Some Integral Inequalities for Polynomial

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***Abstract*—This paper contains the finding of some upper bound estimates for the maximal modulus of a lacunary polynomial of degree n on a circle of radius $0 < r \leq R \leq k$ under the assumption that the polynomial has no zero in a disk of radius $k, k > 0$. Our result extends some known inequalities concerning derivative of a polynomial into integral analogues and it further generalizes as well as sharpens some other results in this direction.**

***Index Terms*—polynomial, zero, integral inequality, maximum modulus.**

I. Introduction

Let $p(z)$ be a polynomial of degree n. We define

$$\|p\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, 0 < \gamma < \infty. \quad (1)$$

If we let $\gamma \to \infty$ in the above equality and make use of the well-known fact from analysis [21] that

$$\lim_{\gamma \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, one can define $\|p\|_0 = exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|p(e^{i\theta})|d\theta \right\}$ and show that $\lim_{\gamma \to 0^+} \|p\|_\gamma = \|p\|_0$. It would be of further interest that by taking limits as $\gamma \to 0^+$ that the stated results holding for $\gamma > 0$, hold for $\gamma = 0$ as well.

For $r > 0$, we denote $M(p, r) = \max_{|z|=r} |p(z)|$.

A famous result due to Bernstein [16], [22] states that if $p(z)$ is a polynomial of degree n, then

$$\|p^{'}\|_\infty \leq n\|p\|_\infty. \quad (2)$$

Inequality (2) can be obtained by letting $\gamma \to \infty$ in the inequality

$$\|p^{'}\|_\gamma \leq n\|p\|_\gamma, \gamma > 0. \quad (3)$$

Inequality (3) for $\gamma \geq 1$ is due to Zygmund [23] and Arestov [1] proved that it remains valid for $0 < \gamma < 1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (2) and (3) can be respectively improved by

$$\|p^{'}\|_\infty \leq \frac{n}{2}\|p\|_\infty \quad (4)$$

and

$$\|p^{'}\|_\gamma \leq \frac{n}{\|1 + z\|_\gamma}\|p\|_\gamma, \gamma > 0. \quad (5)$$

Inequality (4) was conjectured by Erdös and later verified by Lax [14] , whereas, inequality (5) was proved by de-Bruijn [9] for $\gamma \geq 1$. Rahman and Schmeisser [19] showed that (5) remains true for $0 < \gamma < 1$. As a generalization of (4), Malik [15] proved that if $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\|p^{'}\|_\infty \leq \frac{n}{1 + k}\|p\|_\infty. \quad (6)$$

Under the same assumptions, Qazi [18, Lemma 1] improved the bound (6) by proving

$$\|p^{'}\|_\infty \leq \frac{n}{1 + \left(\frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}} \right)}\|p\|_\infty. \quad (7)$$

Under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [13] extended inequality (6) to L^γ norm by showing that

$$\|p^{'}\|_\gamma \leq \frac{n}{\|k + z\|_\gamma}\|p\|_\gamma, \gamma \geq 1. \quad (8)$$

It was shown by Gardner and Weems [12] and independently by Rather [20] that (8) also holds for $0 < \gamma < 1$.

While L^γ analogue of (7) was given for $\gamma \geq 1$ by Dewan et al. [10] and independently by Chanam [6] for $\gamma > 0$.

$$\|p^{'}\|_\gamma \leq \frac{n}{\|A + z\|_\gamma}\|p\|_\gamma, \quad (9)$$

where $A = \frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}}$.

Further, as a generalization of (6) Bidkham and Dewan [5] proved that

$$\|p^{'}(rz)\|_\infty \leq \frac{n(r + k)^{n-1}}{(1 + k)^n}\|p\|_\infty, \text{ for } 1 \leq r \leq k. \quad (10)$$

As a generalization of (10), Aziz and Zargar [4] proved that if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$ then for $0 < r \leq R \leq k$,

$$\|p^{'}(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\|p(rz)\|_\infty. \quad (11)$$

Equality holds in (11) for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

Moreover, as an improvement and generalization of (10), Aziz and Shah [3] proved that

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if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \le R \le k$,

$$\|p^{'}(Rz)\|_\infty \le \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \left\{\|p(rz)\|_\infty - m\right\}, \tag{12}$$

The result is best possible and equality in (12) holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

Further, Chanam and Dewan [7] improved (12) by involving certain coefficients of the polynomial. In fact, they proved

Theorem 1. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \le R \le k$,*

$$\|p^{'}(Rz)\|_\infty \le$$

$$n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu + k^{2\mu}R)}\right\} \times$$

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\right\}$$

$$\left\{\|p(rz)\|_\infty - m\right\}. \tag{13}$$

where $m = \min_{|z|=k} |p(z)|$.

