Generalized Bell Collocation Method to Solve Fractional Riccati Differential Equations

Rui Jin, Linjun Wang

Abstract—**In the article, a collocation method based upon generalized Bell functions is applied to study fractional Riccati differential equations numerically. By using this method, the problem is converted to a system of nonlinear algebraic equations. Thereafter, we can obtain the numerical solutions of the original equations by solving the nonlinear equations. The error estimation is devoted to proving the convergence of the method. The numerical results indicate the method is valid and accurate to solve this class of fractional differential equations.**

Index Terms—**Generalized Bell functions, Riccati differential equations, Fractional differential equations, Collocation method.**

I. Introduction

Riccati differential equations play a vital part in science and engineering. For example, they are applied to describe different phenomena in random processes, econometric mathematics, optimal control, network synthesis and solitary wave in [1], [2], [3], [4]. For more detail of the background of Riccati differential equations, we refer to [5], [6] and reference therein.

In recent years, the researchers have combined fractional derivatives with Riccati differential equations. In other words, by replacing the integer order derivative with the fractional order derivative in Riccati differential equations, the fractional Riccati differential equations (FRDEs) are obtained in [7]. As pointed out by [8], some mathematical models using FRDEs can be more reasonable.

In the article, we want to solve fractional Riccati differential equation (FRDE):

$$D^{\beta}y(t) = a(t) + r(t)y(t) + k(t)y^2(t),\ 0 < \beta \leq 1, \quad (1)$$

with initial condition

$$y(0) = \lambda, \quad (2)$$

in which $0 \leq t \leq 1$, $a(t), r(t), k(t)$ are functions on $[0, 1]$ and λ is a given number.

Because it is not easy to solve FRDEs exactly, the numerical methods are of great importance for solving FRDEs. In order to obtain the numerical solutions, people have developed various methods such as modified homotopy perturbation method [9], Adomian decomposition method (ADM) [10], iterative reproducing kernel Hilbert spaces method (IRKHSM) [11] and other algorithms [12], [13].

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In the past few years, the researchers have been trying to employ the collocation method to obtain more accurate approximate solution of FRDEs. Yüzbaşı [14] proposed a collocation method using the Bernstein polynomials. Kashkari and Syam [15] obtained the numerical solutions of FRDEs by combining fractional-order Legendre operational matrix with collocation method. Singh and Srivastava [16] used Jacobi collocation method to solve FRDEs with variable coefficients. Izadi [17] exhibited the solvability of FRDEs by a collocation method using fractional polynomial basis.

Very recently, generalized Bell functions (GBFs) presented in [18] have been extended as a new collocation basis to solve linear fractional differential equations. Motivated by above, the goal of the article is to develop a generalized Bell collocation method to solve FRDEs. By using the method, (1) and (2) can be simplified to a class of nonlinear equations. The numerical solutions can be obtained by the results of the nonlinear equations. We also prove the convergence of the present method.

In Section II, some important definitions of fractional calculus are reviewed and the definitions of the GBFs are introduced. Section III describes the collocation method based on the GBFs to solve FRDEs. We provide the error analysis in Section IV. In final, we give some numerical examples to demonstrate the advantage of our generalized Bell collocation method.

II. Preliminary

At the beginning of this section, some standard definitions of the fractional calculus are reviewed. Moreover, the definitions and some properties of the GBFs are recalled.

Definition 1. ([19], [20]) The Riemann-Liouville fractional integral operator of order α is defined by:

$$J^{\alpha}h(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}h(s)ds, & \alpha > 0, \\ h(t), & \alpha = 0. \end{cases}$$

Definition 2. ([19]) The Caputo's fractional derivative is defined by:

$$D^{\alpha}h(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-s)^{n-\alpha-1}\dfrac{d^n}{ds^n}h(s)ds, & \\ n-1 < \alpha < n, & \\ h^{(n)}(t), & \alpha = n. \end{cases}$$

We collect some properties of the operator J^{α}, which can be found in [21]:

$$J^{\alpha_1}J^{\alpha_2}h(t) = J^{\alpha_1+\alpha_2}h(t), \quad (3)$$

$$J^{\alpha_1}J^{\alpha_2}h(t) = J^{\alpha_2}J^{\alpha_1}h(t). \quad (4)$$

Moreover, we have [21]

$$J^{\alpha}D^{\alpha}h(t) = h(t) - \sum_{i=0}^{n-1} h^{(i)}(0^+)\frac{t^i}{i!}, n-1 < \alpha < n, n \in \mathrm{N}^+. \tag{5}$$

Before we give the definition of GBFs, we provide the definition of Bell polynomials.

