Graphs of Order n with Partition Dimension $n-3$

Debi Oktia Haryeni, Muhammad Ridwan, and Edy Tri Baskoro

***Abstract*—The characterizations of all graphs of order n with partition dimension $2, n-2, n-1$ or n have been completely studied. Recently, all graphs of order $n \geq 11$ and diameter two with partition dimension $n-3$ have been characterized. In this paper, we continue characterizing all graphs on n vertices with partition dimension $n-3$ and diameter either 3 or 4. This completes the characterization of all graphs of order $n \geq 11$ with partition dimension $n-3$.**

***Index Terms*—partition dimension, graph, characterization, diameter.**

I. Introduction

LET $G(V,E)$ be a connected graph, $u, v \in V(G)$ and $S \subset V(G)$. The *distance* between vertices u and v, denoted by $d(u,v)$, is the number of edges in a shortest path connecting u and v in G. The distance of u and S, denoted by $d(u,S)$, is $\min\{d(u,x) : x \in S\}$. The *eccentricity* of u is defined as $\text{ecc}(u) = \max\{d(u,v) : v \in V(G)\}$. The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum eccentricity of the vertices in G, namely $\text{diam}(G) = \max\{\text{ecc}(u) : u \in V(G)\}$. Furthermore, if $\text{ecc}(u) = \text{diam}(G)$, then u is called a *peripheral* vertex of G.

Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of a connected graph G. For any $u \in V(G)$, the *representation* $r(u|\Pi)$ of u with respect to Π is the k-vector $(d(u,S_1), d(u,S_2), \ldots, d(u,S_k))$. Such partition Π is called a *resolving partition* of G if $r(u|\Pi) \neq r(v|\Pi)$ for any two vertices $u, v \in V(G)$. The cardinality of a minimum resolving partition of G is called the *partition dimension* of G and it is denoted by $pd(G)$.

The study of the partition dimension of a connected graph was initiated by Chartrand et al. [5]. They characterized all connected graphs G of order n with $pd(G) \in \{2, n-1, n\}$. They showed that $pd(G) = 2$ if and only if $G = P_n$ and $pd(G) = n$ if and only if $G = K_n$. Furthermore, they showed that $pd(G) = n-1$ if and only if G is one of the graphs $K_{1,n-1}$, $K_n - e$ or $K_1 + (K_1 \cup K_{n-2})$, with e is an edge. In [18] Tomescu proved that there are only 23 connected graphs of order $n \geq 9$ with partition dimension $n-2$. These graphs are $K_{2,n-2}$, $K_2 + \overline{K_{n-2}}$, $K_n - E(P_3)$, $K_n - E(K_3)$, $K_n - E(P_4)$, $K_1 + (K_1 \cup (K_{n-2} - e))$, $K_n - E(C_4)$, $K_{1,n-1} + e$, $K_n - E(2K_2)$, $K_{2,n-2} - e$, $K_n - E(K_{1,3} + e)$, $G_1, G_2, \ldots, G_{12}$, where e is an edge. However, Baskoro and Haryeni [3] revised this characterization. They showed that two of these above graphs, namely $K_{1,n-1} + e$ and $K_n - E(K_{1,3} + e)$, have partition dimension $n-3$ (not $n-2$). Two other graphs, namely G_3 and G_5 are isomorphic to two graphs in [3] namely H_{12} and $(K_1 \cup K_2) + \overline{K_{n-3}}$, respectively. Furthermore, the graph G_4 is isomorphic to G_6. The characterization of Tomescu also missed one graph F constructed from $K_{n-1} - e$ by adding one new vertex x and connecting x with vertex a, where a is one of the end-vertices of e [3]. In addition, in this paper we show that G_{11} and $K_{2,n-2} - e$ in [18] have partition dimension $n-3$ (not $n-2$), where $G_{11} \cong F_{30}$ and $K_{2,n-2} - e \cong (2K_1 + \overline{K_{n-2}}) - e$. This concludes that there are only 17 non-isomorphic graphs of order $n \geq 9$ with partition dimension $n-2$.

Further results on the partition dimension of graphs obtained from unary or binary graphical operations can be seen in [1], [9], [16], [20]. The bounds of the partition dimensions of certain graphs have been studied in [2], [6], [12]–[15], [19]. The study on the partition dimension has been extended so that it can also be applied to disconnected graphs, see [7], [8], [10]. The applications of the concept of resolving partition of graphs can be seen in [11], [17] and [12].

For any connected graph G of order n, we have $pd(G) \leq n - \text{diam}(G) + 1$ [5]. This implies that if $pd(G) = n-3$, then $\text{diam}(G) \in \{2,3,4\}$. The characterization of graphs of order $n \geq 11$ with $pd(G) = n-3$ has been completed for $\text{diam}(G) = 2$ [3]. There are **114** non-isomorphic such graphs G on $n \geq 11$ vertices with $pd(G) = n-3$ and diameter 2. In this paper, we characterize all graphs G of order $n \geq 11$ and $\text{diam}(G) \in \{3,4\}$ with $pd(G) = n-3$. We show that there are **46** non-isomorphic such graphs, **41** of them with diameter 3 and the remaining **5** such graphs with diameter 4.

II. Main Results

In the following result, Chartrand et al. [5] showed that any two vertices of G having the same distance to all other vertices belong to distinct elements of a resolving partition of G.

Lemma 2.1: [5] Let Π be a resolving partition of G and $u, v \in V(G)$. If $d(u,x) = d(v,x)$ any $x \in V(G) \setminus \{u,v\}$, then u and v belong to distinct elements of Π.

Baskoro and Haryeni [3] generated some conditions of graphs so that forming certain graphs, as follows.

Lemma 2.2: [3] For $n \geq 8$, let G be a graph on n vertices. If G does not contain the following three configurations:

(C1) five vertices a, t_1, t_2, t_3 and t_4 forming $at_1, at_2 \in E(G)$ and $at_3, at_4 \notin E(G)$, as depicted in Figure 1(a),
(C2) six vertices a, b, t_1, t_2, t_3 and t_4 forming $at_1, bt_3 \in E(G)$ and $at_2, bt_4 \notin E(G)$, as depicted in Figure 1(b), and
(C3) four vertices t_1, t_2, t_3 and t_4 forming $t_1t_2 \in E(G)$ and $t_1t_4, t_2t_3, t_3t_4 \notin E(G)$, as depicted in Figure 1(c),

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then G is isomorphic to either $\overline{K_n}$, K_n, $K_{1,n-1}$, $K_{n-1} \cup K_1$, $K_n - E(K_{1,n-2})$, or $K_n - e$ for an edge $e \in E(K_n)$.

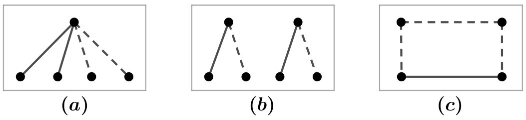

Fig. 1. (a) Configuration C1, (b) Configuration C2, and (c) Configuration C3

In the following theorem, we prove that there are 46 non-isomorphic graphs G of order $n \geq 11$, $\mathrm{diam}(G) \in \{3, 4\}$ and $pd(G) = n - 3$. In particular, there are 41 non-isomorphic graphs of $pd(G) = n - 3$ with $\mathrm{diam}(G) = 3$, namely $(2K_1 + \overline{K_{n-2}}) - e$, $F_1, F_2, \ldots, F_{40}$, and 5 non-isomorphic such graphs with $\mathrm{diam}(G) = 4$, namely $H_1, H_2, \ldots, H_5$. Note that the graphs $F_1, F_2, \cdots, F_{40}$ and $H_1, H_2, \cdots, H_5$ are presented in Appendix A.

Theorem 2.3: Let G be a connected graph of order $n \geq 11$ and $\mathrm{diam}(G) \in \{3, 4\}$. Then, $pd(G) = n - 3$ if and only if G is one of the graphs $(2K_1 + \overline{K_{n-2}}) - e$, $F_1, F_2, \ldots, F_{40}$, H_1, H_2, H_3, H_4 or H_5.

Proof: It is easy to verify that the graphs $(2K_1 + \overline{K_{n-2}}) - e$, F_i and H_j for each i and j have partition dimension $n - 3$. Now, we will show for the reverse direction. Let G be a connected graph of order $n \geq 11$ where $pd(G) = n - 3$ and $\mathrm{diam}(G) \in \{3, 4\}$. Let x be a peripheral vertex of G with $\mathrm{ecc}(x) \in \{3, 4\}$. Denote $N_i(x)$ as the set of all vertices of G at distance i from x and let $n_i = |N_i(x)|$, for any $i \in [1, \mathrm{diam}(G)]$. We divide into two cases based on the diameter of G.

(A) $\mathrm{diam}(G) = 3$.