Inequality (13) is sharp for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

Mir and Dar [17] proved the following inequality for the same class of polynomials by involving some more parameters, which they claimed that their result was a generalization and refinement of Theorem 1. But if we analyse closely, it is noticed that their result is just a weak generalization of Theorem 1.

Theorem 2. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for every $l > 1, 0 < r \le R \le k$ and $0 \le \lambda \le 1$,*

$$\|p(lRz) - p(Rz)\|_\infty \le$$

$$\frac{(l^n - 1)\left\{\left(\frac{l^\mu-1}{l^n-1}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^\mu + R^{\mu+1}\right\}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{l^\mu-1}{l^n-1}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)} \times$$

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\right\}$$

$$\left\{\|p(rz)\|_\infty - \lambda m\right\}, \tag{14}$$

where $m = \min_{|z|=k} |p(z)|$.

Dividing both sides of (14) by $R(l-1)$ and making limit as $l \to 1$, inequality (14) reduces to

$$\|p^{'}(Rz)\|_\infty \le$$

$$n\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)} \times$$

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\right\}$$

$$\left\{\|p(rz)\|_\infty - \lambda m\right\}, \tag{15}$$

where $0 \le \lambda \le 1$.

For $\lambda = 1$, inequality (15) immediately assumes (13) of Theorem 1. For each $\lambda \in (0, 1)$, inequality (15) does not set to any significant result having implications to the related existing results. For example, for $\lambda = \frac{1}{3}$, it is obvious from Lemma 5 that the first two factors in it are respectively less than or equal to that of inequality (13) of Theorem 1, whereas in the last factors, the situation is reverse. That is,

$$\|p(rz)\|_\gamma - \frac{m}{3} \ge \|p(rz)\|_\gamma - m.$$

Hence, as mentioned earlier, inequality (14) of Theorem 2 is just a weak generalization. Thus it would have been better for the authors [17] to set $\lambda = 1$ in the proof of Theorem 2 in order not to arise these ambiguities.

Extensions of (11) and (12) into L^γ norm were done very recently by Chanam et al. [8] by proving the following two results.

Theorem 3. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \le R \le k$, and $\gamma > 0$,*

$$\|p^{'}(Rz)\|_\gamma \le \frac{n}{R}F_\gamma \times$$

$$\left[\int_0^{2\pi}\left\{|p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu}M(p,t)dt\right\}^\gamma d\theta\right]^{\frac{1}{\gamma}} \tag{16}$$

where $M(p,t) = \max_{|z|=t} |p(z)|$

and $F_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^\mu + e^{i\alpha}\right|^\gamma d\alpha\right\}^{\frac{-1}{\gamma}}$.

Theorem 4. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \le R \le k$, and $\gamma > 0$,*

$$\|p^{'}(Rz)\|_\gamma \le \frac{n}{R}F_\gamma \times$$

$$\left[\int_0^{2\pi}\left\{|p(re^{i\theta})| + n\left\{\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}mdt\right\} - m\right\}^\gamma d\theta\right]^{\frac{1}{\gamma}}, \tag{17}$$

where F_γ and $M(p,t)$ are as defined in Theorem 3 and $m = \min_{|z|=k} |p(z)|$.

II. Lemmas

For the proof of the theorem, we require the following lemmas.

Lemma 5. *For $\mu = 1, 2, 3, ...n, n \in \mathbb{N}$, any complex number $a_\mu \ne 0$, and for every $0 < R \le k$, the function*

$$f(x) = \frac{\frac{\mu}{n}\frac{|a_\mu|}{x}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{x}(k^{\mu+1}R^\mu + k^{2\mu}R)} \tag{18}$$

is a non-increasing function of $x > 0$.