Definition 3. ([18]) The Bell polynomials of degree n are defined as

$$B_n(t) = \sum_{k=0}^{n} S(n,k)t^k, \tag{6}$$

where $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$ denotes the Stirling number of the second kind.

By Definition 3, we mention the result of [18]

$$B_N(t) = S_N X_N(t), \tag{7}$$

where

$$\begin{aligned} B_N(t) &= [B_0(t), B_1(t), \ldots, B_N(t)]^T, \\ X_N(t) &= [1, t, \ldots, t^N]^T, \end{aligned} \tag{8}$$

and

$$S_N = \begin{pmatrix} S(0,0) & 0 & \cdots & 0 \\ S(1,0) & S(1,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S(N,0) & S(N,1) & \cdots & S(N,N) \end{pmatrix}. \tag{9}$$

Due to the fact that matrix S_N is a lower triangular matrix with nonzero diagonal elements, the matrix is nonsingular. Together with (7), it is easy to get

$$X_N(t) = S_N^{-1} B_N(t). \tag{10}$$

According to [18], the GBFs are obtained by replacing t with $t^{\alpha}(0 < \alpha \leq 1)$ in the Bell polynomials $B_n(t)$. We denote the GBFs by $B_n^{\alpha}(t)$. That is

$$B_n^{\alpha}(t) = \sum_{k=0}^{n} S(n,k)t^{k\alpha}. \tag{11}$$

Consider the vector of GBFs

$$B_{N,\alpha}(t) = [B_0^{\alpha}(t), B_1^{\alpha}(t), \ \ldots, B_N^{\alpha}(t)]^T. \tag{12}$$

Lemma 1. [18] The vector $B_{N,\alpha}(t)$ can be represented by

$$B_{N,\alpha}(t) = S_N X_{N,\alpha}(t), \tag{13}$$

where

$$X_{N,\alpha}(t) = [1, t^{\alpha}, \cdots, t^{N\alpha}]^T, \tag{14}$$

$$S_N = \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0N} \\ s_{10} & s_{11} & \cdots & s_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N0} & s_{N1} & \cdots & s_{NN} \end{pmatrix}, \tag{15}$$

and

$$s_{ij} = \begin{cases} \displaystyle\sum_{n=0}^{j} \frac{(-1)^n}{j!}\binom{j}{n}(j-n)^i, & i \geq j, \\ 0, & i < j. \end{cases} \tag{16}$$

III. Generalized Bell collocation method for FRDEs

In the section, we propose a collocation method using the GBFs to solve (1) and (2).

The approximate solution of (1) and (2) is supposed as

$$y(t) \simeq y_{N,\alpha}(t) = \sum_{n=0}^{N} c_n B_n^{\alpha}(t), \tag{17}$$

where $c_n(n = 0, 1, \cdots, N)$ are unknown.

By means of matrix operation, we rewrite (17) as

$$y_{N,\alpha}(t) = C_N B_{N,\alpha}(t), \tag{18}$$

where

$$C_N = [c_0, c_1, \cdots, c_N]. \tag{19}$$

By (13) and (18), we have

$$y_{N,\alpha}(t) = C_N S_N X_{N,\alpha}(t). \tag{20}$$

Using (20), we can conveniently determine that the Caputo fractional derivative D^{β} of (20) is written as

$$D^{\beta}y_{N,\alpha}(t) = C_N S_N X_{N,\alpha}^{(\beta)}(t), \ \beta \leq \alpha, \tag{21}$$

where

$$\begin{aligned} X_{N,\alpha}^{(\beta)}(t) &= [D^{\beta}1, D^{\beta}t^{\alpha}, D^{\beta}t^{2\alpha}, \ldots, D^{\beta}t^{N\alpha}] \\ &= \left[0, \frac{\Gamma(\alpha+1)t^{\alpha-\beta}}{\Gamma(\alpha+1-\beta)}, \ldots, \frac{\Gamma(N\alpha+1)t^{N\alpha-\beta}}{\Gamma(N\alpha+1-\beta)}\right]. \end{aligned} \tag{22}$$