Let x be a peripheral vertex of G with $\mathrm{ecc}(x) = 3$. Let $N_1(x) = \{u_j : 1 \leq j \leq n_1\}$, $N_2(x) = \{v_j : 1 \leq j \leq n_2\}$, and $N_3(x) = \{w_j : 1 \leq j \leq n_3\}$. If each of $\{n_1, n_2, n_3\}$ is at least 2, then $(x)(u_1, v_1, w_1)(u_2, v_2, w_2)\pi$ is a resolving partition of G having $(n - 4)$ classes, where π is a singleton partition consisting of a single vertex, which contradicts the hypothesis. Therefore, there are at most two of $\{n_1, n_2, n_3\}$ greater than or equal 2. However, only one of $\{n_1, n_2, n_3\}$ is greater than 2. Since otherwise, without loss of generality suppose that $n_1, n_2 \geq 3$. Then one deduces that $(x)(u_1, v_1, w_1)(u_2, v_2)(u_3, v_3)\pi$, where π is a singleton partition, is also an $(n - 4)$-resolving partition of G, a contradiction. Therefore, based on the values of (n_1, n_2, n_3) we have the following 9 subcases: (A1) $(1, 1, n - 3)$, (A2) $(1, n - 3, 1)$, (A3) $(n - 3, 1, 1)$, (A4) $(1, 2, n - 4)$, (A5) $(1, n - 4, 2)$, (A6) $(2, 1, n - 4)$, (A7) $(2, n - 4, 1)$, (A8) $(n - 4, 1, 2)$, and (A9) $(n - 4, 2, 1)$.

(A1) $(1, 1, n - 3)$.
Assume that $N_3(x)$ contains one of the configurations (C1), (C2) or (C3) in Lemma 2.2 such that

(C1) $w_1w_3, w_1w_4 \in E(G)$ and $w_1w_5, w_1w_6 \notin E(G)$, or

(C2) $w_1w_3, w_2w_4 \in E(G)$ and $w_1w_5, w_2w_6 \notin E(G)$, or

(C3) $w_3w_4 \in E(G)$ and $w_3w_6, w_4w_5, w_5w_6 \notin E(G)$,

then one deduces that $(x)(w_1)(w_2)(u_1, v_1, w_7)(w_3, w_5)(w_4, w_6)\pi$, where π is a singleton partition of the remaining vertices, is a resolving partition of G having $n - 4$ classes, a contradiction. It follows that $N_3(x)$ induces one of $\{\overline{K_{n-3}}, K_{n-3}, K_{1,n-4}, K_{n-4} \cup K_1, K_{n-3} - E(K_{1,n-5}), K_{n-3} - e\}$ by Lemma 2.2. If $N_3(x)$ induces $\overline{K_{n-3}}$, then the resulting graph is $G \cong F_{30}$ as depicted in Figure 2(a). If $N_3(x)$ induces K_{n-3}, then $G \cong G_{10}$. However $pd(G_{10}) = n - 2$ by [18]. Now suppose that $N_3(x)$ induces $K_{1,n-4}$. Let w_1 be the center of $K_{1,n-4}$. However, the partition $(x, w_2)(u_1, w_3)(v_1, w_4)(w_1, w_5)\pi$, where π is a singleton partition, is an $(n - 4)$-resolving partition of G, which contradicts the hypothesis. Suppose that $N_3(x)$ induces $K_{n-3} - E(K_{1,n-5})$ with the edge set $\{w_iw_j : 1 \leq i < j \leq n - 3\} \setminus \{w_2w_i : 3 \leq i \leq n - 3\}$. However, $(w_2)(w_1, w_3)(x, u_1, v_1, w_4)\pi$, where π is a singleton partition, is an $(n - 4)$-resolving partition of G, a contradiction. Finally assume that $N_3(x)$ induces $K_{n-4} \cup K_1$ or $K_{n-3} - e$. The first case yields that $G \cong F_{32}$ and the second case yields that $G \cong F_{15}$ (Figures 2(b) and 2(c)).

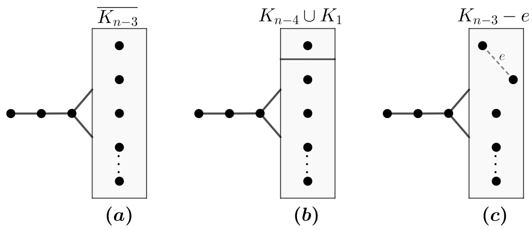

Fig. 2. Graphs (a) F_{30}, (b) F_{32}, and (c) F_{15}

(A2) $(1, n - 3, 1)$.

By a similar reason to Subcase (A1), if $N_2(x)$ contains one of the configurations (C1), (C2) or (C3) such that

(C1) $v_1v_3, v_1v_4 \in E(G)$ and $v_1v_5, v_1v_6 \notin E(G)$, or

(C2) $v_1v_3, v_2v_4 \in E(G)$ and $v_1v_5, v_2v_6 \notin E(G)$, or

(C3) $v_3v_4 \in E(G)$ and $v_3v_6, v_4v_5, v_5v_6 \notin E(G)$,

then one deduces that $(x)(v_1)(v_2)(u_1, v_7, w_1)(v_3, v_5)(v_4, v_6)\pi$, where π is a singleton partition, is an $(n - 4)$-resolving partition of G, a contradiction. Therefore by Lemma 2.2, $N_2(x)$ induces one of graphs (A2.1) $\overline{K_{n-3}}$, (A2.2) K_{n-3}, (A2.3) $K_{1,n-4}$, (A2.4) $K_{n-4} \cup K_1$, (A2.5) $K_{n-3} - E(K_{1,n-5})$, or (A2.6) $K_{n-3} - e$.

(A2.1) $N_2(x)$ induces $\overline{K_{n-3}}$. If $v_1w_1, v_2w_1, v_3w_1 \in E(G)$ and $v_4w_1, v_5w_1 \notin E(G)$, then $(w_1)(x, u_1, v_1)(v_2, v_4)(v_3, v_5)\pi$ is an $(n - 4)$-resolving partition of G, a contradiction. Therefore, the number of neighbors of w_1 in $N_2(x)$, denoted by $d(w_1)$, is either $1, 2, n - 4$ or $n - 3$.

If $d_(w_1) = n - 4$ where $v_1w_1 \notin E(G)$, then one deduces that $(v_1, v_2)(x, v_3)(u_1, v_4)(w_1, v_5)\pi$ is an $(n - 4)$-resolving partition of G, a contradiction. Otherwise, $G \cong F_{30}$ if $d(w_1) = 1$, or $G \cong F_{31}$ if $d_{N_2(x)}(w_1) = 2$, or $G \cong (2K_1 + \overline{K_{n-2}}) - e$ if $d(w_1) = n - 3$, as depicted in Figures 3(a)-3(c).

(A2.2) $N_2(x)$ induces K_{n-3}. By a similar reason to subcase (A2.1), the number of neighbors of w_1 in $N_2(x)$ is either $1, 2, n - 4$ or $n - 3$, since otherwise we have an $(n - 4)$-resolving partition of G. However, $G \cong G_{12}$ if $d(w_1) = 1$ and $G \cong F$ if $d(w_1) = n - 3$. In this case $pd(G_{12}) = n - 2$ [18] and $pd(F) = n - 2$ [3], a contradiction. Hence the resulting graph is $G \cong F_{14}$ if $d(w_1) = 2$ or $G \cong F_1$ if $d_{N_2(x)}(w_1) = n - 4$, as depicted in Figures 3(d) or 3(e), respectively.

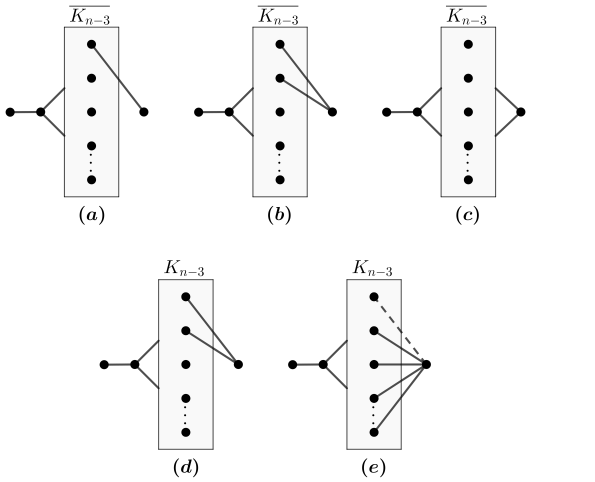

Fig. 3. Graphs (a) F_{30}, (b) F_{31}, (c) $(2K_1 + \overline{K_{n-2}}) - e$, (d) F_{14} and (e) F_1.

(A2.3) $N_2(x)$ induces $K_{1,n-4}$. Let v_1 be the center of $K_{1,n-4}$. We consider the neighbors of w_1 in $N_2(x)$. If $d_{N_2(x)}(w_1) = 1$, then one deduces that $(x, v_2)(u_1, v_3)(v_1, v_4)(w_1, v_5)\pi$ is an $(n-4)$–resolving partition of G. If $2 \le d_{N_2(x)}(w_1) \le n-3$, then one can deduces that $(x, v_2)(u_1, v_3)(v_1, v_4)(w_1, v_5)\pi$, where $w_1v_k \in E(G)$ and v_k is element of a singleton partition, is an $(n-4)$– resolving partition of G, a contradiction.