Proof: The proof follows simply from first the derivative test. For

$$f^{'}(x) = -\frac{\frac{\mu}{n}\frac{|a_\mu|^2}{x^2}k^{2\mu}R^{\mu-1}(k^2-R^2)}{\left[R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{x}(k^{\mu+1}R^{\mu}+k^{2\mu}R)\right]^2} \leq 0.$$

■

Lemma 6. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{1+s_0}\left\{\max_{|z|=1}|p(z)| - m\right\}, \qquad (19)$$

where $m = \min_{|z|=k}|p(z)|$

and

$$s_0 = k^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}+1}\right\}.$$

The above lemma is due to Gardner et al. [11].

Lemma 7. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then on $|z| = 1$*

$$|q^{'}(z)| \geq k^{\mu+1}\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}}|p^{'}(z)|, \qquad (20)$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

This lemma was proved by Qazi [18].

Lemma 8. *If $p(z)$ is a polynomial of degree n and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each α, $0 \leq \alpha < 2\pi$ and $\gamma > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi}|q^{'}(e^{i\theta}) + e^{i\alpha}p^{'}(e^{i\theta})|^\gamma d\theta d\alpha \leq$$

$$2\pi n^\gamma\int_0^{2\pi}|p(e^{i\theta})|^\gamma d\theta. \qquad (21)$$

This lemma was obtained by Aziz [2].

Lemma 9. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$,*

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}} \times \{M(p,t)-m\}\,dt, \qquad (22)$$

and

$$M(p,r) + n\left[\{M(p,t)-m\} \times \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right] \leq$$

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\} \times \{M(p,r)-m\}+m, \qquad (23)$$

where $m = \min_{|z|=k}|p(z)|$, $M(p,t) = \max_{|z|=t}|p(z)|$ and $M(p,r) = \max_{|z|=r}|p(z)|$.

Proof: Since $p(z)$ has no zero in $|z| < k, k > 0$, for $0 < t \leq k, P(z) = p(tz)$ has no zero in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$. Thus using Lemma 6 to $P(z)$, we have

$$\max_{|z|=1}|P^{'}(z)| \leq \frac{n}{1+(\frac{k}{t})^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}t^\mu(\frac{k}{t})^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}t^\mu(\frac{k}{t})^{\mu+1}+1}\right\}}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=\frac{k}{t}}|P(z)|\right\}$$

where

$$m = \min_{|z|=\frac{k}{t}}|P(z)| = \min_{|z|=\frac{k}{t}}|p(tz)| = \min_{|z|=k}|p(z)|.$$

Which gives

$$t\max_{|z|=t}|p^{'}(z)| \leq n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{\mu+1}}{t}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{\mu+1}}{t}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{2\mu}}{t^\mu}+\frac{k^{\mu+1}}{t^{\mu+1}}}\right\}\left\{\max_{|z|=1}|p(tz)| - m\right\},$$

which is equivalent to

$$\max_{|z|=t}|p^{'}(z)| \leq n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}}\right\}\left\{\max_{|z|=t}|p(z)| - m\right\}. \qquad (24)$$

Now, for $0 < r \leq R \leq k$ and $0 \leq \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p^{'}(te^{i\theta})|dt$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p^{'}(te^{i\theta})|dt. \qquad (25)$$

Since

$$\int_r^R |p^{'}(te^{i\theta})|dt \leq \int_r^R \max_{|z|=t}|p^{'}(z)|dt,$$

using inequality (24) in (25), we get the first inequality (22) of Lemma 9.

Further, taking maximum over θ in inequality (22), we have

$$\max_{|z|=R}|p(z)| \leq \max_{|z|=r}|p(z)| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}} \times \{M(p,t)-m\}\,dt. \qquad (26)$$

Now, let us denote the right hand side of inequality (26) by $\phi(R)$. Then

$$\phi^{'}(R) = n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}\right\} \times \{M(p,R)-m\}. \quad (27)$$

Using $M(p,R)\le\phi(R)$, equality (27) can be written as

$$\phi^{'}(R) - n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}\right\} \times \{\phi(R)-m\}\le 0. \quad (28)$$

Multiplying both sides of (28) by

$$exp\left\{-n\times\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}dR\right\}$$

we get

$$\frac{d}{dR}\left[\{\phi(R)-m\}\,exp\left\{-n\times\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}dR\right\}\right] \le 0. \quad (29)$$