To get the approximate solution of the form (17), let us substitute (20) and (21) into (1) and then get

$$\begin{aligned} C_N S_N X_{N,\alpha}^{(\beta)}(t) &= a(t) + r(t)C_N S_N X_{N,\alpha}(t) \\ &\quad + k(t)(C_N S_N X_{N,\alpha}(t))^2 \end{aligned} \tag{23}$$

We choose N roots of shifted Legendre polynomial $P_N(t)$ in [22] as collocation points. After inserting these nodes $t_j, j = 0, 1, ...N-1$ into (23), we obtain a system of matrix equations

$$\begin{aligned} C_N S_N X_{N,\alpha}^{(\beta)}(t_j) &= a(t_j) + r(t_j)C_N S_N X_{N,\alpha}(t_j) \\ &+ k(t_j)(C_N S_N X_{N,\alpha}(t_j))^2, \ j = 0, 1, ...N-1. \end{aligned} \tag{24}$$

Next, we deal with the initial condition (2). By setting $t = 0$ for (20), we have

$$C_N S_N X_{N,\alpha}(0) = \lambda. \tag{25}$$

By adding the initial value condition, $N+1$ nonlinear equations are obtained. After solving these nonlinear equations and getting C_N, we can easily acquire $y_{N,\alpha}(t)$ by (18).

IV. Error analysis

Here, we conduct the error analysis of the generalized Bell collocation method for solving FRDEs. We suppose

$$\|f(t)\|_{\infty} = \sup_{t\in[0,1]} |f(t)|.$$

Firstly, an existing result of Riemann-Liouville fractional integral operator will be listed.

Lemma 2. ([23]) Let $w : [0,1] \to R$ and $J^{\alpha}(\cdot)$ is the Riemann-Liouville fractional integral operator. Then,

$$\|J^{\alpha}(w(t))\|_{\infty} \leq \frac{1}{\Gamma(\alpha+1)}\|w(t)\|_{\infty} \tag{26}$$

We state the results of error analysis.

Theorem 1. Suppose $y(t)$ is the exact solution of (1),(2) and $y_{N,\alpha}(t)$ is the numerical solution of $y(t)$ solved by the generalized Bell collocation method. Furthermore, we assume
(1) There exists positive numbers $\rho_1, \rho_2, \rho_3, \rho_4$, such that $\|r(t)\|_\infty \le \rho_1$, $\|k(t)\|_\infty \le \rho_2$, $\|y(t)\|_\infty \le \rho_3$, $\|y_{N,\alpha}(t)\| \le \rho_4$,
(2) $\rho_1 + \rho_2(\rho_3 + \rho_4) - \Gamma(\beta+1) < 0$.
Then,

$$\|y(t) - y_{N,\alpha}(t)\|_\infty \le \frac{\|Res_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1) - \rho_1 - \rho_2(\rho_3+\rho_4)}.$$

Proof: By this hypothesis, we can easily find that $y(t)$ satisfies

$$D^\beta y(t) = a(t) + r(t)y(t) + k(t)y^2(t), \tag{27}$$

and $y_{N,\alpha}(t)$ makes the following problem set up

$$D^\beta y_{N,\alpha}(t) = a(t) + r(t)y_{N,\alpha}(t) + k(t)y_{N,\alpha}^2(t) + Res_{N,\alpha}(t), \tag{28}$$

where $Res_{N,\alpha}(t)$ is the residual function and represents the error generated by inserting $y_{N,\alpha}(t)$ into (27).