(A2.4) $N_2(x)$ induces $K_{n-4} \cup K_1$. Let v_1 be an isolated vertex of $K_{n-4} \cup K_1$. We consider the neighbors of w_1 in $N_2(x) \setminus \{v_1\}$. If $w_1v_2, w_1v_3 \in E(G)$ and $w_1v_4 \notin E(G)$, then $(w_1)(x, u_1, v_2)(v_3, v_4)(v_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, the number of neighbors of w_1 in $N_2(x) \setminus \{v_1\}$ is either $0, 1$ or $n-4$.

If $w_1v_i \notin E(G)$ for all $i \ge 2$, then $w_1v_1 \in E(G)$ and one deduces $G \cong F_{32}$, as depicted in Figure 4(a). If $w_1v_2 \in E(G)$ and $w_1v_i \notin E(G)$ for all other $i \ge 3$, then we have two cases. First, if $w_1v_1 \in E(G)$ then it follows that $(x, w_1)(v_2, v_3)(u_1, v_1, v_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Second, if $v_1w_1 \notin E(G)$ one deduces $G \cong F_{36}$, as depicted in Figure 4(b). Now, let $w_1v_i \in E(G)$ for all $i \ge 2$, and one deduces $G \cong F_{16}$ if $w_1v_1 \notin E(G)$ or $G \cong F_{19}$ if $w_1v_1 \in E(G)$, as depicted in Figures 4(c) or 4(d), respectively.

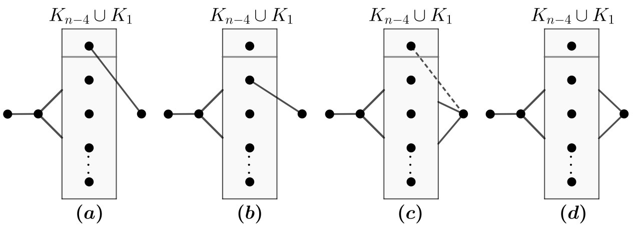

Fig. 4. Graphs (a) F_{32}, (b) F_{36}, (c) F_{16}, and (d) F_{19}.

(A2.5) $N_2(x)$ induces $K_{n-3} - E(K_{1,n-5})$. Assume the edge set of $K_{n-3} - E(K_{1,n-5})$ is $\{v_iv_j : 1 \le i < j \le n-3\} \setminus \{v_2v_i : 3 \le i \le n-3\}$. We consider the neighbors of w_1 in $N_2(x) \setminus \{v_1, v_2\}$. If $w_1v_3 \in E(G)$ and $w_1v_4 \notin E(G)$, then $(v_2)(w_1)(v_3, v_4)(x, u_1, v_5)(v_1, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. In addition, if $w_1v_i \notin E(G)$ for all $i \ge 3$ or $w_1v_2 \in E(G)$, then $(v_2)(x, u_1, v_3)(v_1, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $w_1v_i \in E(G)$ for all $i \ge 3$ and $w_1v_2 \notin E(G)$. This case produces $G \cong F_{29}$ if $w_1v_1 \notin E(G)$ or $G \cong F_{20}$ if $w_1v_1 \in E(G)$, as depicted in Figures 5(a) or 5(b), respectively.

(A2.6) $N_2(x)$ induces $K_{n-3} - e$. Let $e = v_1v_2$. If $w_1v_3, w_1v_4 \in E(G)$ and $w_1v_5 \notin E(G)$, then $(w_1)(v_1)(x, u_1, v_3)(v_4, v_5)(v_2, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that the number of neighbors of w_1 in $N_2(x) \setminus \{v_1, v_2\}$ is $0, 1$ or $n-5$.

If $w_1v_i \notin E(G)$ for all $i \ge 3$, then one deduces $G \cong F_{17}$ if $w_1v_1 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_{21}$ if $w_1v_1, w_1v_2 \in E(G)$, as depicted in Figures 5(c) or 5(d), respectively.

If $w_1v_3 \in E(G)$ and $w_1v_i \notin E(G)$ for all other $i \ge 4$, then $w_1v_i \notin E(G)$ for at least one of $i \in \{1, 2\}$, since otherwise $(w_1)(x, u_1, v_3)(v_1, v_4)(v_2, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. In this case one deduces $G \cong F_{18}$ if $w_1v_1, w_1v_2 \notin E(G)$, and otherwise $G \cong F_{22}$, see Figures 5(e) and 5(f).

Now assume that $w_1v_i \in E(G)$ for all $i \ge 3$. It follows that $w_1v_i \in E(G)$ for at least one of $i \in \{1, 2\}$, since otherwise $(w_1)(x, u_1, v_3)(v_1, v_4)(v_2, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. One deduces $G \cong F_4$ if $w_1v_1 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_8$ if $w_1v_1, w_1v_2 \in E(G)$, as depicted in Figures 5(g) or 5(h), respectively.

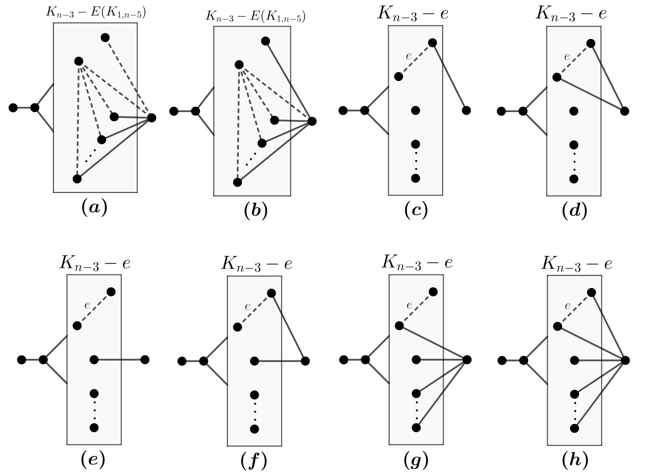

Fig. 5. Graphs (a) F_{29}, (b) F_{20}, (c) F_{17}, (d) F_{21}, (e) F_{18}, (f) F_{22}, (g) F_4, and (h) F_8.

(A3) $(n-3, 1, 1)$.
By a similar reason to Subcase (A1), if that $N_1(x)$ contains one of the configurations (C1), (C2) or (C3) in Lemma 2.2. Without loss of generality, we may assume:

(C1) $u_1u_3, u_1u_4 \in E(G)$ and $u_1u_5, u_1u_6 \notin E(G)$, or

(C2) $u_1u_3, u_2u_4 \in E(G)$ and $u_1u_5, u_2u_6 \notin E(G)$, or

(C3) $u_3u_4 \in E(G)$ and $u_3u_6, u_4u_5, u_5u_6 \notin E(G)$.

Then one deduces that $(x)(u_1)(u_2)(u_7, v_1, w_1)(u_3, u_5)(u_4, u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. It follows that $N_1(x)$ induces one of graphs (A3.1) $\overline{K_{n-3}}$, (A3.2) K_{n-3}, (A3.3) $K_{1,n-4}$, (A3.4) $K_{n-4} \cup K_1$, (A3.5) $K_{n-3} - E(K_{1,n-5})$, or (A3.6) $K_{n-3} - e$, by Lemma 2.2.

(A3.1) If $N_1(x)$ induces $\overline{K_{n-3}}$, then v_1 is adjacent to all vertices of $N_1(x)$, since otherwise $\mathrm{diam}(G) = 4$. One deduces $G \cong (2K_1 + \overline{K_{n-2}}) - e$, as depicted in Figure 6(a).

(A3.2) $N_1(x)$ induces K_{n-3}. By a similar reason to subcase (A2.1), the number of neighbors of v_1 in $N_1(x)$, denoted by $d_{N_1(x)}(v_1)$, is either $1, 2, n-4$ or $n-3$, since otherwise we have an $(n-4)$-resolving partition of G.

If $d_{N_1(x)}(v_1) = 1$ or $d_{N_1(x)}(v_1) = n-3$, then $G \cong G_{10}$ or $G \cong F$, respectively. However, $pd(G_{10}) = n-2$ [18] and $pd(F) = n-2$ [3], a contradiction. If $d_{N_1(x)}(v_1) = 2$, then $G \cong F_{13}$. Otherwise, $d_{N_1(x)}(v_1) = n-4$ and one deduces $G \cong F_2$, see Figures 6(b) and 6(c).

(A3.3) $N_1(x)$ induces $K_{1,n-4}$. Let u_1 be the centre of $K_{1,n-4}$. Now we consider the neighbors of v_1 in $N_1(x) \setminus \{u_1\}$. If $v_1u_2 \in E(G)$, then $(u_2)(x, u_3)(u_1, u_4)$ $(v_1, u_5)(w_1, u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $v_1u_i \notin E(G)$ for all $i \neq 1$ and one deduces $G \cong F_{40}$ with $v_1u_1 \in E(G)$, as depicted in Figure 6(d).