It is concluded from (29) that the function

$$exp\left\{-n\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}dR\right\} \times\{\phi(R)-m\}$$

is a non-increasing function of R in $(0,k]$. Hence for $0<r\le R\le k$,

$$exp\left\{-n\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}}dr\right\} \times\{\phi(r)-m\}\ge$$

$$exp\left\{-n\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}}dR\right\} \times\{\phi(R)-m\},$$

which is equivalent to

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}}dR\right\} \times\{\phi(r)-m\}\ge\{\phi(R)-m\}. \quad (30)$$

Since $\phi(r)=M(p,r)$ and using the value of $\phi(R)$ in (30), we get

$$M(p,r)+n\left[\{M(p,t)-m\}\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right]\le$$

$$m+\{M(p,r)-m\}\times exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\}.$$

This completes the proof of inequality (23) of Lemma 9. ■

Lemma 10. *If* $p(z)=a_0+\sum_{\nu=\mu}^n a_\nu z^\nu$, $1\le\mu\le n$, *is a polynomial of degree* n *having no zero in* $|z|<k, k>0$, *then*

$$\frac{\mu}{n}\frac{|a_\mu|k^\mu}{|a_0|-m}\le 1, \quad (31)$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [11].

Lemma 11. *If* $p(z)=a_0+\sum_{\nu=\mu}^n a_\nu z^\nu$, $1\le\mu\le n$, *is a polynomial of degree* n *having no zero in* $|z|<k, k>0$, *then* $0<R\le k$,

$$\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu}+k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu+R^{\mu+1}}\ge 1. \quad (32)$$

Proof: Since $p(z)\ne 0$ in $|z|<k, k>0$, for $0<R\le k$, the polynomial $P(z)=p(Rz)\ne 0$ in $|z|<\frac{k}{R}$, $\frac{k}{R}\ge 1$. If we apply Lemma 10 to the polynomial $P(z)$, we have

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^\mu\le 1. \quad (33)$$

Since $R\le k$, we have

$$0\le R^\mu k-Rk^\mu\le k^{\mu+1}-R^{\mu+1}. \quad (34)$$

Multiplying (33) and (34) sidewise, we have

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^\mu(R^\mu k-Rk^\mu)\le(k^{\mu+1}-R^{\mu+1}),$$

which is equivalent to (32) and the proof of Lemma 11 is completed. ■

Lemma 12. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z|<k, k>0$, *then*

$$|p(z)|\ge m \qquad for\ |z|\le k, \quad (35)$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [11].

Lemma 13. *The function*

$$g(x)=k^{t+1}\left\{\frac{\frac{t}{n}\frac{|a_t|}{x}k^{t-1}+1}{\frac{t}{n}\frac{|a_t|}{x}k^{t+1}+1}\right\} \quad (36)$$

where $k \geq 1, t > 0, n \in \mathbb{N}$, is a non-decreasing function of $x > 0$.

Proof: The proof follows simply by the first derivative test. ■

Lemma 14. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$,*

$$\{M(p,t) - m\} \times \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt \leq \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}\{M(p,t) - m\}\,dt, \quad (37)$$

where $M(p,t) = \max_{|z|=t}|p(z)|$, $m = \min_{|z|=k}|p(z)|$.

Proof: Since $p(z) \neq 0$ in $|z| < k, k > 0$, the polynomial $P(z) = p(tz) \neq 0$ in $|z| < \frac{k}{t}, \frac{k}{t} \geq 1$ where $0 < t \leq k$. Hence applying Lemma 10 to $P(z)$, we get

$$\frac{\mu}{n}\frac{|a_\mu|t^\mu}{|a_0| - m}\left(\frac{k}{t}\right)^\mu \leq 1, \quad (38)$$

where $m = \min_{|z|=\frac{k}{t}}|P(z)| = \min_{|z|=\frac{k}{t}}|p(tz)| = \min_{|z|=k}|p(z)|$.