Let us subtract (28) from (27) and denote the error as

$$e_{N,\alpha}(t) = y(t) - y_{N,\alpha}(t).$$

Then we can write

$$\begin{aligned} D^\beta e_{N,\alpha}(t) =& r(t)e_{N,\alpha}(t) + k(t)(y(t) + y_{N,\alpha}(t))e_{N,\alpha}(t) \\ & - Res_{N,\alpha}(t). \end{aligned} \tag{29}$$

By (2) and (25), we have

$$e_{N,\alpha}(0) = 0.$$

Implementing the Riemann-Liouville fractional integral operator to (29) leads to

$$\begin{aligned} e_{N,\alpha}(t) =& J^\beta(r(t)e_{N,\alpha}(t)) + J^\beta(k(t)(y(t) + y_{N,\alpha}(t))e_{N,\alpha}(t)) \\ & - J^\beta(Res_{N,\alpha}(t)). \end{aligned} \tag{30}$$

It follows from (30) that

$$\|e_{N,\alpha}(t)\|_\infty \le \|E_1(t)\|_\infty + \|E_2(t)\|_\infty + \|E_3(t)\|_\infty, \tag{31}$$

where

$$\begin{aligned} E_1(t) &= J^\beta(r(t)e_{N,\alpha}(t)), \\ E_2(t) &= J^\beta(k(t)(y(t) + y_{N,\alpha}(t))e_{N,\alpha}(t)), \\ E_3(t) &= J^\beta(Res_{N,\alpha}(t)). \end{aligned}$$

The above three items will be estimated one after another. For $\|E_1(t)\|_\infty$, through Lemma 2, we have

$$\begin{aligned} \|E_1(t)\|_\infty &\le \frac{1}{\Gamma(\beta+1)}\|r(t)e_{N,\alpha}(t)\|_\infty \\ &\le \frac{\rho_1\|e_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1)}. \end{aligned} \tag{32}$$

In the same manner, we can handle $\|E_2(t)\|_\infty$ and $\|E_3(t)\|_\infty$. That is

$$\begin{aligned} \|E_2(t)\|_\infty &\le \frac{1}{\Gamma(\beta+1)}\|k(t)(y(t) + y_{N,\alpha}(t))e_{N,\alpha}(t)\|_\infty \\ &\le \frac{\rho_2(\rho_3+\rho_4)\|e_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1)}, \end{aligned} \tag{33}$$

and

$$\|E_3(t)\|_\infty \le \frac{1}{\Gamma(\beta+1)}\|Res_{N,\alpha}(t)\|_\infty. \tag{34}$$

Noting (32), (33) and (34), we conclude

$$\begin{aligned} \|e_{N,\alpha}(t)\|_\infty \le& \frac{\rho_1\|e_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1)} + \frac{\rho_2(\rho_3+\rho_4)\|e_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1)} \\ & + \frac{1}{\Gamma(\beta+1)}\|Res_{N,\alpha}(t)\|_\infty. \end{aligned} \tag{35}$$

Consequently,

$$\|e_{N,\alpha}(t)\|_\infty \le \frac{\|Res_{N,\alpha}(t)\|_\infty}{\Gamma(\beta+1) - \rho_1 - \rho_2(\rho_3+\rho_4)}. \tag{36}$$

Following the idea of [24], if $Res_{N,\alpha}(t)$ approaches 0, then we conclude $\|e_{N,\alpha}(t)\|_\infty = \|y(t) - y_{N,\alpha}(t)\|_\infty \to 0$.

Remark 1. In theory, $Res_{N,\alpha}(t)$ equals zero at the collocation point, i.e. $Res_{N,\alpha}(t_j) = 0,\ j = 0, 1, 2, \cdots, N-1$.

Remark 2. Since the exact solutions are often not available when $0 < \beta < 1$, Theorem 1 gives a method to measure the accuracy of numerical solutions.

V. Illustrative Examples

Example 1. At first, Let's solve the simple FRDE from [10]:

$$\begin{aligned} & D^\beta y(t) = t^3y^2(t) - 2t^4y(t) + t^5 + 1, \\ & y(0) = 0. \end{aligned}$$

When $\beta = 1$, it is easy to check the exact solution is $y(t) = t$.

