On Constructing Edge Irregular q-Labelling of Several Book Graphs

Lucia Ratnasari, Sri Wahyuni, Yeni Susanti, and Diah Junia Eksi Palupi

***Abstract*—Given a graph $B = (V(B), E(B))$ with the vertex set $V(B)$ and the edge set $E(B)$. A vertex labelling of a simple, connected, and undirected B, $g : V(B) \to \{1, 2, \dots, q\}$ is called an edge irregular q-labelling on B if two arbitrary edges $uv, u'v'$ in B have different weights, where the weight of edge uv is defined as $\omega_g(uv) = g(u) + g(v)$. Then the edge irregularity strength of B, denoted by $es(B)$ is defined as the smallest integer q such that B can be labelled by an edge irregular q-labelling. Furthermore, the edge irregularity strength in the graph B with maximum degree $\Delta(B)$ satisfies $es(B) \geq max\left\{\lceil\frac{|E(B)|+1}{2}\rceil, \Delta(B)\right\}$. In this study, we develop an edge irregular q-labelling and determine the exact value of the edge irregularity strength of triangular book graph $B_p(C_3)$, rectangular book graph $B_p(C_4)$, pentagonal book graph $B_p(C_5)$. We show that the exact value of the $es(B)$ of $B_p(C_3)$, $B_p(C_4)$, $B_p(C_5)$ is equal to $p + 2$, $\lceil\frac{3p+2}{2}\rceil$, $\lceil\frac{4p+2}{2}\rceil$, respectively. We also investigate an edge irregular q-labelling and determine the exact value of the edge irregularity strength of book graph $B_p(C_m)$ with additional $(m - 2)p$ pendant edges and with additional p pendant edges for $m \geq 6$. For any book graph $B_p(C_m)$ for $m \geq 6$, we obtain that $\lceil\frac{(m-1)p+2}{2}\rceil \leq es(B_p(C_m)) \leq \lceil\frac{mp+2}{2}\rceil$.**

***Index Terms*—edge irregularity strength, edge irregular q-labelling, book graphs.**

I. Introduction

A graph labelling is a mapping assigning numbers, particularly positive numbers, to graph elements. There are several types of graph labelling associated with the weights of the edges or vertices. Respect to a labelling, the weight of vertex v is the sum of label of a vertex v and labels of all edges incident to the vertex v. Meanwhile, the sum of label of an edge and all labels of vertices that incident to that edge, is called the edge weight.

According to [1], an edge irregular q-labelling of the graph B was proposed as a function g from the edge set $E(B)$ to the positive integers $\{1, 2, ..., q\}$, such that the weights of the vertices are different. An integer q is called the irregularity strength of B, if q is the smallest number such that we can be labelled the graph B with an irregular q-labelling.

Bača et al. introduced a new type of labelling in [2], that is irregular and labels can be used more than once. These new types of labelling can be edge or vertex irregular total q-labelling. An edge (vertex) irregular total q-labelling of B is a mapping g from $V(B) \cup E(B)$ to $\{1, 2, \dots, q\}$, where the weights of all edges (vertices) are different. The weight of edge uv denoted $\omega_g(uv)$ is defined as $\omega_g(uv) = g(uv) + g(u) + g(v)$, while the weight of vertex u denoted $\omega_g(u)$ is defined as $\omega_g(u) = g(u) + \Sigma_{uv\in E(B)} g(uv)$. Also, a positive integer q is called the total edge (vertex) irregularity strength of B and denoted by $tes(B)$ ($tvs(B)$), if q is the smallest number such that the graph B can be labelled with edge (vertex) irregular total q-labelling.

The lower bound of the tes of any graph $B = (V(B), E(B))$ of maximal vertex degree $\Delta(B)$ is $max\{\lceil\frac{|E(B)|+2}{3}\rceil, \lceil\frac{\Delta(B)+1}{2}\rceil\}$. In [2], Bača et al. determine that $tes(K_3) = 2, tes(K_4) = 3$, and $tes(K_5) = 5$. They also provide the conjecture that $tes(K_p) = \lceil\frac{p^2-p+4}{6}\rceil$ for $p \geq 6$ and have proven for $6 \leq p \leq 20$.

Ahmad et al. in [3] introduced a new parameter of a graph called edge irregularity strength. The edge irregularity strength of any graph B, denoted by $es(B)$, is defined as the smallest integer q, such that B can be labelled with an edge irregular vertex q-labelling respect to edge weight defined as the sum of labels of two end vertices of an edge. Mathematically, for any vertex q-labelling $g : V(B) \to \{1, 2, \dots, q\}$, the weight of edge uv respect to g is $g(u) + g(v)$. Ahmad et al. also stated that the lower bound of $es(B)$ for any graph B with maximum degree $\Delta(B)$ is $max\left\{\lceil\frac{|E(B)|+1}{2}\rceil, \Delta(B)\right\}$. They also observed the exact value of edge irregularity strength $es(B)$ for some class of graphs as path graphs, star graphs, double star graphs, and the graph obtained from the cartesian product of two paths.