Now, (38) becomes

$$\frac{\mu}{n}\frac{|a_\mu|k^\mu}{|a_0| - m} \leq 1,$$

which is equivalent to

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)} \leq \frac{t^{\mu-1}}{t^\mu + k^\mu}. \quad (39)$$

Since $0 < t \leq k$, in particular, by Lemma 12, we have

$$\max_{|z|=t}|p(z)| \geq m,$$

that is,

$$M(p,t) - m \geq 0. \quad (40)$$

Multiplying both sides of (39) by $\{M(p,t) - m\}$ and integrating both sides of the resulting inequality with respect to t from r to R, we obtain inequality (37) of Lemma 14. ■

Lemma 15. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$,*

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)}\right\} \leq \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}, \quad (41)$$

where $m = \min_{|z|=k}|p(z)|$.

This lemma was obtained by Chanam and Dewan [7].

III. MAIN RESULT

In this paper, under the same set of hypotheses, by involving certain coefficients of the polynomial, we improve both the Theorems 3 and 4 proved recently by Chanam et al. [8] by extending Theorem 1 into L^γ norm. More precisely, we prove

Theorem 16. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$, and $\gamma > 0$,*

$$\|p'(Rz)\|_\gamma \leq \frac{n}{R}T_\gamma\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[|p(re^{i\theta})| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt \{M(p,t) - m\} - m\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \quad (42)$$

where

$$T_\gamma = \frac{1}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|A + e^{i\alpha}|^\gamma\right\}^{\frac{1}{\gamma}}}, \quad (43)$$

$$m = \min_{|z|=k}|p(z)|,$$

and

$$A = \frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu + R^{\mu+1}}.$$

Proof: Since $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, does not vanish in $|z| < k, k > 0$, for any λ with $|\lambda| < 1$ by Rouche's theorem, the polynomial $p(z) - \lambda m$ has no zero in $|z| < k, k > 0$. Hence for $0 < R \leq k$, the polynomial $P(z) = p(Rz) - \lambda m$ has no zero in $|z| < \frac{k}{R}, \frac{k}{R} \geq 1$.

Applying Lemma 7 to the polynomial $P(z)$, we have for $|z| = 1$,

$$B|P'(z)| \leq |Q'(z)|, \quad (44)$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ and

$$B = \left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0-\lambda m|}(\frac{k}{R})^{\mu-1} + 1}{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0-\lambda m|}(\frac{k}{R})^{\mu+1} + 1}\right\}.$$

Using Lemma 12, $|p(z)| > m$ for $|z| < k$, i.e., in particular, $|a_0| > m$. Since $|\lambda| < 1$, we have $|\lambda|m < m < |a_0|$, and therefore

$$|a_0 - \lambda m| \geq |a_0| - |\lambda|m > |a_0| - m.$$

Using the fact of Lemma 13, we have $B \geq A$, where

$$A = \frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu + R^{\mu+1}}. \quad (45)$$

From (44), we have for $|z| = 1$,

$$A|P'(z)| \leq |Q'(z)|. \quad (46)$$

and by Lemma 11, $A \geq 1$.

We can easily verify that for every real number α and $R^{'} \geq r^{'} \geq 1$,

$$|R^{'} + e^{i\alpha}| \geq |r^{'} + e^{i\alpha}|. \quad (47)$$

This implies for each $\gamma > 0$,

$$\int_0^{2\pi} |R^{'} + e^{i\alpha}|^\gamma d\alpha \geq \int_0^{2\pi} |r^{'} + e^{i\alpha}|^\gamma d\alpha. \quad (48)$$

For point $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, for which $P^{'}(e^{i\theta}) \neq 0$, we denote

$$R^{'} = \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} \right|,$$

and $r^{'} = A$, then from (47), $R^{'} \geq r^{'} \geq 1$.

Now, we have for each $\gamma > 0$,

$$\begin{aligned}
&\int_0^{2\pi} |Q^{'}(e^{i\theta}) + e^{i\alpha} P^{'}(e^{i\theta})|^\gamma d\alpha \\
&= |P^{'}(e^{i\theta})|^\gamma \int_0^{2\pi} \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} + e^{i\alpha} \right|^\gamma d\alpha \\
&= |P^{'}(e^{i\theta})|^\gamma \int_0^{2\pi} \left| \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} \right| + e^{i\alpha} \right|^\gamma d\alpha \\
&\geq |P^{'}(e^{i\theta})|^\gamma \int_0^{2\pi} |A + e^{i\alpha}|^\gamma d\alpha. [\text{by } (48)] \quad (49)
\end{aligned}$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P^{'}(e^{i\theta}) = 0$, inequality (49) trivially holds.