In Figure 1, we see that the numerical solution approaches the exact solution when $\beta = 1$. Table I demonstrates the absolute errors for $\beta = \alpha = 1$. In Table II, we compare the absolute errors of the present method with those of IRKHSM in [11], ADM in [10] and other methods in [12], [25].

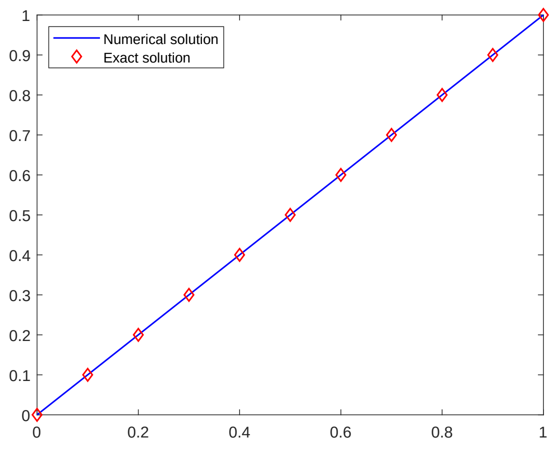

Fig. 1. The numerical results for Example 1

Example 2. We consider the FRDE from [9]:

$$\begin{aligned} & D^\beta y(t) = 1 - y^2(t), \\ & y(0) = 0. \end{aligned}$$

For this example, the exact solution is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ when $\beta = 1$.

The numerical solutions with $\beta = 0.5, 0.75, 0.85, 0.95$ are shown in Figure 2. In Figure 3, the numerical solution is

TABLE II
THE ABSOLUTE ERRORS OBTAINED BY VARIOUS METHODS FOR EXAMPLE 1

t	IRKHSM[11]	ADM[10]	JCM[16]	VIM[12]	OHAM[25]	Present method
0.1	5.59e-07	0	7.45e-07	1.98e-08	3.20e-05	8.80e-15
0.2	1.11e-06	1.13e-14	8.51e-07	1.03e-06	2.90e-04	1.45e-14
0.3	1.67e-06	7.47e-12	9.30e-07	8.85e-06	1.10e-03	1.05e-14
0.4	2.23e-06	7.45e-10	1.08e-06	3.33e-05	2.50e-03	5.01e-15
0.5	2.79e-06	2.64e-08	1.14e-06	7.26e-05	4.40e-03	9.75e-16
0.6	3.30e-06	4.89e-07	1.14e-06	9.98e-05	5.50e-03	3.62e-14
0.7	3.95e-06	5.76e-06	1.21e-06	8.84e-05	5.50e-03	1.81e-13
0.8	4.56e-06	4.87e-05	1.04e-06	1.54e-05	3.80e-03	5.90e-13
0.9	5.24e-06	3.20e-04	1.13e-06	4.99e-04	3.20e-03	1.51e-12
1	6.07e-06	1.71e-03	4.84e-07	3.47e-03	3.40e-03	3.32e-12

TABLE I
THE ABSOLUTE ERRORS WITH $N = 4, 5$ FOR EXAMPLE 1

t	$N = 4$	$N = 5$
0.1	2.6321e-13	8.8016e-15
0.2	6.2621e-13	1.4488e-14
0.3	8.7334e-13	1.0468e-14
0.4	1.0270e-12	5.0118e-15
0.5	1.3479e-12	9.7485e-16
0.6	2.3345e-12	3.6178e-14
0.7	4.7236e-12	1.8110e-13
0.8	9.4901e-12	5.9029e-13
0.9	1.7847e-11	1.5146e-12
1	3.1245e-11	3.3232e-12