Another study on edge irregular strength (es) can be found in [4], where it is determined the es for several classes of Toeplitz graphs. Tarawneh et al. in [5] studied es for the corona product of cycles with isolated vertices; they also investigated the es of grid graphs in [6] and the es for some classes of plane graphs in [7]. Imran et al. in [8] determined the es for caterpillar graphs, star graphs, (n-t) kite, cycle chains, and friendship graphs, while Ahmad et al. in [9] discussed the es for some join graphs. Moreover, in [10] Suparta and Suharta studied the es of joint graph $P_m + \bar{K}_n$ for $m, n \geq 3$. Furthermore, Alrawajfeh et al. in [11] determine the es of graph $K_{n,2}$, $P_n \odot P_6$, and $P_n \odot C_3$.

The aim of this research is to investigate the edge irregularity strength of several book graphs by constructing an edge irregular q-labelling of these graphs, including triangular book graphs, rectangular book graphs, pentagonal book graphs, and book graphs with additional $(m-2)p$ and p pendant edges for $m \geq 6$.

Manuscript submition December 2, 2021; revised November 9, 2022.

L. Ratnasari is a Lecturer at the Mathematics Department of the Faculty of Science and Mathematics, Universitas Diponegoro, Semarang, Indonesia. (corresponding author's e-mail: ratnasari.lucia@gmail.com).

S. Wahyuni is a Professor at the Mathematics Department of the Faculty of Mathematics and Natural Science, Universitas Gadjah Mada, Yogyakarta, 55281 Indonesia (e-mail: swahyuni@ugm.ac.id).

Y. Susanti is an Associate Professor at the Mathematics Department of the Faculty of Mathematics and Natural Science, Universitas Gadjah Mada, Yogyakarta, 55281 Indonesia (email: yeni_math@ugm.ac.id.

D. J. E. Palupi is an Associate Professor at the Mathematics Department of the Faculty of Mathematics and Natural Science, Universitas Gadjah Mada, Yogyakarta, 55281 Indonesia (email:diah_yunia@ugm.ac.id).

II. Result

We constructed some edge irregular q-labellings for determining the edge irregularity strength of the triangular book graphs $B_p(C_3)$, rectangular book graphs $B_p(C_4)$, pentagonal book graphs $B_p(C_5)$, book graphs with additional $(m-2)p$ pendant edges $B_p(C_m(m-2)pK_1)$, and book graphs with additional p pendant edges $B_p(C_m pK_1)$, for $m \geq 6$. Before we give the es of these book graphs, we give definitions of book graphs as follows.

Definition 2.1 Suppose C_m^r, $r = 1, 2, ..., p$ are cycle graphs with vertex sets $V(C_m^r) = \{u, v\} \cup \{x_{r,s} : s = 1, ..., m-2\}$ and edge sets $E(C_m^r) = \{uv, ux_{r,1}, x_{r,s}x_{r,s+1}, x_{r,m-2}v : s = 1, ..., m-3\}$. A book graph with m sides and p sheets, denoted by $B_p(C_m)$, is a graph generated from cycle graphs C_m^r, $r = 1, ..., p$ by merging edge uv from each cycle. Therefore, the set of vertex is $V(B_p(C_m)) = \{u, v\} \cup \{x_{r,s} : r = 1, ..., p, s = 1, ...m-2\}$ and the set of edge is $E(B_p(C_m)) = \{uv\} \cup \{ux_{r,1}, x_{r,s}x_{r,s+1}, x_{r,m-2}v : r = 1, ..., p, s = 1, ..., m-3\}$.

In the following propositions we give some constructions of edge irregular q-labelling for book graphs.

Proposition 2.1 Let $B_p(C_3)$ be a book graphs of 3 sides and p sheets. Then $es(B_p(C_3)) = p + 2$.

Proof: The book graph $B_p(C_3)$ has p sheets C_3, hence the maximum degree is $\Delta(B_p(C_m)) = p + 1$ and the number of edges are $|E(B_p(C_3))| = 2p + 1$. Therefore, by $es(B_p(C_3)) \geq max\{\lceil \frac{|E(B_p(C_3))|+1}{2} \rceil, \lceil \frac{\Delta(B_p(C_3))+1}{2} \rceil\}$, clearly $es(B_p(C_3)) \geq p+1$. For the upper bound, we define an edge irregular q-labelling for $B_p(C_3)$ with $q = p + 2$ in the following way.

The vertices are labelled as follows:

$g(u) = 1, \quad g(v) = q$
$g(x_{r,1}) = r + 1, \quad 1 \leq r \leq p.$

The weight of the edges are

$\omega_g(ux_{r,1}) = r + 2, \quad 1 \leq r \leq p$
$\omega_g(uv) = p + 3$
$\omega_g(x_{r,1}v) = p + r + 3, \quad 1 \leq r \leq p.$

For all edges triangular book graph $E(B_p(C_3))$, the weights are different, and the vertex labels are less than equal to $p+2$, hence the edge irregular strength of the triangular graph is $es(B_p(C_3)) = p + 2$.

■

Proposition 2.2 Let $B_p(C_4)$ be a book graph of 4 sides and p sheets. Then $es(B_p(C_4)) = \lceil \frac{3p+2}{2} \rceil$.

Proof: The book graph $B_p(C_4)$ is obtained from the p cycle graphs C_4 by uniting one edge in each cycle graph, implying $\Delta(B_p(C_4)) = p + 1$. Each book graph $B_p(C_4)$ has $|E(B_p(C_4))| = 3p + 1$ edges. Based on the lower bound given in [3], we have $es(B_p(C_4)) \geq \lceil \frac{3p+2}{2} \rceil$. The upper bound is proven by constructing an edge irregular q-labelling for the graph $B_p(C_4)$ with $q = \lceil \frac{3p+2}{2} \rceil$.

