_{11}d_{22}\frac{(\rho k - \mu)v_*}{\alpha k u_*}} + \frac{\gamma d_{22}}{\alpha k}.
$$

By the necessary conditions for instability above, let $(k^-)^2$ and $(k^+)^2$ be the two roots of $Det(M_k)$. They can be calculated as

$$
(k^{\pm})^2 = \frac{d_{22}a_{11} - d_{21}a_{12} \pm \Lambda_1}{2d_{11}d_{22}}, \tag{7}
$$

where $\Lambda_1 = \sqrt{(d_{22}a_{11} - d_{21}a_{12})^2 - 4d_{11}d_{22}Det(J)}$. From (6), we deduce that

$$
0 < (k^-)^2 < (k^+)^2.
$$

Thus, in order to get the instability of (u_*, v_*), we must have $(k^-)^2 < k^2 < (k^+)^2$ for some positive integer k, and a necessary condition is $k^+ - k^- \geq 1$, which results in

$$
\begin{aligned}
&2[(d_{22}a_{11} - d_{21}a_{12})^2 - 4d_{11}d_{22}Det(J)] \\
&\geq 2d_{11}d_{22} * (d_{22}a_{11} - d_{21}a_{12} + \sqrt{\Lambda_2}) \\
&= 2d_{11}d_{22} * (d_{22}a_{11} - d_{21}a_{12} + 2\sqrt{d_{11}d_{22}Det(J)}),
\end{aligned}
$$

where $\Lambda_2 = (d_{22}a_{11} - d_{21}a_{12})^2 - [(d_{22}a_{11} - d_{21}a_{12})^2 - 4d_{11}d_{22}Det(J)]$. Let $d_{11} = d_{22} = 1$. Then $2[(a_{11} - d_{21}a_{12})^2 - 4Det(J)] \geq 2(a_{11} - d_{21}a_{12} + 2\sqrt{Det(J)})$, and we have

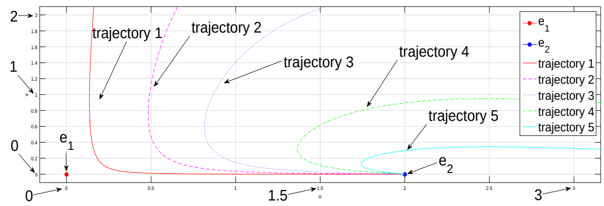
$$
\begin{aligned}
&(a_{11} - d_{21}a_{12})^2 - 4Det(J) \\
&\geq a_{11} - d_{21}a_{12} + 2\sqrt{Det(J)} \\
&\Leftrightarrow a_{12}^2 \cdot d_{21}^2 + a_{12}(1 - 2a_{11}) \cdot d_{21} \\
&\quad + (a_{11}^2 - a_{11} - 4Det(J) - 2\sqrt{Det(J)}) \geq 0 \\
&\Rightarrow d_{21} \geq \tfrac{a_{12}(2a_{11}-1)+\sqrt{\Lambda_3}}{2a_{12}^2} \\
&\Rightarrow d_{21} \geq \tfrac{a_{12}(2a_{11}-1)-a_{12}\sqrt{1+8(2Det(J)+\sqrt{Det(J)})}}{2a_{12}^2} \\
&\Rightarrow d_{21} \geq \tfrac{(2a_{11}-1)-\sqrt{1+8(2Det(J)+\sqrt{Det(J)})}}{2a_{12}},
\end{aligned}
$$

where $\Lambda_3 = a_{12}^2(1-2a_{11})^2 - 4a_{12}^2(a_{11}^2 - a_{11} - 4Det(J) - 2\sqrt{Det(J)})$. This concludes the proof of the theorem. ■

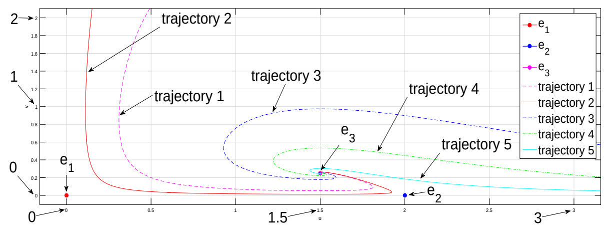
III. Numerical simulation

In this section, we using numerical simulations methods illustrate to verify our theoretical findings. To illustrate the results given by Theorem 2.1, we choose parameters $K = 2$, $\gamma = 1$, $\alpha = 1$ and $\rho = 1$.