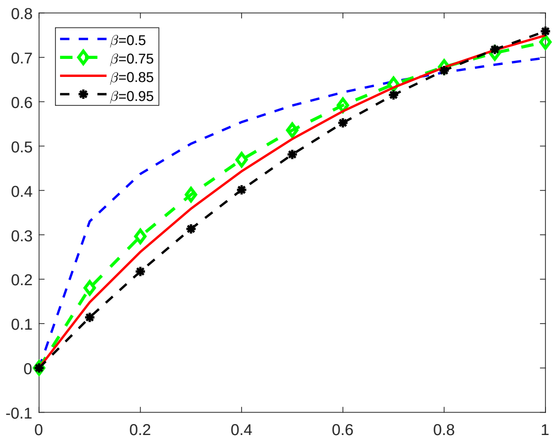

Fig. 2. The numerical solutions with different β for Example 2

observed to approach the exact solution when $\beta = 1$. The difference between the exact and the numerical solution of Figure 3 is shown in Figure 4. Figure 5 draws the absolute error by selecting $\beta = 1$. Table III gives the absolute errors for different N with $\beta = 1$. Moreover, we also compare the absolute errors made by present method with those in [9], [26], [11] and present the results in Table IV.

The exact solution is not available when $0 < \beta < 1$. Therefore, we use (28) and compute residual function $Res_{N,\alpha}(t)$ to report our results. The results are demonstrated in Figure 6 for different α and $\beta = 0.8$. The values of residual function at $0.2, 0.4, 0.6, 0.8, 1$ with different α and $\beta = 0.8$ are also presented in Table V. By observing the values of residual

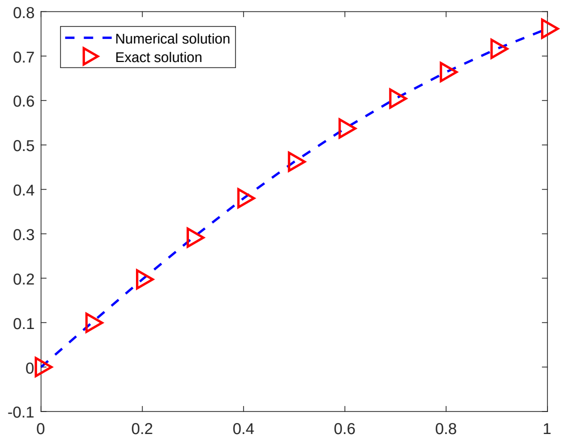

Fig. 3. The numerical results with $\beta = 1$ for Example 2

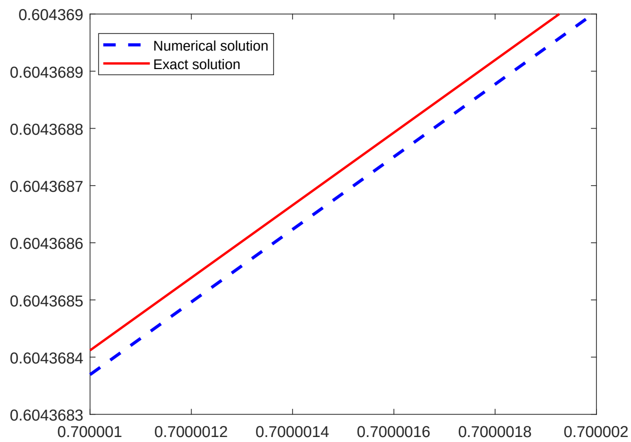

Fig. 4. The "zoom in" figure of Figure 3 for Example 2

function $Res_{N,\alpha}(t)$, we find that when $\alpha = \beta$ the error is smallest.

Example 3. The last example is from [27]:

$$D^\beta y(t) = \left(\frac{t^{\beta+1}}{\Gamma(\beta+2)}\right)^2 + t - y(t)^2,$$
$$y(0) = 0.$$

We find $y(t) = \frac{t^{\beta+1}}{\Gamma(\beta+2)}$ is the exact solution.

Without losing generality, we first choose $\beta = 0.5$. The

TABLE V
THE RESULTS OF RESIDUAL FUNCTION AT SOME POINTS WITH VARIOUS CHOICES OF α FOR EXAMPLE 2

t	$Res_{6,0.8}(t)$	$Res_{6,0.85}(t)$	$Res_{6,0.9}(t)$	$Res_{6,1}(t)$
0.2	3.1974e-14	4.8783e-13	2.9310e-14	3.0261e-13
0.4	2.5269e-13	3.9810e-12	2.5258e-13	2.7141e-12
0.6	1.1129e-12	1.9636e-11	1.3413e-12	1.6664e-11
0.8	3.5905e-12	6.9176e-11	5.0435e-12	7.1474e-11
1	9.5760e-12	1.9909e-10	1.5417e-11	2.4714e-10