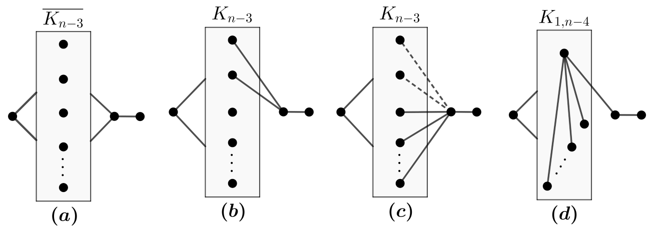

Fig. 6. Graphs (a) $(2K_1 + \overline{K_{n-2}}) - e$, (b) F_{13}, (c) F_2, and (d) F_{40}.

(A3.4) $N_1(x)$ induces $K_{n-4} \cup K_1$. Let u_1 be an isolated vertex of $K_{n-4} \cup K_1$. Note that $v_1u_1 \in E(G)$ and $v_1u_i \in E(G)$ for at least one $i \geq 2$, since otherwise $\text{diam}(G) = 4$. Now, we consider the neighbors of v_1 in $N_1(x) \setminus \{u_1\}$. If $v_1u_2, v_1u_3 \in E(G)$ and $v_1u_4 \notin E(G)$, then $(v_1)(x, u_2)(u_3, u_4)(u_1, u_5)(w_1, u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, we have two cases. First, if v_1 is only adjacent to exactly one vertex of K_{n-4}, namely $v_1u_2 \in E(G)$, then it is follows that $(u_1, v_1, w_1)(u_2, u_3)(x, u_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Second, if v_1 is adjacent to all vertices of K_{n-4} then one deduces $G \cong F_{19}$, as in Figure 7(a).

(A3.5) $N_1(x)$ induces $K_{n-3} - E(K_{1,n-5})$. Assume the edge set of $K_{n-3} - E(K_{1,n-5})$ is $\{u_iu_j : 1 \leq i < j \leq n-3\} \setminus \{u_2u_i : 3 \leq i \leq n-3\}$. Note that $v_1u_2 \notin E(G)$, since otherwise $(u_2)(u_1, u_3)(x, u_4)(u_5, v_1, w_1)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $v_1u_1 \in E(G)$, since otherwise $\text{diam}(G) = 4$, a contradiction. Furthermore, if $v_1u_3 \in E(G)$ and $v_1u_4 \notin E(G)$, then $(u_2)(v_1)(u_3, u_4)(x, u_5)(u_1, u_6)(w_1, u_7)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, we have two cases. First, if $v_1u_i \notin E(G)$ for all $i \geq 3$, then it is follows that $(x, u_4)(u_1, u_3)(u_2, v_1, w_1)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Second, if $v_1u_i \in E(G)$ for all $i \geq 3$, then one deduces $G \cong F_{23}$, as depicted in Figure 7(b).

(A3.6) $N_1(x)$ induces $K_{n-3} - e$. Let $e = u_1u_2$. By a similar reason to subcase (A2.6), the number of neighbors of v_1 in $N_1(x) \setminus \{u_1, u_2\}$ is either $0, 1$ or $n-5$.

If $v_1u_i \notin E(G)$ for all $i \geq 3$, then $v_1u_1, v_1u_2 \in E(G)$, since otherwise $\text{diam}(G) = 4$, a contradiction. This case yields that $G \cong F_{24}$ as depicted in Figure 7(c).

If $v_1u_3 \in E(G)$ and $v_1u_i \notin E(G)$ for all $i \geq 4$, then $v_1u_i \notin E(G)$ for at least one of $i \in \{1, 2\}$, since otherwise $(v_1)(x, u_1)(u_2, u_4)(u_3, u_5)(w_1, u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This case produces $G \cong F_{15}$ if $v_1u_1, v_1u_2 \notin E(G)$, or $G \cong F_{25}$ if $v_1u_1 \in E(G)$ and $v_1u_2 \notin E(G)$, as depicted in Figure 7(d) or 7(e), respectively.

Now assume that $v_1u_i \in E(G)$ for all $i \geq 3$. It follows that $v_1u_i \in E(G)$ for at least one of $i \in \{1, 2\}$, since otherwise $(u_1, u_3)(u_2, u_4)(x, u_5)(v_1, w_1)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. One deduces $G \cong F_5$ if $v_1u_1 \in E(G)$ and $v_1u_2 \notin E(G)$, or $G \cong F_8$ if $v_1u_1, v_1u_2 \in E(G)$, as depicted in Figure 7(f) or 7(g), respectively.

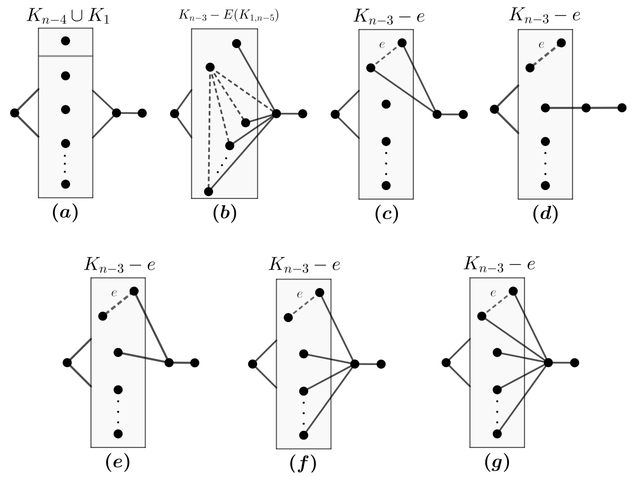

Fig. 7. Graphs (a) F_{19}, (b) F_{23}, (c) F_{24}, (d) F_{15}, (e) F_{25}, (f) F_5, and (g) F_8

(A4) $(1, 2, n-4)$.
If there exist three vertices $w_1, w_2, w_3 \in N_3(x)$ such that $w_1w_2 \in E(G)$ and $w_1w_3 \notin E(G)$, then $(x)(w_1)(u_1, v_1, w_4)$ $(w_2, w_3)(v_2, w_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $N_3(x)$ induces (A4.1) $\overline{K_{n-4}}$ or (A4.2) K_{n-4}.

(A4.1) $N_3(x)$ induces $\overline{K_{n-4}}$. In this case, $(x, w_1)(u_1, w_2)$ $(v_1, w_3)(v_2, w_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, there exists no graph G with $pd(G) = n-3$ satisfying this condition.

(A4.2) $N_3(x)$ induces K_{n-4}. If $v_1w_1, v_1w_2 \in E(G)$ and $v_1w_3, v_1w_4 \notin E(G)$, then $(x)(v_1)(u_1, v_2, w_5)(w_1, w_3)$ $(w_2, w_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, the number of neighbors of v_i in $N_3(x)$ for any $i \in \{1, 2\}$, denoted by $d_{N_3(x)}(v_i)$, is either $0, 1, n-5$ or $n-4$.

If $d_{N_3(x)}(v_1) = 0$, then $d_{N_3(x)}(v_2) = n-4$. This implies that $G \cong F_{33}$ if $v_1v_2 \notin E(G)$ or $G \cong F_{34}$ if $v_1v_2 \in E(G)$, as depicted in Figures 8(a) or 8(b). Now let $d_{N_3(x)}(v_1) = 1$ with $v_1w_1 \in E(G)$. If $d_{N_3(x)}(v_2) = n-5$ with $v_2w_1 \notin E(G)$, then $(w_1)(x, w_2)(u_1, v_1, w_3)(v_2, w_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. If $d_{N_3(x)}(v_2) = n-4$, then $(v_1)(w_1, w_2)(v_2, w_3)(x, w_4)(u_1, w_5)\pi$ or $(x, u_1, v_1)(w_1, w_2)$

$(v_2, w_3)\pi$ is an $(n-4)$-resolving partition of G for $v_1v_2 \in E(G)$ or $v_1v_2 \notin E(G)$, respectively. However this leads to a contradiction. Next, let $d_{N_3(x)}(v_1) = n-5$ with $v_1w_1 \notin E(G)$. If $d_{N_3(x)}(v_2) = n-5$ with $v_2w_2 \notin E(G)$, then $(v_1)(v_2)(w_1, w_3)(w_2, w_4)(x, w_5)(u_1, w_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. If $d_{N_3(x)}(v_2) = n-4$, then $v_1v_2 \in E(G)$, since otherwise $(v_1)(w_1, w_2)(v_2, w_3)(x, w_4)(u_1, w_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. One deduces $G \cong F_{25}$, as depicted in Figure 8(c).

For the remaining case, let $d_{N_3(x)}(v_1) = d_{N_3(x)}(v_2) = n-4$. This condition yields $G \cong F_{24}$ if $v_1v_2 \notin E(G)$ or $G \cong F_{13}$ if $v_1v_2 \in E(G)$, as depicted in Figures 8(d) or 8(e), respectively.