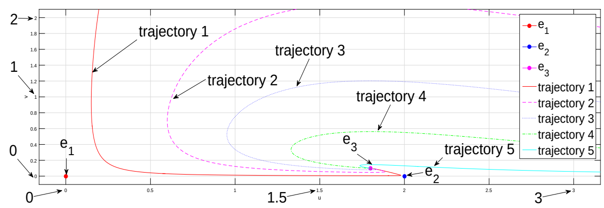
We plot the phase portraits of system (1) for different β in Fig.1. The equilibrium point e_1 is unstable saddle point and the equilibrium e_2 is a asymptotically stable node when $\beta = 2.5$ (see Fig.1(a)). The equilibrium e_3 is a asymptotically stable node and e_1, e_2 are unstable saddle points when $\beta = 1.5$ (see Fig.1(b)). The equilibrium e_3 is a asymptotically stable spiral point and e_1, e_2 are unstable saddle points when $\beta = 1.5$ (see Fig.1(c)).



(a)



(b)

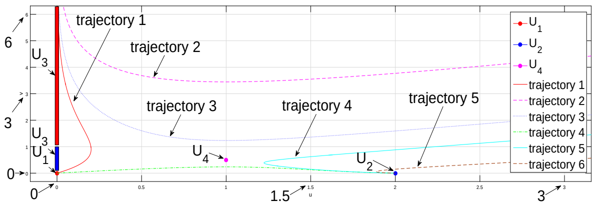


(c)

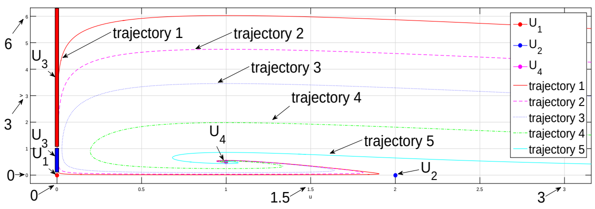
Fig. 1. The phase portraits of system (1) with fixed $K = 2$, $\gamma = 1$, $\alpha = 1$ and $\rho = 1$. e_i $(i = 1, 2, 3)$ represents the constant equilibrium. (a): $\beta = 2.5$; (b): $\beta = 1.5$; (c): $\beta = 1.8$.

In Fig.2, we plot the phase portraits of system (2). The equilibrium point U_1 is unstable node when $\mu > \beta$ (see Fig.2(a, c)) or unstable saddle point when $\mu < \beta$ (see Fig.2 (b, d)). The equilibrium U_2 is a asymptotically stable node when $\beta > 2$ (see Fig.2(a, d)) and unstable saddle point when $\beta < 2$ (see Fig.2(b, c)). Black and green rectangular line represent equilibrium $U_3 = (0, c)$ (where c is an arbitrary positive constant) and asymptotically stable node when $c > 1$, $\beta > \mu$ (see Fig. 2(b, d) black rectangular line) and unstable when $c < 1$, $\beta < \mu$ (see Fig. 2(a, c) green rectangular line). The equilibrium U_4 is unstable when $\mu > \beta > 2$ (see Fig.2(a)) and stable when $2 > \beta > \mu$ (see Fig.2 (b)), the positive equilibrium U_4 does not exist when $2 > \mu > \beta$ and $\beta > \mu > 2$ (see Fig.2(c, d)).

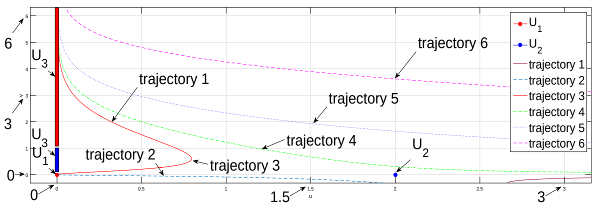
Here, we use parameters as $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$. By Theorem 2.1, we have $d_{21}^c = 1.366$. According to Theorem 2.1 there exists an unbounded region $d_{21} > d_{21}^c$ in which Turing instability occurs. Taking cross-diffusion coefficient $d_{21} = 1.066$, a value less than the critical diffusion coefficient d_{21}^c, we observe that $Re(\lambda) < 0$ and $Det(M_k) > 0$ (red curve in Fig.3) for all k. If we take $d_{21} = 1.866$, a value that is greater



(a)



(b)



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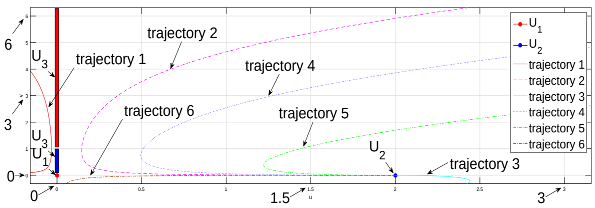
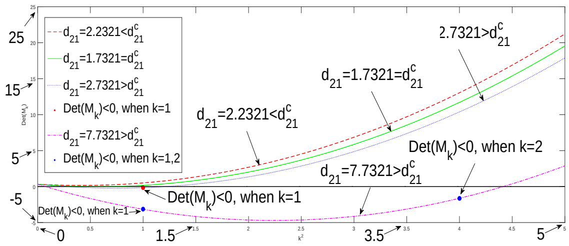