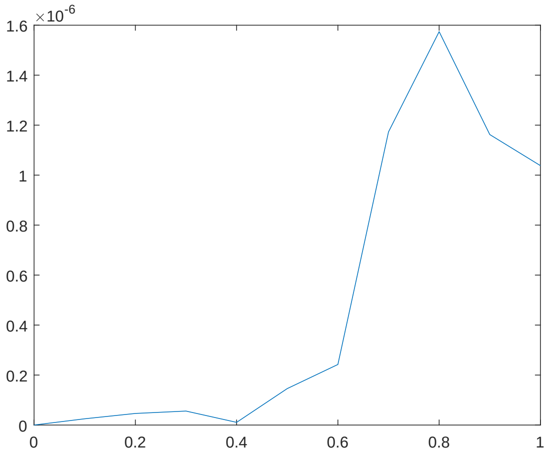

Fig. 5. The absolute error with $\beta = 1$ for Example 2

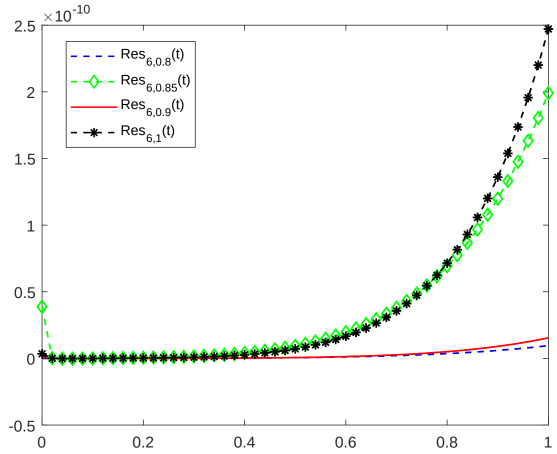

Fig. 6. The graph of residual functions for Example 2

TABLE III
THE ABSOLUTE ERRORS WITH DIFFERENT N FOR EXAMPLE 2

t	$N=6$	$N=7$	$N=8$
0.1	5.2719e-07	2.2227e-07	2.5061e-08
0.2	1.0914e-07	2.3672e-07	4.6382e-08
0.3	1.3636e-06	7.2482e-08	5.5961e-08
0.4	7.6236e-07	5.1583e-07	1.0774e-08
0.5	3.1802e-06	3.3107e-07	1.4589e-07
0.6	2.6364e-06	2.5785e-06	2.4235e-07
0.7	1.9919e-05	4.1888e-06	1.1731e-06
0.8	3.7585e-05	3.9204e-06	1.5742e-06
0.9	3.7149e-05	3.0788e-06	1.1624e-06
1	2.9273e-05	2.7732e-06	1.0381e-06

TABLE IV
THE COMPARISON OF ABSOLUTE ERRORS OBTAINED BY VARIOUS METHODS FOR EXAMPLE 2

t	MHPM[9]	IDM[26]	IRKHSM[11]	Present method
0.1	0	1.00e-11	9.05e-06	2.51e-08
0.2	0	0.00e-10	1.72e-05	4.64e-08
0.3	1.00e-06	2.50e-09	2.38e-05	5.60e-08
0.4	5.00e-06	5.61e-08	2.85e-05	1.08e-08
0.5	3.90e-05	6.03e-07	3.11e-05	1.46e-07
0.6	1.93e-04	4.09e-06	3.17e-05	2.42e-07
0.7	7.37e-04	2.01e-05	3.07e-05	1.17e-06
0.8	2.33e-03	7.78e-05	2.81e-05	1.57e-06
0.9	6.37e-03	2.50e-04	2.32e-05	1.16e-06
1	1.55e-02	6.99e-04	1.19e-05	1.04e-06

absolute errors with $\alpha = 0.5$ are shown in Table VI. The results are also compared with those of fractional-order Bernoulli wavelets (FBW) in [28]. Figure 7 displays the absolute error where $\alpha = \beta = 0.5$ and $N = 5$. To verify the effect of α in the proposed method, we give the absolute errors for different α in Table VII. By the results in Table VII, we can observe that the smallest error is made when $\alpha = \beta$.