The vertices are labelled as follows:

$g(u) = 1, \quad g(v) = q,$
$g(x_{1,1}) = 1, \quad g(x_{1,2}) = 2,$
$g(x_{r,1}) = r + 1, \quad 2 \leq r \leq p$
$g(x_{r,2}) = q - \frac{r-1}{2}, \quad 2 \leq r \leq p - 1$ and r odd
$g(x_{r,2}) = p - \frac{r-2}{2}, \quad 2 \leq r \leq p - 1$ and r even
$g(x_{p,2}) = q.$

Based on the vertex labelling g, the weight of the edges are:

$\omega_g(ux_{1,1}) = 2 \quad \omega_g(x_{1,1}x_{1,2}) = 3$
$\omega_g(ux_{r,1}) = r + 2, \quad 2 \leq r \leq p$
$\omega_g(uv) = 1 + q$
$\omega_g(x_{r,1}x_{r,2}) = \frac{r+3}{2} + q, \quad 2 \leq r \leq p - 1$ and r odd
$\omega_g(x_{r,1}x_{r,2}) = \frac{r+4}{2} + p, \quad 2 \leq r \leq p - 1$ and r even
$\omega_g(x_{p,1}x_{p,2}) = p + q + 1$
$\omega_g(x_{r,2}v) = 2q - \frac{r-1}{2}, \quad 2 \leq r \leq p - 1$ and r odd
$\omega_g(x_{r,2}v) = p + q - \frac{r-2}{2}, \quad 2 \leq r \leq p - 1$ and r even
$\omega_g(x_{p,2}v) = 2q.$

Respect to the labelling g above, obviously all edge weights are different and the vertex labels do not exceed $q = \lceil \frac{3p+2}{2} \rceil$. Therefore, $es(B_p(C_4)) = \lceil \frac{3p+2}{2} \rceil$.

■

Proposition 2.3 Let $B_p(C_5)$ be a book graph of 5 sides and p sheets. Then $es(B_p(C_5)) = \lceil \frac{4p+2}{2} \rceil$.

Proof: According to Definition 2.1. we have $\Delta(B_p(C_5)) = p + 1$ and $|E(B_p(C_5))| = 4p + 1$. Based on the lower bound given by Ahmad et al., that is $es(B) \geq max\{\lceil \frac{|E(B)|+1}{2} \rceil, \Delta(B)\}$, $es(B_p(C_5)) \geq \lceil \frac{4p+2}{2} \rceil$ is obtained. We show that there is an edge irregular q-labelling of the graph $B_p(C_5)$ with $q = \lceil \frac{4p+2}{2} \rceil$.

The vertices are labelled as follows:

$g(u) = 1, \quad g(v) = q$
$g(x_{r,1}) = 2r - 1, \quad 1 \leq r \leq p$
$g(x_{r,2}) = 2r, \quad 1 \leq r \leq p$
$g(x_{r,3}) = 2r + 1, \quad 1 \leq r \leq p.$

So that the weight of the edges are as below:

$\omega_g(ux_{r,1}) = 2r, \quad 1 \leq r \leq p$
$\omega_g(x_{r,1}x_{r,2}) = 4r - 1, \quad 1 \leq r \leq p$
$\omega_g(uv) = 1 + q$
$\omega_g(x_{r,2}x_{r,3}) = 4r + 1, \quad 1 \leq r \leq p$
$\omega_g(x_{r,3}v) = 2r + q + 1 \quad 1 \leq r \leq p$

Respect to labelling g, clearly the edge weights are different and the maximum vertex label is $q = \lceil \frac{4p+2}{2} \rceil$. Hence the edge irregularity strength of $B_p(C_5)$ is $es(B_p(C_5)) = \lceil \frac{4p+2}{2} \rceil$.

■

Example 2.1 On this example we make an edge irregular k-labeling of $B_4(C_5)$ with $q = 9$ as shown in Figure 2. We can see that the weight of each edge is different.

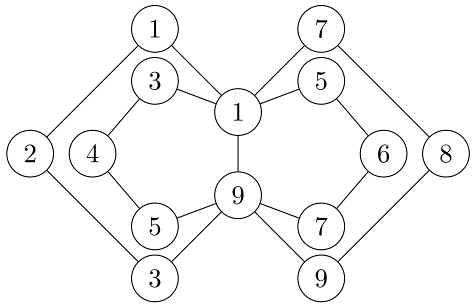

Fig. 1. Edge irregularity 9-labeling of book graph $B_4(C_5)$

Next, we investigate the es of the book graph $B_p(C_m)$ for $m \geq 6$ with some additional pendant edges.

Definition 2.2 Let $B_p(C_m)$ be a book graph of m sides and p sheets. A book graph with additional $(m-2)p$ pendant edges, denoted by $B_p(C_m(m-2)pK_1)$ is a graph obtained from the book graph $B_p(C_m)$ by adding one pendant edge $x_{r,s}y_{r,s}$ at each vertex $x_{r,s}$ with $1 \leq r \leq p,\ 1 \leq s \leq m-2$. Therefore, the graph $B_p(C_m(m-2)pK_1)$ has $V(B_p(C_m(m-2)pK_1)) = \{u, v, x_{r,s}, y_{r,s} \mid 1 \leq r \leq p,\ 1 \leq s \leq m-2\}$, and $E(B_p(C_m(m-2)pK_1)) = E(B_p(C_m)) \cup \{x_{r,s}, y_{r,s} \mid 1 \leq r \leq p,\ 1 \leq s \leq m-2\}$ as vertex and edge sets, respectively.