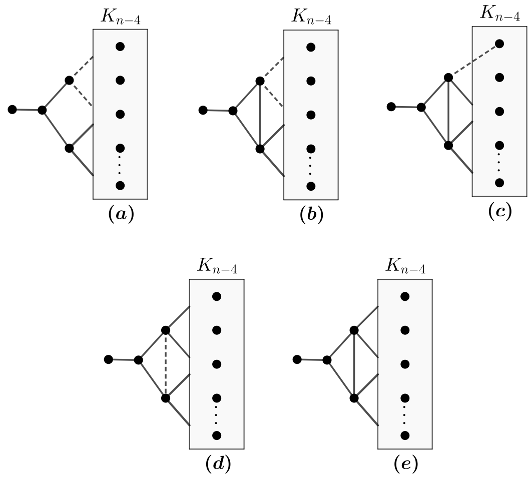

Fig. 8. Graph (a) F_{33}, (b) F_{34}, (c) F_{25}, (d) F_{24}, (e) F_{13}

(A5) $(1, n-4, 2)$.
By a similar reason to Case (A4), $N_2(x)$ also induces one of (A5.1) $\overline{K_{n-4}}$ or (A5.2) K_{n-4}.

(A5.1) $N_2(x)$ induces $\overline{K_{n-4}}$. If $w_1v_1 \in E(G)$ and $w_1v_2 \notin E(G)$, then $(x)(u_1, v_3)(v_4, w_2)(v_5, w_1)(v_1, v_2)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, any vertex of $N_3(x)$ is adjacent to all vertices $N_2(x)$. However, $(v_1)(v_2, w_2)(v_3, w_1)(x, u_1, v_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies there exists no graphs satisfying this condition.

(A5.2) $N_2(x)$ induces K_{n-4}. If $w_1v_1, w_1v_2 \in E(G)$ and $w_1v_3, w_1v_4 \notin E(G)$, then $(x)(w_1)(v_1, v_3)(v_2, v_4)(u_1, v_5)$ $(v_6, w_2)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, the number of neighbors of w_i in $N_2(x)$ for any $i \in \{1, 2\}$ is either $1, n-5$ or $n-4$.

If $d_{N_2(x)}(w_1) = d_{N_2(x)}(w_2) = 1$ and w_1 and w_2 are adjacent to the same vertex in $N_2(x)$, then $G \cong F_{36}$ if $w_1w_2 \notin E(G)$ or $G \cong F_{37}$ if $w_1w_2 \in E(G)$, as depicted in Figure 9(a) or (b). If w_1 and w_2 are not adjacent to the same vertex in $N_2(x)$, say $w_1v_1, w_2v_2 \in E(G)$, then $(x, u_1, v_1)(v_2, v_3)(w_1, w_2)\pi$ is an $(n-4)$-resolving partition of G, a contradiction.

Let $d_{N_2(x)}(w_1) = 1$ and $w_1v_1 \in E(G)$. If $d_{N_2(x)}(w_2) = n-5$ and w.l.o.g. $w_2v_1 \notin E(G)$ or $w_2v_2 \notin E(G)$, then $(w_2)(v_1, v_2)(u_1, v_3)(x, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. If $d_{N_2(x)}(w_2) = n-4$, then one can deduce $G \cong F_{17}$ if $w_1w_2 \notin E(G)$ or $G \cong F_{23}$ if $w_1w_2 \in E(G)$, as depicted in Figures 9(c) or 9(d)).

Let $d_{N_2(x)}(w_1) = n-5$ and $w_1v_1 \notin E(G)$. If $d_{N_2(x)}(w_2) = n-5$, then $w_2v_1 \notin E(G)$, since otherwise for $w_2v_2 \notin E(G)$ we have $(w_1)(w_2)(v_1, v_3)(v_2, v_4)$ $(x, v_5)(u_1, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Furthermore, w_1 must be adjacent to w_2, since otherwise $(w_1)(v_1, v_2)(x, v_3)(u_1, v_4)(w_2, v_5)\pi$ is also an $(n-4)$-resolving partition of G, a contradiction. This condition yields $G \cong F_{11}$, as depicted in Figure 9(e). If $d_{N_2(x)}(w_2) = n-4$ then w_1 must be adjacent to w_2, since otherwise $(w_1)(v_1, v_2)(x, v_3)(u_1, v_4)(w_2, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This condition yields $G \cong F_5$, as depicted in Figure 9(f).

For the remaining case, let $d_{N_2(x)}(w_1) = d_{N_2(x)}(w_2) = n-4$. This case produces $G \cong F_9$ if $w_1w_2 \notin E(G)$ or $G \cong F_2$ if $w_1w_2 \in E(G)$, as depicted in Figures 9(g) or 9(h), respectively.

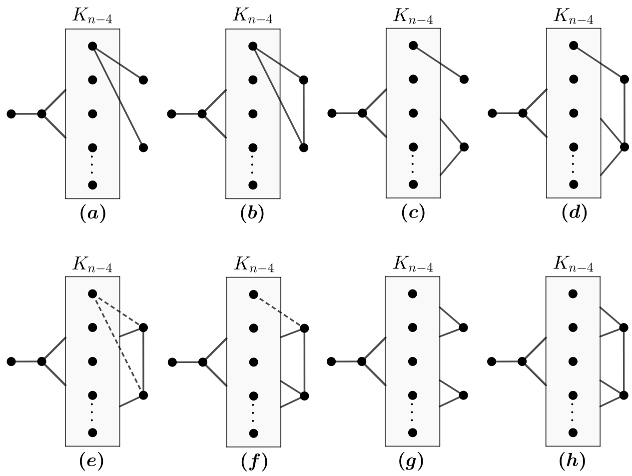

Fig. 9. Graphs (a) F_{36}, (b) F_{37}, (c) F_{17}, (d) F_{23}, (e) F_{11}, (f) F_5, (g) F_9, and (h) F_2

(A6) $(2, 1, n-4)$.
By a similar reason to Case (A4), $N_3(x)$ also induces one of (A6.1) $\overline{K_{n-4}}$ or (A6.2) K_{n-4}, since otherwise we have an $(n-4)$-resolving partition of G, a contradiction. Note that for these two subcases, $v_1u_1, v_1u_2 \in E(G)$ or (one of $\{v_1u_1, v_1u_2\}$ is in $E(G)$ and $u_1u_2 \in E(G)$), since otherwise diam$(G) = 4$.

(A6.1) $N_3(x)$ induces $\overline{K_{n-4}}$. If $v_1u_1, v_1u_2 \in E(G)$, or one of $\{v_1u_1, v_1u_2\}$ is in $E(G)$ and $u_1u_2 \in E(G)$, then $(x, w_1)(u_1, w_2)(u_2, w_3)(v_1, w_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, we conclude that there exists no graph G satisfying this condition.

(A6.2) $N_3(x)$ induces K_{n-4}. In this case, one deduces $G \cong F_{38}$ if $v_1u_1, u_1u_2 \in E(G)$ and $v_1u_2 \notin E(G)$, or $G \cong F_{35}$ if $v_1u_1, v_1u_2 \in E(G)$ and $u_1u_2 \notin E(G)$, or $G \cong F_{39}$ if $v_1u_1, v_1u_2, u_1u_2 \in E(G)$, as depicted in Figure 10.

(A7) $(2, n-4, 1)$.
By a similar reason to Case (A4), $N_2(x)$ induces (A7.1) $\overline{K_{n-4}}$ or (A7.2) K_{n-4}.

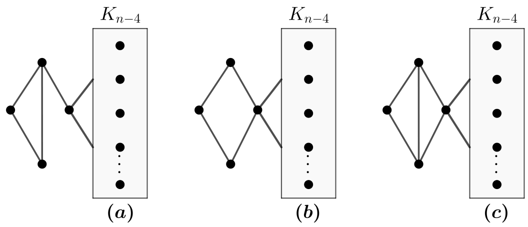

Fig. 10. Graphs (a) F_{38}, (b) F_{35}, and (c) F_{39}

(A7.1) $N_2(x)$ induces $\overline{K_{n-4}}$. Suppose that $w_1v_1 \in E(G)$. However, $(x, v_1)(u_1, v_2)(u_2, v_3)(w_1, v_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This concludes that there exists no graph G satisfying this condition.

(A7.2) $N_2(x)$ induces K_{n-4}. If $u_1v_1, u_1v_2 \in E(G)$ and $u_1v_3, u_1v_4 \notin E(G)$, then $(x)(u_1)(v_1, v_3)(v_2, v_4)(u_2, v_5)(w_1, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore the number of neighbors of u_i in $N_2(x)$ for any $i \in \{1, 2\}$ is either $0, 1, n-5$ or $n-4$.

If $d_{N_2(x)}(u_1) = 0$, then $d_{N_2(x)}(u_2) = n-4$. This implies that $u_1u_2 \in E(G)$, since otherwise diam$(G) = 4$. Now we consider the number of neighbors of w_1 in $N_2(x)$. If $w_1v_1, w_1v_2 \in E(G)$ and $w_1v_3 \notin E(G)$, then $(w_1)(v_1, v_3)(u_2, v_2)(u_1, v_4)(x, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $d(w_1) = 1$ or $d(w_1) = n-4$. The first case produces $G \cong F_{37}$ and the second case yields $G \cong F_{26}$ (Figures 11(a) and 11(b)).