TABLE VI
THE ABSOLUTE ERRORS FOR EXAMPLE 3

t	Our method $\alpha = 0.5$	FBW[28]
0.1	8.6736e-16	1.42121e-09
0.2	1.0131e-15	1.70884e-09
0.3	1.4794e-14	1.95085e-08
0.4	8.7930e-14	1.47103e-08
0.5	2.4547e-13	1.13654e-08
0.6	5.0115e-13	8.98023e-09
0.7	8.6287e-13	7.46806e-09
0.8	1.3342e-12	6.97788e-09
0.9	1.9169e-12	7.78465e-09
1	2.6106e-12	1.02211e-08

At last, $\beta = \frac{1}{3}$ is selected. We show the values of residual function and absolute errors with various choices of N in Table VIII. We can see that the numerical results are consistent with the theoretical error estimation (36).

VI. CONCLUSIONS

In this article, the generalized Bell collocation method based on the GBFs is developed to solve nonlinear FRDEs. Through the GBFs together with the choose of collocation points, the original FRDEs are converted into some nonlinear equations. Numerical examples are shown to indicate the

TABLE VII
THE ABSOLUTE ERRORS WITH $\beta = 0.5$,$N = 5$ AND VARIOUS VALUES OF α FOR EXAMPLE 3

t	$\alpha = 0.5$	$\alpha = 0.55$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
0.2	1.01e-15	1.30e-05	1.25e-05	1.49e-06	1.23e-05
0.4	8.79e-14	1.53e-05	2.29e-05	1.51e-05	2.47e-05
0.6	5.01e-13	1.44e-05	4.23e-05	6.51e-05	1.69e-04
0.8	1.33e-12	1.08e-05	3.17e-05	5.02e-05	1.36e-04
1	2.61e-12	3.93e-06	1.35e-05	2.35e-05	6.60e-05

TABLE VIII
THE VALUES OF RESIDUAL FUNCTION AND ABSOLUTE ERRORS FOR $\beta = \frac{1}{3}$ WITH VARIOUS CHOICES OF N FOR EXAMPLE 3

t	$Res_{4,\frac{1}{3}}(t)$	$e_{4,\frac{1}{3}}(t)$	$Res_{5,\frac{1}{3}}(t)$	$e_{5,\frac{1}{3}}(t)$
0.1	3.3924e-11	1.4623e-11	3.1983e-13	1.2926e-13
0.3	1.0987e-10	5.3579e-11	1.0085e-12	4.9075e-13
0.5	1.9185e-10	9.2521e-11	1.7577e-12	8.4677e-13
0.7	2.8279e-10	1.2505e-10	2.6055e-12	1.1497e-12
0.9	3.8109e-10	1.5005e-10	3.5394e-12	1.3896e-12

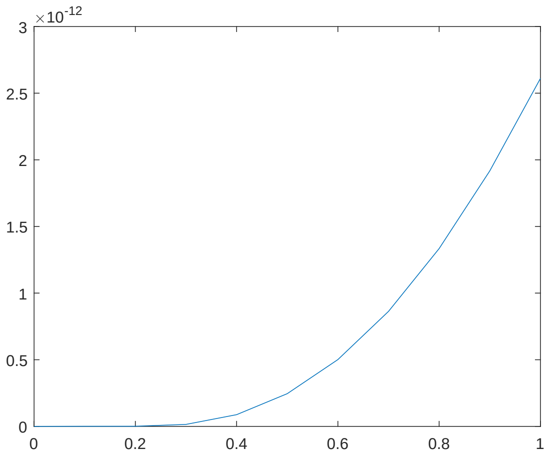

Fig. 7. The graph of absolute error with $\alpha = \beta = 0.5$ for Example 3

validity and accuracy of the method. By the results of numerical examples, we can see that the approximate solutions acquired by the generalized Bell collocation method are more accurate than those of several existing methods. In addition, it is shown that when the order of the GBFs is the same as the fractional order of the FRDEs, a more accurate numerical results can be obtained.

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