Now using (49) in Lemma 8, we obtain for each $\gamma > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^\gamma d\alpha \int_0^{2\pi} |P^{'}(e^{i\theta})|^\gamma d\theta \leq 2\pi n^\gamma \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta. \quad (50)$$

Since $P(z) = p(Rz) - \lambda m$, inequality (50) can be written as

$$\int_0^{2\pi} |A + e^{i\alpha}|^\gamma d\alpha \int_0^{2\pi} |Rp^{'}(Re^{i\theta})|^\gamma d\theta \leq 2\pi n^\gamma \int_0^{2\pi} |p(Re^{i\theta}) - \lambda m|^\gamma d\theta. \quad (51)$$

Now, we choose the argument of λ suitably such that

$$|p(Re^{i\theta}) - \lambda m| = |p(Re^{i\theta})| - |\lambda| m. \quad (52)$$

Using (52) in (51), we have

$$\int_0^{2\pi} |A + e^{i\alpha}|^\gamma d\alpha \int_0^{2\pi} |Rp^{'}(Re^{i\theta})|^\gamma d\theta \leq 2\pi n^\gamma \int_0^{2\pi} \left\{ |p(Re^{i\theta})| - |\lambda| m \right\}^\gamma d\theta. \quad (53)$$

By applying inequality (22) of Lemma 9 to inequality (53), we obtain

$$R^\gamma \int_0^{2\pi} |A + e^{i\alpha}|^\gamma d\alpha \int_0^{2\pi} |P^{'}(Re^{i\theta})|^\gamma d\theta \leq 2\pi n^\gamma \times$$

$$\int_0^{2\pi} \left[|p(re^{i\theta})| - |\lambda| m + n \left\{ \max_{|z|=t} |p(z)| - m \right\} \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t) + k^{\mu+1}} dt \right]^\gamma d\theta$$

or equivalently,

$$\|p^{'}(Rz)\|_\gamma \leq \frac{n}{R} T_\gamma \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[|p(re^{i\theta})| + n \times \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right.\right.$$
$$\left.\left. \{M(p,t) - m\} - |\lambda| m \right]^\gamma d\theta \right\}^{\frac{1}{\gamma}}. \quad (54)$$

where T_γ is as defined in (43).

Taking limit as $|\lambda| \to 1$, inequality (54) becomes (42) of Theorem 16 and this completes the proof of Theorem 16. ■

Remark 17. Both the ordinary inequalities (11) and (12) are best possible for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ. It may be expected that inequality (42) of Theorem 16 is sharp for this polynomial. But it is not so, as is discussed below:

It is obvious that for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ, $m = \min_{|z|=k} |p(z)| = 0$, and hence inequality (42) of Theorem 16 equivalently takes

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k^\mu + R^\mu e^{i\alpha} \right|^\gamma d\alpha \right\} \times \left\{ \int_0^{2\pi} \left| R^\mu e^{i\theta\mu} + k^\mu \right|^{\gamma(\frac{n}{\mu}-1)} d\theta \right\} \leq$$

$$\left[\int_0^{2\pi} \left\{ \left| r^\mu e^{i\theta\mu} + k^\mu \right|^{\frac{n}{\mu}} + (R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}} \right\}^\gamma d\theta \right]. \quad (55)$$

In particular, if we set $k = R = r$, and $\mu = 1$, then inequality (55) assumes

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha \right\} \left\{ \int_0^{2\pi} \left| e^{i\theta} + 1 \right|^{\gamma(n-1)} d\theta \right\} \leq \left\{ \int_0^{2\pi} \left| e^{i\theta} + 1 \right|^{n\gamma} d\theta \right\}. \quad (56)$$

Now, we have for $p > -1$,

$$\int_0^{\frac{\pi}{2}} \cos^p \theta d\theta = \frac{\sqrt{\pi}\Gamma(\frac{p}{2} + \frac{1}{2})}{2\Gamma(\frac{p}{2} + 1)}. \quad (57)$$