By a routine counting we have $|E(B_p(C_m(m-2)pK_1))| = (2m-3)p+1$.

The edge irregular strength of the book graph $B_p(C_m)$ for $m \geq 6$ with additional $(m-2)p$ pendant edges are given in the following two lemmas:

Lemma 2.1 For any positive integer $p \geq 2$ and any odd positive integer $m \geq 6$, $es(B_p(C_m(m-2)pK_1)) = \lceil \frac{(2m-3)p+2}{2} \rceil$.

Proof: The maximum degree of graph $B_p(C_m(m-2)pK_1)$ is obviously $p+1$ and since $|E(B_p(C_m(m-2)pK_1))| = (2m-3)p+1$ based on the given lower bound given in [3], we obtain that $es(B_p(C_m(m-2)pK_1)) \geq \lceil \frac{(2m-3)p+2}{2} \rceil$.

We construct an edge irregular q-labelling g_1 with $q = \lceil \frac{(2m-3)p+2}{2} \rceil$ to show the upper bound as follows:

$$
\begin{aligned}
&g_1(u) = 1, \quad g_1(v) = q\\
&g_1(x_{r,s}) = r + (s-1)p, \quad 1 \leq r \leq p,\ 1 \leq s \leq \tfrac{m-1}{2}\\
&g_1(x_{r,s}) = q + r - (m-s-1)p, \quad 1 \leq r \leq p, \tfrac{m+1}{2} \leq r \leq m-2\\
&g_1(y_{r,s}) = sp + r + 1, \quad 1 \leq r \leq p, 1 \leq s \leq \tfrac{m-3}{2}\\
&g_1(y_{r,s}) = sp + 1, \quad 1 \leq r \leq \tfrac{p}{2},\ s = \tfrac{m-1}{2}\\
&g_1(y_{r,s}) = \tfrac{(m-2)p}{2} + r + 2, \quad \tfrac{p+2}{2} \leq r \leq p,\ s = \tfrac{m-1}{2}, \text{ and } p \text{ even}\\
&g_1(y_{r,s}) = \tfrac{((m-2)p+1)p}{2} + r + 1, \quad \tfrac{p+3}{2} \leq r \leq p,\ s = \tfrac{m-1}{2}, \text{ and } p \text{ odd}\\
&g_1(y_{r,s}) = (s-1)p + r + 1, \quad \tfrac{p+2}{2} \leq r \leq p,\ s = \tfrac{m+1}{2}, \text{ and } p \text{ even}\\
&g_1(y_{r,s}) = \tfrac{mp+1}{2}, \quad 1 \leq r \leq \tfrac{p}{2},\ s = \tfrac{m+1}{2}, \text{ and } p \text{ even}\\
&g_1(y_{r,s}) = (s-1)p + r, \quad 1 \leq r \leq \tfrac{p+1}{2},\ s = \tfrac{m+1}{2}, \text{ and } p \text{ odd}\\
&g_1(y_{r,s}) = \tfrac{mp}{2} + 1, \quad \tfrac{p+3}{2} \leq r \leq p,\ s = \tfrac{m+1}{2}, \text{ and } p \text{ odd}\\
&g_1(y_{r,s}) = q + r - (m-s)p - 1, \quad 1 \leq r \leq p, \tfrac{m+3}{2} \leq s \leq m-2
\end{aligned}
$$

The weights of the edges relative to the labelling g_1 are the followings:

$$
\begin{aligned}
&\omega_{g_1}(ux_{r,1}) = r + 1, \quad 1 \leq r \leq p\\
&\omega_{g_1}(x_{r,s}x_{r,s+1}) = 2r + (2s-1)p, \quad 1 \leq r \leq p,\ 1 \leq s \leq \tfrac{m-1}{2}\\
&\omega_{g_1}(x_{r,s}x_{r,s+1}) = 2q + 2r - (2m-2s-3)p, \quad 1 \leq r \leq p, \tfrac{m+1}{2} \leq s \leq m-3\\
&\omega_{g_1}(x_{r,s}v) = 2q + r - (m-s-1)p, \quad 1 \leq r \leq p,\ s = m-2\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = 2r + (2s-1)p + 1, \quad 1 \leq r \leq p,\ 1 \leq s \leq \tfrac{m-3}{2}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = r + (2s-1)p + 1, \quad 1 \leq r \leq \tfrac{p}{2},\ s = \tfrac{m-1}{2}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = 2r + \tfrac{(2s+m-4)p+3}{2}, \quad \tfrac{p+3}{2} \leq r \leq p,\ s = \tfrac{m-1}{2}, \text{ and } p \text{ odd}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = 2r + \tfrac{(2s+m-4)p+4}{2}, \quad \tfrac{p+2}{2} \leq r \leq p,\ s = \tfrac{m-1}{2}, \text{ and } p \text{ even}\\
&\omega_{g_1}(uv) = 1 + q\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = q + 2r + (2s-m)p, \quad 1 \leq r \leq \tfrac{p+1}{2},\ s = \tfrac{m+1}{2}, \text{ and } p \text{ odd}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = q + 2r + (2s-m)p + 1, \quad 1 \leq r \leq \tfrac{p}{2},\ s = \tfrac{m+1}{2}, \text{ and } p \text{ even}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = q + r + \tfrac{(2s-m+2)p+1}{2}, \quad \tfrac{p+3}{2} \leq r \leq p,\ s = \tfrac{m+1}{2}, \text{ and } p \text{ odd}\\
&\omega_{g_1}(x_{r,s}y_{r,s}) = q + i + \tfrac{(2s-m+2)p+2}{2}, \quad \tfrac{p+2}{2} \leq r \leq p,\ s = \tfrac{m+1}{2,}, \text{ and } p \text{ even}\\
&\omega_{g_1}(x_{r,4}y_{r,s}) = 2q + 2r - (2m-2s-1)p - 1 \quad 1 \leq r \leq p,\ \tfrac{m+3}{2} \leq s \leq m-2.
\end{aligned}
$$