Now assume that $d_{N_2(x)}(u_1) = 1$ with $u_1v_1 \in E(G)$. If $d_{N_2(x)}(u_2) = n-5$, then $u_2v_i \in E(G)$ for all $i \neq 1$. However, one deduces that $(u_2)(v_1, v_2)(x, v_3)(u_1, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $d_{N_2(x)}(u_2) = n-4$. To see the number of neighbors of w_1 in $N_2(x)$, by a similar reason to the previous case we also can conclude that $d(w_1) = 1$ or $d(w_1) = n-4$. However, in the first case one deduces that $(u_1)(v_1, v_2)(x, v_3)(u_2, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G if $u_1u_2 \in E(G)$, or $(x, u_1)(v_1, v_2, w_1)(u_2, v_3)\pi$ is an $(n-4)$-resolving partition of G if $u_1u_2 \notin E(G)$, a contradiction. In the second case $G \cong F_{27}$ if $u_1u_2 \notin E(G)$ or $G \cong F_{28}$ if $u_1u_2 \in E(G)$ (Figures 11(c) or 11(d)).

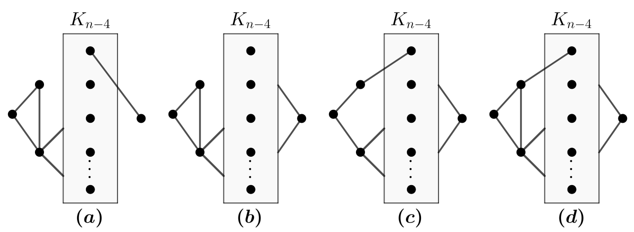

Fig. 11. Graphs (a) F_{37}, (b) F_{26}, (c) F_{27}, and (d) F_{28}

Let $d_{N_2(x)}(u_1) = n-5$ where $u_1v_1 \notin E(G)$. If $d_{N_2(x)}(u_2) = n-5$ where $u_2v_2 \notin E(G)$, then $(x, u_1)(u_2)(v_1, v_3)(v_2, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $d_{N_2(x)}(u_2) = n-4$. In this case $u_1u_2 \in E(G)$, since otherwise $(u_1)(v_1, v_2)(x, v_3)(u_2, v_4)(w_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Furthermore, we consider the number of neighbors of w_1 in $N_2(x) \setminus \{v_1\}$. If $w_1v_2, w_1v_3 \in E(G)$ and $w_1v_4 \notin E(G)$, then $(w_1)(v_1)(v_2, v_4)(u_2, v_3)(u_1, v_5)(x, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that the number of neighbors of w_1 in $N_2(x) \setminus \{v_1\}$ is either $0, 1$ or $n-5$. If $w_1v_i \notin E(G)$ for all $i \neq 1$, then $w_1v_1 \in E(G)$ and one deduces $G \cong F_{23}$ as depicted in Figure 12(a). If $w_1v_2 \in E(G)$ and $w_1v_i \notin E(G)$ for all $i \neq 1, 2$. Then $w_1v_1 \notin E(G)$, since otherwise $(x)(w_1)(v_1, v_3)(v_2, v_4)(u_1, v_5)(u_2, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. We deduce $G \cong F_{22}$, as depicted in Figure 12(b). Otherwise assume that $w_1v_i \in E(G)$ for all $i \neq 1$. Then $w_1v_1 \in E(G)$, since otherwise $(w_1)(v_1, v_2)(x, v_3)(u_1, v_4)(u_2, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. We deduce $G \cong F_6$, as depicted in Figure 12(c).

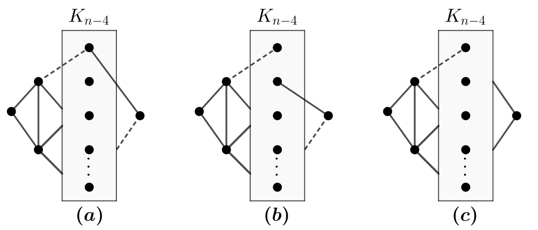

Fig. 12. Graphs (a) F_{23}, (b) F_{22}, and (c) F_6

For the remaining case, let $d_{N_1(x)}(u_1) = d_{N_1(x)}(u_2) = n-4$. We consider the number of neighbors of w_1 in $N_2(x)$. If $w_1v_1, w_1v_2 \in E(G)$ and $w_1v_3, w_1v_4 \notin E(G)$, then $(x)(w_1)(v_1, v_3)(v_2, v_4)(u_1, v_5)(u_2, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. If $w_1v_1, w_1v_2, w_1v_3 \in E(G)$ and $w_1v_4 \notin E(G)$, then $(w_1)(u_1, v_1)(u_2, v_2)(v_3, v_4)(x, v_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $d(w_1) = 1$ or $d(w_1) = n-4$. In the first case $G \cong F_{21}$ if $u_1u_2 \notin E(G)$ or $G \cong F_{14}$ if $u_1u_2 \in E(G)$. In the second case $G \cong F_{10}$ if $u_1u_2 \notin E(G)$ or $G \cong F_3$ if $u_1u_2 \in E(G)$ (Figure 13).

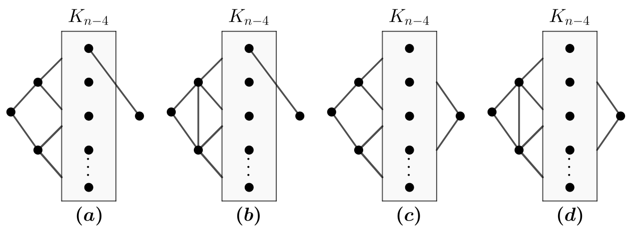

Fig. 13. Graphs (a) F_{21}, (b) F_{14}, (c) F_{10}, and (d) F_3

(A8) $(n-4, 1, 2)$.
By a similar reason to Case (A4), $N_1(x)$ induces $(A8.1)$ $\overline{K_{n-4}}$ or (A8.2) K_{n-4}.

(A8.1) $N_1(x)$ induces $\overline{K_{n-4}}$. However, $(x, u_1)(u_2, v_1)(u_3, w_1)(u_4, w_2)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, no graph G satisfying this condition.

(A8.2) $N_1(x)$ induces K_{n-4}. We consider the number of neighbors of vertex v_1 in $N_1(x)$. If $v_1u_1, v_1u_2 \in E(G)$ and $v_1u_3 \notin E(G)$, then $(v_1)(x, u_1)(u_2, u_3)(u_4, w_1)(u_5, w_2)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $d_{N_1(x)}(v_1) = 1$ or $d_{N_1(x)}(v_1) = n-4$. In the first case one deduces $G \cong F_{33}$ if $u_1u_2 \notin E(G)$ or $G \cong F_{38}$ if $u_1u_2 \in E(G)$, as depicted in Figures 14(a) or 14(b). In the second case one deduces $G \cong F_{16}$ if $u_1u_2 \notin E(G)$ or $G \cong F_{26}$ if $u_1u_2 \in E(G)$ (Figure 14(c) or 14(d)).

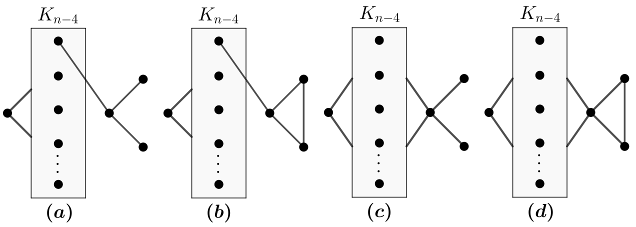

Fig. 14. Graphs (a) F_{33}, (b) F_{38}, (c) F_{16}, and (d) F_{26}

(A9) $(n-4, 2, 1)$.
By a similar reason to Case (A4), $N_1(x)$ induces (A9.1) $\overline{K_{n-4}}$ or (A9.2) K_{n-4}.

(A9.1) $N_1(x)$ induces $\overline{K_{n-4}}$. If $v_1u_1, v_2u_2 \in E(G)$, then $(u_1)(u_2)(x,u_3)(v_1,u_4)(v_2,u_5)(w_1,u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, no graph G satisfying this condition.

(A9.2) $N_1(x)$ induces K_{n-4}. If $v_1u_1, v_1u_2 \in E(G)$ and $v_1u_3, v_1u_4 \notin E(G)$, then $(x)(v_1)(u_1,u_3)(u_2,u_4)(v_2,u_5)$ $(w_1,u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that the number of neighbors of v_i in $N_1(x)$ for any $i \in \{1,2\}$ is either $1, n-5$ or $n-4$.

Let $d_{N_1(x)}(v_1) = 1$ with $v_1u_1 \in E(G)$. If $d_{N_1(x)}(v_2) = 1$ with $v_2u_2 \in E(G)$, then $w_1v_1, w_1v_2 \in E(G)$, or $w_1v_i \in E(G)$ for some i and $v_1v_2 \in E(G)$, since otherwise $\text{diam}(G) = 4$. However, for the first case one deduces that $(x)(w_1)(u_1,u_3)(u_2,u_4)(v_1,u_5)(v_2,u_6)\pi$ is an $(n-4)$-resolving partition of G, and for the second case $(x,u_1)(u_2,u_3)(v_1,v_2,w_1)\pi$ is also an $(n-4)$-resolving partition of G, a contradiction. If $d_{N_1(x)}(v_2) = 1$ with $v_2u_1 \in E(G)$, then $G \cong F_{32}$ if $w_1v_1 \in E(G)$ and $w_1v_2, v_1v_2 \notin E(G)$, or $G \cong F_{34}$ if $w_1v_1, v_1v_2 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_{35}$ if $w_1v_1, w_1v_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong F_{39}$ if $w_1v_1, w_1v_2, v_1v_2 \in E(G)$, as depicted in Figure 15.