For $\gamma > 0$, by a simple calculation, we have

$$\int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha = 2^{\gamma+2} \int_0^{\frac{\pi}{2}} \cos^\gamma \alpha d\alpha,$$

which on using (57) gives

$$\int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha = 2^{\gamma+1} \sqrt{\pi} \frac{\Gamma(\frac{\gamma}{2} + \frac{1}{2})}{\Gamma(\frac{\gamma}{2} + 1)}. \quad (58)$$

Applying equality (58) in inequality (56), we have

$$\frac{1}{2\pi} \times 2^{\gamma(n-1)+1} \sqrt{\pi} \frac{\Gamma(\frac{\gamma(n-1)}{2} + \frac{1}{2})}{\Gamma(\frac{\gamma(n-1)}{2} + 1)} \times 2^{\gamma+1} \sqrt{\pi} \frac{\Gamma(\frac{\gamma}{2} + \frac{1}{2})}{\Gamma(\frac{\gamma}{2} + 1)} \leq 2^{n\gamma+1} \sqrt{\pi} \frac{\Gamma(\frac{n\gamma}{2} + \frac{1}{2})}{\Gamma(\frac{n\gamma}{2} + 1)},$$

that is,

$$\frac{1}{\sqrt{\pi}} \times \frac{\Gamma(\frac{\gamma(n-1)}{2} + \frac{1}{2})}{\Gamma(\frac{\gamma(n-1)}{2} + 1)} \times \frac{\Gamma(\frac{\gamma}{2} + \frac{1}{2})}{\Gamma(\frac{\gamma}{2} + 1)} \leq \frac{\Gamma(\frac{n\gamma}{2} + \frac{1}{2})}{\Gamma(\frac{n\gamma}{2} + 1)}. \quad (59)$$

Further, when $n = 3, \gamma = 4$, inequality (59) becomes

$$\frac{1}{\sqrt{\pi}} \times \frac{\Gamma(4+\frac{1}{2})}{\Gamma(5)} \times \frac{\Gamma(2+\frac{1}{2})}{\Gamma(3)} \leq \frac{\Gamma(6+\frac{1}{2})}{\Gamma(7)}$$

which on simplification gives

$$5 \leq 11,$$

in which equality does not hold. This shows that inequality (42) of Theorem 16 is not sharp.

Remark 18. Since $(\frac{k}{R})^\mu \leq A$, where A is as defined in Theorem 16, and by Lemma 14, the bound given by Theorem 16 is better than both the bounds given by Theorems 3 and 4 recently proved by Chanam et al. [8].

Remark 19. Using $|p(re^{i\theta})| \leq M(p,r)$ in Theorem 16, we have the following interesting result.

Corollary 20. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $\gamma > 0$,

$$\|p^{'}(Rz)\|_\gamma \leq \frac{n}{R} T_\gamma \left[M(p,r) + n\times \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)} dt \{M(p,t) - m\} - m\right], \quad (60)$$

where T_γ is as defined in Theorem 16 and $m = \min_{|z|=k} |p(z)|$.

Remark 21. By the same argument of Remark 18, it is evident that Corollary 20 yields a better bound than that of the bounds given by Chanam et al. [7, Corollaries 3.5 and 3.10].

In addition, using inequality (23) of Lemma 9 in inequality (60) of Corollary 20, we have the following L^γ version of Theorem 1

Corollary 22. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $\gamma > 0$,

$$\|p^{'}(Rz)\|_\gamma \leq \frac{n}{R} T_\gamma \times exp\left\{ n \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu}t)} \right\} \{M(p,r) - m\} \quad (61)$$

where T_γ is as defined in Theorem 16 and $m = \min_{|z|=k} |p(z)|$.

Letting $\gamma \to \infty$ in inequality (61) we get inequality (13) of Theorem 1.

Remark 23. Using Lemma 15 and considering limit as $\gamma \to \infty$, we see that inequality (61) of Corollary 22 reduces to inequality (12) proved by Aziz and Shah [3].

Further, if we let $\mu = 1$ and $r = 1$ in Corollary 22, we obtain an improved L^γ version of inequality (10) due to Bidkham and Dewan [5].

Also, when $\mu = 1 = R = r$ in Corollary 22, it gives an improvement of L^γ inequality (8) due to Govil and Rahman [13] of inequality (6) for ordinary derivative proved by Malik [15].

In addition, if we use Lemma 15 for $\mu = 1 = R = r = k$, then Corollary 20 gives an improved version in L^γ setting of inequality (4) due to Erdös and Lax [14].