Then it is confirmed that there is a vertex q-labelling with $q = \lceil \frac{(2m-3)p+2}{2} \rceil$ so that the edge weights of $B_p(C_m(m-2)pK_1)$ with $m \geq 6$ and m odd are all different. ■

The following lemma shows the es of the graph $B_p(C_m(m-2)pK_1)$ for $m \geq 6$ and m is even.

Lemma 2.2 For any positive integer $p \geq 2$ and any positive even integer $m \geq 6$, $es(B_p(C_m(m-2)pK_1)) = \lceil \frac{(2m-3)p+2}{2} \rceil$.

Proof: Similar to Lemma 2.1. we obtain that $es(B_p(C_m(m-2)pK_1)) \geq \lceil \frac{(2m-3)p+2}{2} \rceil$. We construct an edge irregular q-labelling g_2 with $q = \lceil \frac{(2m-3)p+2}{2} \rceil$ to show the upper bound.

Define the vertex labeling g_2 as the following:

$$
\begin{aligned}
&g_2(u) = 1, \quad g_2(v) = q\\
&g_2(x_{r,s}) = r + (s-1)p, \quad 1 \leq r \leq p,\ 1 \leq s \leq \tfrac{m-2}{2}\\
&g_2(x_{r,s}) = r + (s-1)p,\\
&\quad 1 \leq r \leq \tfrac{3p}{4},\ s = \tfrac{m}{2},\ p \equiv 0 (mod\ 4)\\
&\quad 1 \leq r \leq \tfrac{3p+1}{4},\ s = \tfrac{m}{2},\ p \equiv 1 (mod\ 4)\\
&\quad 1 \leq r \leq \tfrac{3p+2}{4},\ s = \tfrac{m}{2},\ p \equiv 2 (mod\ 4)\\
&\quad 1 \leq r \leq \tfrac{3p+3}{4},\ s = \tfrac{m}{2},\ p \equiv 3 (mod\ 4)\\
&g_2(x_{r,s}) = r + (s-1)p + 1,\\
&\quad \tfrac{3p+4}{4} \leq r \leq p,\ s = \tfrac{m}{2},\ p \equiv 0 (mod\ 4)\\
&\quad \tfrac{3p+5}{4} \leq r \leq p,\ s = \tfrac{m}{2},\ p \equiv 1 (mod\ 4)\\
&\quad \tfrac{3p+6}{4} \leq r \leq p,\ s = \tfrac{m}{2},\ p \equiv 2 (mod\ 4)\\
&\quad \tfrac{3p+7}{4} \leq r \leq p,\ s = \tfrac{m}{2},\ p \equiv 3 (mod\ 4)
\end{aligned}
$$