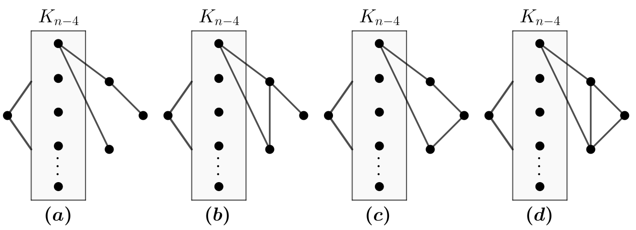

Fig. 15. Graphs (a) F_{32}, (b) F_{34}, (c) F_{35}, and (d) F_{39}

If $d_{N_1(x)}(v_2) = n-5$ with $v_2u_1 \notin E(G)$, then $(v_2)(u_1,u_2)(x,u_3)(v_1,u_4)(w_1,u_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Otherwise, assume that $d_{N_1(x)}(v_2) = n-4$. One deduces $G \cong F_{15}$ if $w_1v_1 \in E(G)$ and $w_1v_2, v_1v_2 \notin E(G)$, or $G \cong F_{25}$ if $w_1v_1, v_1v_2 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_{17}$ if $w_1v_2 \in E(G)$ and $w_1v_1, v_1v_2 \notin E(G)$, or $G \cong F_{20}$ if $w_1v_2, v_1v_2 \in E(G)$ and $w_1v_1 \notin E(G)$, or $G \cong F_{27}$ if $w_1v_1, w_1v_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong F_{28}$ if $w_1v_1, w_1v_2, v_1v_2 \in E(G)$ (Figure 16).

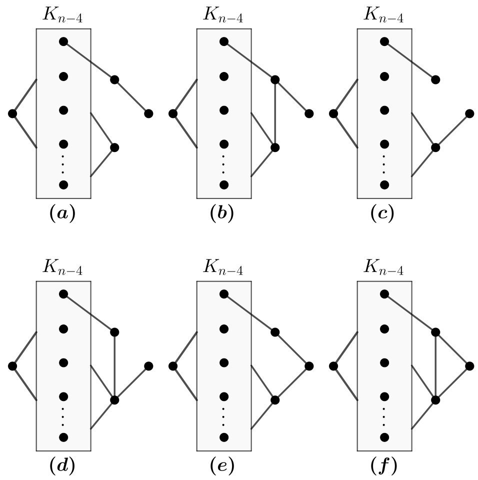

Fig. 16. Graphs (a) F_{15}, (b) F_{25}, (c) F_{17}, (d) F_{20}, (e) F_{27}, and (f) F_{28}

Let $d_{N_1(x)}(v_1) = n-5$ with $v_1u_1 \notin E(G)$. If $d_{N_1(x)}(v_2) = n-5$ with $v_2u_2 \notin E(G)$ or $v_1v_2 \notin E(G)$, then $(v_1)(u_2)(u_1,u_3)(x,u_4)(v_2,u_5)(w_1,u_6)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. It follows that if $d_{N_1(x)}(v_2) = n-5$ with $v_2u_i \notin E(G)$ for some i, then $i = 1$ and $v_1v_2 \in E(G)$. One deduces $G \cong F_{11}$ if $w_1v_1 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_{12}$ if $w_1v_1, w_1v_2 \in E(G)$. Otherwise, $d_{N_1(x)}(v_2) = n-4$. By a similar reason to the previous case, $v_1v_2 \in E(G)$, since otherwise we have an $(n-4)$-resolving partition of G. We deduce $G \cong F_5$ if $w_1v_1 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_4$ if $w_1v_2 \in E(G)$ and $w_1v_1 \notin E(G)$, or $G \cong F_7$ if $w_1v_1, w_1v_2 \in E(G)$.

For the remaining case, let $d_{N_1(x)}(v_1) = d_{N_1(x)}(v_2) = n-4$. We deduce $G \cong F_9$ if $w_1v_1 \in E(G)$ and $w_1v_2, v_1v_2 \notin E(G)$, or $G \cong F_1$ if $w_1v_1, v_1v_2 \in E(G)$ and $w_1v_2 \notin E(G)$, or $G \cong F_{10}$ if $w_1v_1, w_1v_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong F_3$ if $w_1v_1, w_1v_2, v_1v_2 \in E(G)$ (Figure 17).

(B) $\text{diam}(G) = 4$.
Let x be a peripheral vertex of G with $\text{ecc}(x) = 4$. Let $u \in N_1(x)$, $v \in N_2(x)$, $w \in N_3(x)$ and $z \in N_4(x)$. If there exist two other vertices p and q such that $p \in N_1(x)$ and $q \in N_2(x)$, then $(x)(u,v,w,z)(p,q)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that only one of $\{n_1, n_2, n_3, n_4\}$ is greater than or equal to 2. Therefore, based on the values of (n_1, n_2, n_3, n_4) we have the following subcases: (B1) $(1,1,1,n-4)$, (B2) $(1,1,n-4,1)$, (B3) $(1,n-4,1,1)$, and (B4) $(n-4,1,1,1)$. Now w.l.o.g., assume that $n_1 = n-4$. Since $|V(G)| \geq 11$, then there exist three other vertices $a, b, c \in N_1(x) \setminus \{u\}$. If $ab \in E(G)$ and $ac \notin E(G)$, then $(x)(a)(u,v,w,z)(b,c)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $N_1(x)$ induces either $\overline{K_{n-4}}$ or K_{n-4}. Hence we can conclude that if $n_i \geq n-4$, then $N_i(x)$ induces either $\overline{K_{n-4}}$ or K_{n-4}.

Let the set of vertices of $N_i(x)$ for all $i \in \{1,2,3,4\}$ be $N_1(x) = \{u_i : 1 \leq i \leq n_1\}$, $N_2(x) = \{v_i : 1 \leq i \leq n_2\}$, $N_3(x) = \{w_i : 1 \leq i \leq n_3\}$, and $N_4(x) = \{z_i : 1 \leq i \leq n_4\}$.

(B1) $(1,1,1,n-4)$.

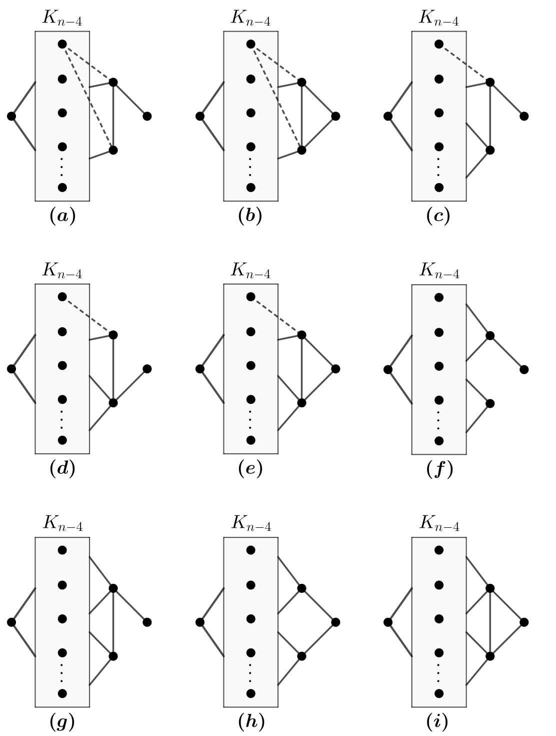

Fig. 17. Graph (a) F_{11}, (b) F_{12}, (c) F_5, (d) F_4, (e) F_7, (f) F_9, (g) F_1, (h) F_{10}, and (i) F_3

If $N_4(x)$ induces $\overline{K_{n-4}}$, then $(x, z_1)(u_1, z_2)(v_1, z_3)(w_1, z_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $N_4(x)$ induces K_{n-4} and it follows that $G \cong H_5$, as dipected in Figure 18(a).

(B2) $(1, 1, n-4, 1)$.
If $N_3(x)$ induces $\overline{K_{n-4}}$, then $(x, w_1)(u_1, w_2)(v_1, w_3)(z_1, w_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $N_3(x)$ induces K_{n-4}. If $w_1z_1, w_2z_1 \in E(G)$ but $w_3z_1, w_4z_1 \notin E(G)$, then $(x)(z_1)(w_1, w_3)(w_2, w_4)(u_1, v_1, w_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, the number of neighbors of z_1 in $N_3(x)$ is either $1, n-5$ or $n-4$. However, if $d(z_1) = 1$, namely $w_1z_1 \in E(G)$ and $w_iz_1 \notin E(G)$ for all other $i \neq 1$, then $(x, u_1, z_1)(w_1, w_2)(v_1, w_3)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Hence the resulting graph is $G \cong H_3$ if $d(z_1) = n-5$ or $G \cong H_1$ if $d(z_1) = n-4$, as depicted in Figures 18(b) or 18(c), respectively.