(d)

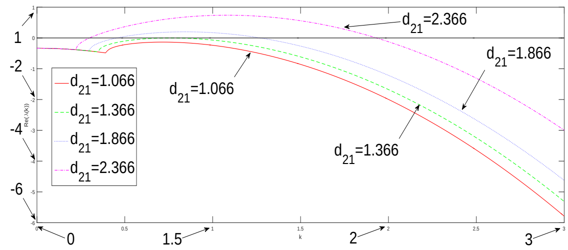
Fig. 2. The phase portraits of system (2) with fixed $K = 2, \gamma = 1, \alpha = 1$ and $\rho = 1$. U_i $(i = 1, 2, 4)$ represents the constant equilibrium. (a): $\beta = 3$, $\mu = 4$; (b): $\beta = 1.5$, $\mu = 1$; (c): $\beta = 1$, $\mu = 1.5$; (c): $\beta = 4$, $\mu = 3$.

than the critical diffusion coefficient d_{21}^c, then $Re(\lambda) > 0$ and $Det(M_k) < 0$ (blue curve and red point in Fig.3) for $k = 1$. If we take $d_{21} = 2.366$, a value that is greater than the critical diffusion coefficient d_{21}^c, then $Re(\lambda) > 0$ and $Det(M_k) < 0$ (magenta curve and blue point in Fig.3) for $k = 1, 2$. Fig.3 implies that $Re(\lambda) < 0$ and $Det(M_k) > 0$ for all k (green curve in Fig.3) when $d_{21} > d_{21}^c$ and $Re(\lambda) > 0$ and $Det(M_k) < 0$ for some k when $d_{21} > d_{21}^c$.

We consider the effect of μ on system (3). For the fixed other parameters, the system (3) admits patterns when $d_{21}(\mu) > d_{21}^c(\mu)$ (see red curve in Fig.4(a)), and stable when $d_{21}(\mu) < d_{21}^c(\mu)$, where $d_{21}^c(\mu)$ is given by Theorem 2.1. Let $\gamma = 1, K = 2, \alpha = 1, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$. It is easy to check that the system (3) has positive equilibriums for all μ. From Fig.4(a), we can see pattern formation in green region and homogenous state exists in region yellow. Next, we fix parameters $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$, and consider the effect of ρ on system (3). According to the Fig 4(b), it is easy to deduce that positive equilibrium exist when $\rho > 1$ and does not exist when $\rho < 1$. The system (3) admits patterns when $d_{21}(\rho) > d_{21}^c(\rho)$ (see red curve in Fig .4(b)), and stable when $d_{21}(\rho) < d_{21}^c(\rho)$, where $d_{21}^c(\rho)$ is given by Theorem 2.1. We consider the effect of ρ on system (3). To be more precise, pattern formation occurs in black solid lines and does not emerge in black dashed lines when $d_{21} = 0.3, 1.366, 2.366$.

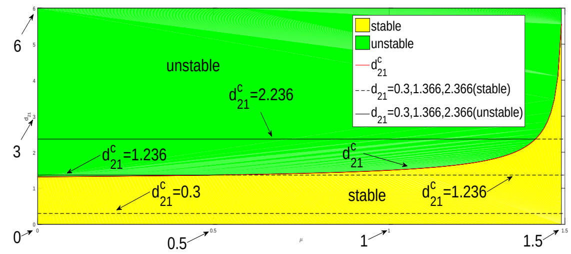


(a)

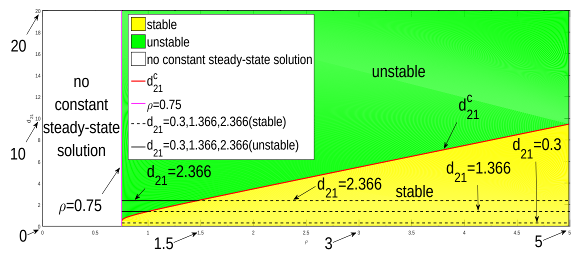


(b)

Fig. 3. Plots of $Det(M(k))$ and $Re(\lambda)$ with fixed reaction parameter values $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$.