$$\begin{aligned}
g_2(x_{r,s}) &= q+r-(m-s-1)p,\\
&\quad 1\le r\le p, \tfrac{m+2}{2}\le s\le m-2\\
g_2(y_{r,s}) &= sp+r+1,\\
&\quad 1\le r\le p, 1\le s\le \tfrac{m-4}{2}\\
g_2(y_{r,s}) &= sp+r+1,\\
&\quad 1\le r\le \tfrac{3p}{4},\ s=\tfrac{m-2}{2},\ p\equiv 0(mod\ 4)\\
&\quad 1\le r\le \tfrac{3p+1}{4},\ s=\tfrac{m-2}{2},\ p\equiv 1(mod\ 4)\\
&\quad 1\le r\le \tfrac{3p-2}{4},\ s=\tfrac{m-2}{2},\ p\equiv 2(mod\ 4)\\
&\quad 1\le r\le \tfrac{3p-1}{4},\ s=\tfrac{m-2}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= sp+r+2,\\
&\quad \tfrac{3p+4}{4}\le r\le p,\ s=\tfrac{m-2}{2},\ p\equiv 0(mod\ 4)\\
&\quad \tfrac{3p+5}{4}\le r\le p,\ s=\tfrac{m-2}{2},\ p\equiv 1(mod\ 4)\\
&\quad \tfrac{3p+2}{4}\le r\le p,\ s=\tfrac{m-2}{2},\ p\equiv 2(mod\ 4)\\
&\quad \tfrac{3p+3}{4}\le r\le p,\ s=\tfrac{m-2}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,j}) &= sp+2,\\
&\quad 1\le r\le \tfrac{p}{2},\ s=\tfrac{m}{2},\ p\equiv 0(mod\ 4)\\
&\quad 1\le r\le \tfrac{p+1}{2},\ s=\tfrac{m}{2},\ p\equiv 1(mod\ 4)\\
&\quad 1\le r\le \tfrac{p}{2},\ s=\tfrac{m}{2},\ p\equiv 2(mod\ 4)\\
&\quad 1\le r\le \tfrac{p+1}{2},\ s=\tfrac{m}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p}{2}+r+2,\\
&\quad \tfrac{p+2}{2}\le r\le \tfrac{3p}{4},\ s=\tfrac{m}{2},\ p\equiv 0(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p-1}{2}+r+2,\\
&\quad \tfrac{p+3}{2}\le r\le \tfrac{3p+1}{4},\ s=\tfrac{m}{2},\ p\equiv 1(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p}{2}+r+2,\\
&\quad \tfrac{p+2}{2}\le r\le \tfrac{3p+2}{4},\ s=\tfrac{m}{2},\ p\equiv 2(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p-1}{2}+r+2,\\
&\quad \tfrac{p+3}{2}\le r\le \tfrac{3p+3}{4},\ s=\tfrac{m}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p}{2}+r+1,\\
&\quad \tfrac{3n+4}{4}\le r\le p,\ s=\tfrac{m}{2},\ p\equiv 0(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p-1}{2}+r+1,\\
&\quad \tfrac{3n+5}{4}\le r\le p,\ s=\tfrac{m}{2},\ p\equiv 1(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p}{2}+r+1,\\
&\quad \tfrac{3n+6}{4}\le r\le p,\ s=\tfrac{m}{2},\ p\equiv 2(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m-2)p-1}{2}+r+1,\\
&\quad \tfrac{3n+7}{4}\le r\le p,\ s=\tfrac{m}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r+1,\\
&\quad 1\le r\le \tfrac{p}{4},\ s=\tfrac{m+2}{2},\ p\equiv 0(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r,\\
&\quad 1\le r\le \tfrac{p+3}{4},\ s=\tfrac{m+2}{2},\ p\equiv 1(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r+1,\\
&\quad 1\le r\le \tfrac{p+2}{4},\ s=\tfrac{m+2}{2},\ p\equiv 2(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r,\\
&\quad 1\le r\le \tfrac{p+5}{4},\ s=\tfrac{m+2}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r,\\
&\quad \tfrac{p+4}{4}\le r\le \tfrac{p+2}{2},\ s=\tfrac{m+2}{2},\ p\equiv 0(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r-1,\\
&\quad \tfrac{p+7}{4}\le r\le \tfrac{p+3}{2},\ s=\tfrac{m+2}{2},\ p\equiv 1(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r,\\
&\quad \tfrac{p+6}{4}\le r\le \tfrac{p+2}{2},\ s=\tfrac{m+2}{2},\ p\equiv 2(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{mp}{2}+r-1,\\
&\quad \tfrac{p+9}{4}\le r\le \tfrac{p+3}{2},\ s=\tfrac{m+2}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m+2)p}{2}+1,\\
&\quad \tfrac{p+4}{2}\le r\le p,\ s=\tfrac{m+2}{2},\ p\equiv 0(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m+2)p}{2}+1,\\
&\quad \tfrac{p+5}{2}\le r\le p,\ s=\tfrac{m+2}{2},\ p\equiv 1(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m+2)p}{2}+1,\\
&\quad \tfrac{p+4}{2}\le r\le p,\ s=\tfrac{m+2}{2},\ p\equiv 2(mod\ 4)\\
g_2(y_{r,s}) &= \tfrac{(m+2)p}{2}+1,\\
&\quad \tfrac{p+5}{2}\le r\le p,\ s=\tfrac{m+2}{2},\ p\equiv 3(mod\ 4)\\
g_2(y_{r,s}) &= q+r-(m-s)p-1,\\
&\quad 1\le r\le p, \tfrac{m+4}{2}\le s\le m-2.
\end{aligned}$$

Based on the vertex labelling g_2, the edge weights are obtained as follows:

$$\begin{aligned}
\omega_{g_2}(ux_{r,1}) &= r+1,\ 1\le r\le p\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+(2s-1)p,\\
&\quad 1\le r\le p,\ 1\le s\le \tfrac{m-2}{2}\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p,\\
&\quad 1\le r\le \tfrac{3n}{4}, s=\tfrac{m}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p,\\
&\quad 1\le r\le \tfrac{3n+1}{4}, s=\tfrac{m}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p,\\
&\quad 1\le r\le \tfrac{3n+2}{4}, s=\tfrac{m}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p,\\
&\quad 1\le r\le \tfrac{3n+3}{4}, s=\tfrac{m}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p+1,\\
&\quad \tfrac{3n+4}{4}\le r\le p, s=\tfrac{m}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p+1,\\
&\quad \tfrac{3n+5}{4}\le r\le p, s=\tfrac{m}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p+1,\\
&\quad \tfrac{3n+6}{4}\le r\le p, s=\tfrac{m}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+q+(2s-m-1)p+1,\\
&\quad \tfrac{3n+7}{4}\le r\le p, s=\tfrac{m}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}x_{r,s+1}) &= 2r+2q+(2s-2m+1)p,\\
&\quad 1\le r\le p,\\
&\quad \tfrac{m+2}{2}\le s\le m-3\\
\omega_{g_2}(x_{r,s}v) &= 2q+r+(s-m+1)p,\\
&\quad 1\le r\le p, s=m-2\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+1,\\
&\quad 1\le r\le p, 1\le s\le \tfrac{m-4}{2}\\
\omega_{g_2}(uv) &= 1+q\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+1,\\
&\quad 1\le r\le \tfrac{3n}{4},\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+1,\\
&\quad 1\le r\le \tfrac{3n+1}{4},\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+1,\\
&\quad 1\le r\le \tfrac{3n-2}{4},\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+1,\\
&\quad 1\le r\le \tfrac{3n-1}{4},\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+2,\\
&\quad \tfrac{3n+4}{4}\le r\le p,\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+2,\\
&\quad \tfrac{3n+4}{4}\le r\le p,\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+2,\\
&\quad \tfrac{3n+4}{4}\le r\le p,\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 2(mod\ 4)
\end{aligned}$$