(B3) $(1, n-4, 1, 1)$.
Let $N_2(x)$ induces $\overline{K_{n-4}}$. If $w_1v_1, w_1v_2 \in E(G)$ and $w_1v_3 \notin E(G)$, then $(x, v_1)(u_1, v_2)(z_1, v_3)(w_1, v_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Furthermore, if $w_1v_i \in E(G)$ for all i, then $(x, v_1)(u_1, v_2)(w_1, v_3)(z_1, v_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that $d_{N_2(x)}(w_1) = 1$ and one deduces $G \cong H_4$, as depicted in Figure 18(d).

Let $N_2(x)$ induces K_{n-4}. If $w_1v_1, w_1v_2 \in E(G)$ and $w_1v_3 \notin E(G)$, then $(w_1)(v_1, v_3)(u_1, v_2)(x, v_4)(z_1, v_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that the number of neighbors of w_1 in $N_2(x)$ is either 1 or $n-4$. However, in the first case by considering $v_1w_1 \in E(G)$ one deduces that $(x, w_1, z_1)(v_1, v_2)(u_1, v_3)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. In the second case, the resulting graph is $G \cong H_2$, as depicted in Figure 18(e).

(B4) $(n-4, 1, 1, 1)$.
If $N_1(x)$ induces $\overline{K_{n-4}}$, then v_1 is adjacent to all vertices of $N_1(x)$ since otherwise $\text{diam}(G) = 5$. However, $(x, u_1)(v, u_2)(w, u_3)(z, u_4)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. Therefore, $N_2(x)$ induces K_{n-4}. If $u_1v_1, u_2v_1 \in E(G)$ but $u_3v_1, u_4v_1 \notin E(G)$, then $(v_1)(u_1, u_3)(x, u_2)(w_1, u_4)(z_1, u_5)\pi$ is an $(n-4)$-resolving partition of G, a contradiction. This implies that the number of neighbors of v_1 in $N_1(x)$ is either 1 or $n-4$. One deduces $G \cong H_5$ for the first case or $G \cong H_1$ for the second case, as depicted in Figures 18(f) or 18(g), respectively.

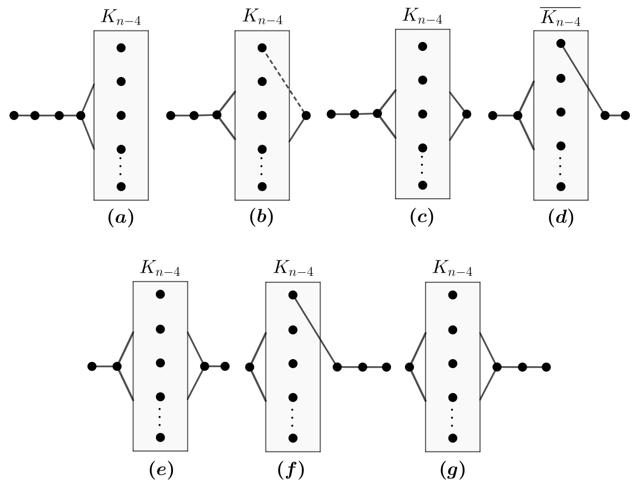

Fig. 18. Graph (a) H_5, (b) H_3, (c) H_1, (d) H_4, (e) H_2, (f)H_5, (g) H_1

■

III. Conclusion

In this paper, we give the characterization of all graphs G of order $n \geq 11$ and $\text{diam}(G) \in \{3, 4\}$ with $pd(G) = n-3$, as stated in Theorem 2.3. There are 46 non-isomorphic such graphs, 41 of them with diameter 3 and the remaining 5 such graphs with diameter 4. By combining Theorem 2.3 and the results of [3], we have a full characterization of all graphs on $n \geq 11$ vertices with partition dimension $n-3$, namely there are exactly 160 non-isomorphic such graphs.

Appendix

Graphs F_i and H_j, for $i \in [1, 40]$ and $j \in [1, 5]$, obtained by Theorem 2.3 can be classified in the following manner.

Graphs of order n obtained from $K_{n-1} - E(P_3)$ by adding one new vertex adjacent to:
F_1: one end vertex of P_3;
F_2: a center vertex of P_3;
F_3: two end vertices of P_3;

Graphs of order n obtained from $K_{n-1} - E(P_4)$ by adding one new vertex adjacent to:
F_4: one end vertex of P_4;
F_5: one vertex of P_4 with degree two;

F_6: two vertices of P_4 with degree two;
F_7: two vertices of P_4 with different degree;

Graphs of order n obtained from $K_{n-1} - E(2K_2)$ by adding one new vertex adjacent to:
F_8: one end vertex of $2K_2$;

Graphs of order n obtained from $K_{n-1} - E(C_3)$ by adding one new vertex adjacent to:
F_9: one vertex of C_3;
F_{10}: two vertices of C_3;

Graphs of order n obtained from $K_{n-1} - E(C_4)$ by adding one new vertex adjacent to:
F_{11}: one vertex of C_4;
F_{12}: two vertices of C_4;

Graphs of order n obtained from K_{n-2} by connecting two new vertices x and y with:
F_{13}: exactly two vertices a and b in K_{n-2} such that (a, x), $(b, x), (x, y)$ are new edges;
F_{14}: exactly three vertices a, b and c in K_{n-2} such that $(a, x), (b, x), (c, y)$ are new edges;

Graphs of order n obtained from $K_{n-2} - e$ by connecting two new vertices x and y with:
F_{15}: two new edges $(c, x), (x, y)$, where c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{16}: two new edges $(a, x), (a, y)$, where a is one of the end points of e;
F_{17}: two new edges $(a, x), (c, y)$, where a is one of the end vertex of e and c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{18}: two new edges $(c, x), (d, y)$, where c and d are two vertices of $K_{n-2} - e$ with maximum degree;
F_{19}: three new edges $(a, x), (a, y), (b, y)$, where a and b are the end points of e;
F_{20}: three new edges $(a, x), (a, y), (c, y)$, where a is one of the end points of e and c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{21}: three new edges $(a, x), (b, x), (c, y)$, where a and b are the end points of e and c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{22}: three new edges $(a, x), (c, x), (d, y)$, where a is one of the end points of e, and c and d are two vertices of $K_{n-2} - e$ with maximum degree;
F_{23}: three new edges $(a, x), (b, y), (c, y)$, where a and b are the end points of e, and c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{24}: three new edges $(a, x), (b, x), (x, y)$, where a and b are the end points of e;
F_{25}: three new edges $(a, x), (c, x), (x, y)$, where a is one of the end points of e and c is a vertex of $K_{n-2} - e$ with maximum degree;
F_{26}: F_{16} by adding new edge (x, y);
F_{27}: F_{17} by adding new edge (x, y);
F_{28}: F_{27} by adding new edge (a, y);
H_1: two new edges $(a, x), (x, y)$, where a is one of the end points of e;
H_2: two new edges $(a, x), (b, y)$, where a and b are end points of e;

Graphs of order n obtained from $K_{n-2} - E(P_3)$ by adding two new vertices x and y with:
F_{29}: three new edges $(a, x), (c, x), (c, y)$, where a and c are end points of P_3;
H_3: two new edges $(a, x), (x, y)$, where a is an end point of P_3;

Graphs of order n obtained from $\overline{K_{n-2}}$:
F_{30}: $K_1 + \overline{K_{n-2}}$ and added by one new vertex adjacent to one vertex of $\overline{K_{n-2}}$;
F_{31}: $K_1 + \overline{K_{n-2}}$ and added by one new vertex adjacent to two vertices of $\overline{K_{n-2}}$;
H_4: $K_1 + \overline{K_{n-2}} - e$ and added by one new vertex adjacent to two vertices of $\overline{K_{n-2}}$ with different degrees;

Graphs of order n obtained from K_{n-3} by connecting three new vertices x, y, and z with:
F_{32}: exactly one vertex a in K_{n-3} such that $(a, x), (a, y)$, (y, z) are new edges;
F_{33}: exactly one vertex a in K_{n-3} such that $(a, x), (x, y)$, (x, z) are new edges;
F_{34}: exactly one vertex a in K_{n-3} such that $(a, x), (a, y)$, $(x, y), (x, z)$ are new edges;
F_{35}: exactly one vertex a in K_{n-3} such that $(a, x), (a, y)$, $(x, z), (y, z)$ are new edges;
F_{36}: exactly two vertices a and b in K_{n-3} such that (a, x), $(a, y), (b, z)$ are new edges;
F_{37}: F_{36} by adding new edge (x, y);
F_{38}: F_{33} by adding new edge (y, z);
F_{39}: F_{34} by adding new edge (y, z);
H_5: exactly one vertex a in K_{n-3} such that $(a, x), (x, y)$, (y, z) are new edges;

Graphs of order n obtained from $\overline{K_{n-3}}$:
F_{40}: $(K_2 + \overline{K_{n-3}}) - e$ where e is an edge connecting K_2 and $\overline{K_{n-3}}$, and added by one new vertex adjacent to one end point of e with minimum degree;

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