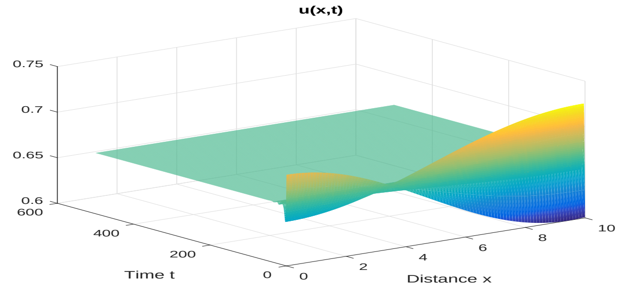
(a)



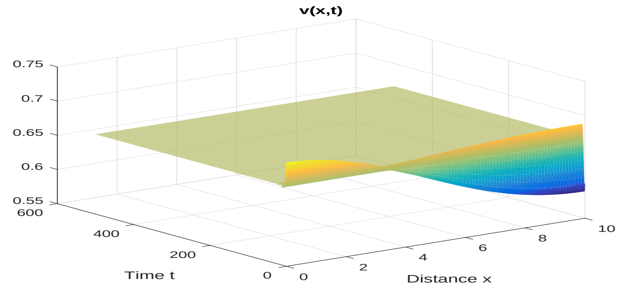
(b)

Fig. 4. The effects of parameter μ or ρ on pattern formation for the model (3). (a): The parameter μ is vary and other parameters: $\gamma = 1, K = 2, \alpha = 1, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$; (b): The parameter ρ is vary and other parameters: $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$.

To investigate the impact of cross-diffusion on Turing pattern, we fixed $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$ in system 3, which means the equilibrium U_4 of the corresponding kinetic and diffusion systems is asymptotically stable (see Fig.5(a,b)). We illustrate the change in the pattern form as $d_{21} = 0, 1.066, 1.866$ and 2.366. By Fig.6(a,b), when the cross-diffusion rate $d_{21} = 0, 1.066$, the equilibrium U_4 is stable, which is did not occurs Turing instability. When cross-diffusion rate $d_{21} = 1.866$ and $d_{21} = 2.366$, the equilibrium U_4 is unstable (Fig.6(c-f)), which occurs Turing pattern state. It is easy to see that small values of the time t, the system resides in a stable homogeneously state (Fig.6(c)). As time increases to a value great, the homogeneous state becomes Turing unstable (Fig.6(c-f)). In Fig.6, we observe that the system (3) may generates pattern formation for large diffusivity d_{21}.

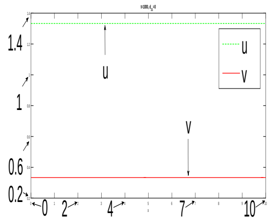


(a)

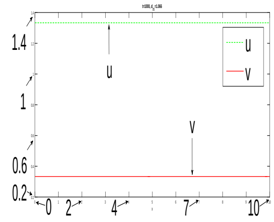


(b)

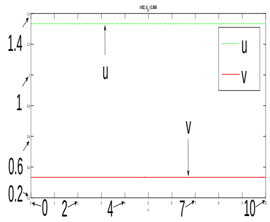
Fig. 5. Numerical solutions of the system (3) for $d_{21} = 0$ in the region $0 \leq x \leq 10$, and initial data U_4, $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \rho = 1, \beta = 1, d_{11} = 1$ and $d_{22} = 1$.



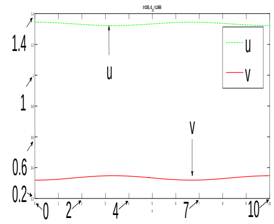
(a)



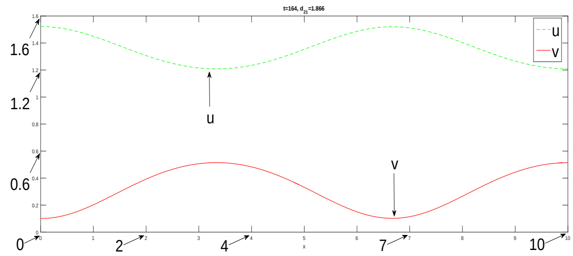
(b)



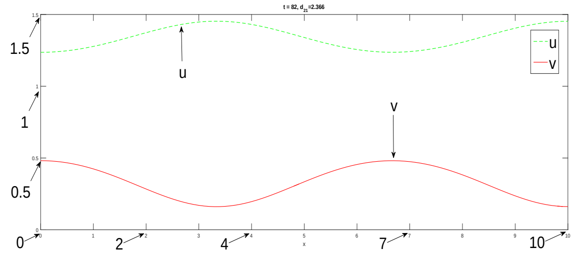
(c)



(d)



(e)



(f)

Fig. 6. One dimensional numerical solutions of the system (1.3) for different d_{21} in the region $0 \leq x \leq 10$, and initial data U_4, the green curve for u and the red curve is v. The parameter d_{21} is vary and other parameter: $\gamma = 1, K = 2, \alpha = 1, \mu = 0.5, \rho = 1, \beta = 1.5, d_{11} = 1$ and $d_{22} = 1$. (a): $d_{21} = 0$; (b): $d_{21} = 1.066$; (c,d,e): $d_{21} = 1.866$; (f): $d_{21} = 2.866$.

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