$$
\begin{aligned}
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+(2s-1)p+2,\\
&\quad \tfrac{3n+4}{4}\le r\le p,\ s=\tfrac{m-2}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= r+(2s-1)p+2,\\
&\quad 1\le r\le \tfrac{p}{2},\ s=\tfrac{m}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= r+(2s-1)p+2,\\
&\quad 1\le r\le \tfrac{p+1}{2},\ s=\tfrac{m}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= r+(2s-1)p+2,\\
&\quad 1\le r\le \tfrac{p}{2},\ s=\tfrac{m}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= r+(2s-1)p+2,\\
&\quad 1\le r\le \tfrac{p+1}{2},\ s=\tfrac{m}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+\tfrac{(2s+m-4)p}{2}+2,\\
&\quad \tfrac{p+2}{2}\le r\le p,\ s=\tfrac{m}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+\tfrac{(2s+m-4)p}{2}+2,\\
&\quad \tfrac{p+3}{2}\le r\le p,\ s=\tfrac{m}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+\tfrac{(2s+m-4)p}{2}+2,\\
&\quad \tfrac{p+2}{2}\le r\le p,\ s=\tfrac{m}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2r+\tfrac{(2s+m-4)p}{2}+2,\\
&\quad \tfrac{p+3}{2}\le r\le p,\ s=\tfrac{m}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p+1,\\
&\quad 1\le r\le \tfrac{p}{4},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p,\\
&\quad 1\le r\le \tfrac{p+3}{4},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p+1,\\
&\quad 1\le r\le \tfrac{p+2}{4},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p,\\
&\quad 1\le r\le \tfrac{p+5}{4},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+\tfrac{(2s-m+2)}{2}p,\\
&\quad \tfrac{p+4}{4}\le r\le \tfrac{p+2}{2},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p-1,\\
&\quad \tfrac{p+7}{4}\le r\le \tfrac{p+3}{2},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+\tfrac{(2s-m+2)}{2}p,\\
&\quad \tfrac{p+6}{4}\le r\le \tfrac{p+2}{2},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+2r+(s-\tfrac{m}{2}+1)p-1,\\
&\quad \tfrac{p+9}{4}\le r\le \tfrac{p+3}{2},\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+r+\tfrac{(2s-m+4)}{2}p+1,\\
&\quad \tfrac{p+4}{2}\le r\le p,\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 0(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+r+\tfrac{(2s-m+4)p+1}{2},\\
&\quad \tfrac{p+5}{2}\le r\le p,\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 1(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+r+\tfrac{(2s-m+4)}{2}p+1,\\
&\quad \tfrac{p+4}{2}\le r\le p,\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 2(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= q+r+\tfrac{(2s-m+4)p+1}{2},\\
&\quad \tfrac{p+5}{2}\le r\le p,\ s=\tfrac{m+2}{2},\\
&\quad p\equiv 3(mod\ 4)\\
\omega_{g_2}(x_{r,s}y_{r,s}) &= 2q+2r-(2m-2s-1)p+1,\\
&\quad 1\le r\le p,\ \tfrac{m+4}{2}\le s\le m-2.
\end{aligned}
$$

Hence there is an edge irregular q-labelling g_2 with $k=\lceil\frac{(2m-3)p+2}{2}\rceil$ such that for $m\ge 6$ the edge weights of $B_p(C_m(m-2)pK_1)$ for $m\ge 6$ and m even are all different. ■

From Lemma 2.1 and Lemma 2.2, we have Proposition 2.4.

Proposition 2.4 If $B_p(C_m(m-2)pK_1)$, $m\ge 6$ is a book graph with additional $(m-2)p$ pendant edges, then $es(B_p(C_m(m-2)pK_1))=\lceil\frac{(2m-3)p+2}{2}\rceil$.

Using the construction of vertex labelling g_1 and vertex labelling g_2 as defined in Lemma 2.1 and Lemma 2.2, the edge weights on the graph $B_p(C_m(m-2)pK_1)$ are all different. If all pendant edges are removed from the graph $B_p(C_m(m-2)pK_1)$ then we get a book graph $B_p(C_m)$ with different edge weights. Based on the lower bound of the irregular strength of the given edge [3] and Proposition 2.4, it is obtained that the edge irregularity strength of the book graph $B_p(C_m)$ for $m\ge 6$ is in the interval $\lceil\frac{(m-1)p+2}{2}\rceil\le es(B_p(C_m))\le\lceil\frac{(2m-3)p+2}{2}\rceil$.

Example 2.2 In Figure 2, we define an edge irregularity q-labeling of $B_2(C_6 8K_1)$ with $q=10$, and we can see each edge of the graph $B_2(C_6 8K_1)$ has a different weight.