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References

[1] V. V. Arestov, "On inequalities for trigonometric polynomials and their derivative", *IZV. Akad. Nauk. SSSR. Ser. Math.*, vol. 45, pp. 3-22, 1981.
[2] A. Aziz and N. A. Rather, "Some Zygmund type L^q inequalities for polynomials", *J. Math. Anal. Appl.*, vol. 289, pp. 14-29, 2004.
[3] A. Aziz and W. M. Shah, "Inequalities for a polynomial and its derivative", *Math. Ineq. Appl.*, vol. 3, pp. 379-391, 2004.
[4] A. Aziz and B. A. Zargar, "Inequalities for a polynomial and its derivative", *Math. Ineq. Appl.*, vol. 4, pp. 543-550, 1998.
[5] M. Bidkham and K. K. Dewan, "Inequalities for polynomial and its derivative", *J. Math. Anal. Appl.*, vol. 166, pp. 319-324, 1992.
[6] B. Chanam, "L^r inequalities for polynomials", *Southeast Asian Bull. Math.*, vol. 42, pp. 825-832, 2018.
[7] B. Chanam and K. K. Dewan, "Inequalities for a polynomial and its derivative", *J. Math. Anal. Appl.*, vol. 336, pp. 171-179, 2007.
[8] B. Chanam, N. Reingachan, K. B. Devi, M. T. Devi and K. Krishnadas, "Some L^q inequalities for polynomial", *Nonlinear Funct. Anal. Appl.*, vol. 26, pp. 331-345, 2021.
[9] N. G. De-Bruijn, "Inequalities concerning polynomials in the complex domain", *Ned-erl. Akad. Wetench. Proc. Ser. A.*, vol. 50, pp. 1265-1272, 1947.
[10] K. K. Dewan, A. Bhat and M. S. Pukhta, "Inequalities concerning the L^p -norm of a polynomial", *J. Math. Anal. Appl.*, vol. 224, pp. 14-21, 1998.
[11] R. B. Gardner, N. K. Govil and A. Weems, "Some results concerning rate of growth of polynomials", *East J. Approx.*, vol. 10, pp. 301-312, 2004.
[12] R. B. Gardner and A. Weems, "A Bernstein-type of L^p inequality for a certain class of polynomials", *J. Math. Anal. Appl.*, vol. 219, pp. 472-478, 1998.
[13] N. K. Govil, and Q. I. Rahman, "Functions of exponential type not vanishing in a half-plane and related polynomials", *Trans. Amer. Math. Soc.*, vol. 137, pp. 501-517, 1969.
[14] P. D. Lax, "Proof of a conjecture of P. Erdös on the derivative of a polynomial", *Bull. Amer. Math. Soc.*, vol. 50, pp. 509-513, 1944.
[15] M. A. Malik, "On the derivative of a polynomial", *J. London Math. Soc.*, vol. 1, pp. 57-60, 1969.
[16] G. V. Milovanović, D. S. Mitrinović and Th. Rassias, "Topics in polynomials: Extremal problems, Inequalities, Zeros", World Scientific, Singapore, 1994.
[17] A. Mir and B. Dar, "Inequalities concerning the rate of growth of polynomials", *Afr. Mat.*, vol. 27, pp. 279-290, 2016.
[18] M. A. Qazi, "On the maximum modulus of polynomials", *Proc. Amer. Math. Soc.*, vol. 115, pp. 337-343, 1992.
[19] Q. I. Rahman and G. Schmeisser, "L^p inequalities for polynomials", *J. Approx. Theory.*, vol. 53, pp. 26-32, 1988.
[20] N. A. Rather, "Extremal properties and location of the zeros of polynomials", Ph.D Thesis, University of Kashmir, 1998.
[21] W. Rudin, "Real and Complex Analysis", Tata McGraw-Hill Publishing Company(reprinted in India), 1977.
[22] A. C. Schaeffer, "Inequalities of A. Markoff and S. N. Bernstein for polynomials and related functions", *Bull. Amer. Math. Soc.*, vol. 47, pp. 565-579, 1941.
[23] A. Zygmund, "A remark on conjugate series", *Proc. London Math. Soc.*, vol. 34, pp. 392-400, 1932.