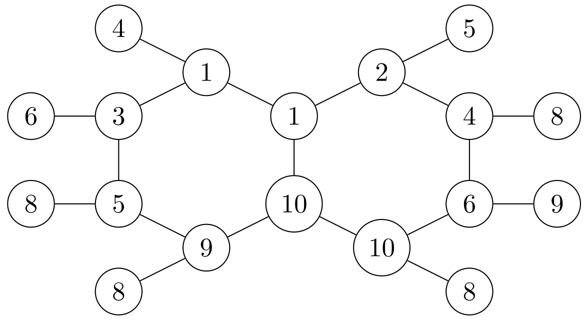

Fig. 2. Edge irregular 10-labeling of book graph $B_2(C_6 8K_1)$

Furthermore, if we remove all pendant edges from the graph $B_2(C_6 8K_1)$, we will get a book graph $B_2(C_6)$ with different weights for all edges, as shown in Figure 3. It indicates that there is an edge irregularity q-labeling with $q=10$.

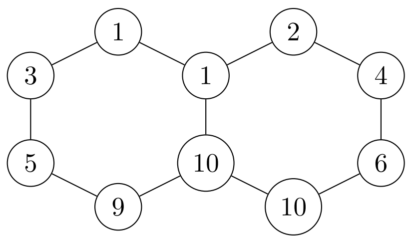

Fig. 3. Edge irregular q-labeling of book graf $B_2(C_6)$ with k=10

Now, we define a book graph by adding p pendant edges as given below.

Definition 2.3 Suppose that $B_p(C_m)$ is a book graph of m sides and p sheets. A book graph with additional p pendant edges, denoted by $B_p(C_m pK_1)$, is a graph that obtained from $B_p(C_m)$ by adding one pendant edge $x_{r,s}y_{r,s}$ at each vertex $x_{r,s}$ with $1 \le r \le p,\ s = m-2$. Therefore, $B_p(C_m pK_1)$ has vertex set $V(B_p(C_m pK_1)) = \{u, v, x_{r,m-2}, y_{r,m-2} \mid 1 \le r \le p\}$ and edge set $E(B_p(C_m pK_1)) = E(B_p(C_m)) \cup \{x_{r,m-2}, y_{r,m-2} \mid 1 \le r \le p\}$.

Proposition 2.5 Let $B_p(C_m pK_1)$ be a book graph with additional one pendant edge at each vertex x_{rs}. Then $es(B_p(C_m pK_1)) = \lceil \frac{pm+2}{2} \rceil$.

Proof: The proof was differentiated into 4 cases.

Case 1 for $m \equiv 2 \ mod\ 4$.
We construct the vertex labeling f and the vertices are labelled as follows:

$$\begin{array}{ll}
f(u) = 1, & f(v) = q \\
f(x_{r,1}) = r, & 1 \le r \le p \\
f(x_{r,s}) = \frac{s}{2}p + 1, & 2 \le s \le m-4 \text{ and } s \text{ even} \\
f(x_{r,s}) = \frac{s-1}{2}p + r, & 3 \le s \le \frac{m-2}{2} \text{ and } s \text{ odd} \\
f(x_{r,s}) = \frac{s-1}{2}p + r, & \frac{m+2}{2} \le s \le m-3 \text{ and } s \text{ odd} \\
f(x_{r,s}) = q - p + r, & s = m-2 \\
f(y_{r,m-2}) = \frac{s}{2}p + r. &
\end{array}$$

For all edges in $E(B_p(C_m pK_1))$ we obtain the weights as given below:

$$\begin{array}{ll}
\omega_f(uv) = 1 + q & \\
\omega_f(ux_{r,1}) = r + 1, & \\
\omega_f(x_{r,s}x_{r,s+1}) = sp + r + 1, & 1 \le s \le \frac{m-2}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) = sp + r + 2, & \frac{m}{2} \le s \le m-4 \\
\omega_f(x_{r,s}x_{r,s+1}) = q + \frac{s-3}{2}n + 2r + 1, & s = m-3 \\
\omega_f(x_{r,s}v) = 2q - p + r, & s = m-2 \\
\omega_f(x_{r,m-2}y_{r,m-2}) = q + \frac{s-2}{2}p + 2r, & \\
\text{with } 1 \le r \le p. &
\end{array}$$

Case 2 for $m \equiv 0 \ mod\ 4$.
The vertices are labelled as follows:

$$\begin{array}{ll}
f(u) = 1, & f(v) = q \\
f(x_{r,1}) & = r, \\
 & 1 \le r \le p \\
f(x_{r,s}) & = \frac{s}{2}p + 1,\ 1 \le r \le p, \\
 & 2 \le s \le m-4,\ s \text{ even},\ s \ne \frac{m+2}{2} \\
f(x_{r,s}) & = \frac{s}{2}p + 2, \\
 & 1 \le r \le p,\ s = \frac{m+2}{2} \\
f(x_{r,s}) & = \frac{s-1}{2}p + r, \\
 & 1 \le r \le p,\ 3 \le s \le m-5 \text{ and } s \text{ odd} \\
f(x_{r,s}) & = \frac{s-1}{2}p + r + 1, \\
 & 1 \le r \le p,\ s = m-3 \\
f(x_{r,s}) & = q - p + r, \\
 & 1 \le r \le p,\ s = m-2 \\
f(y_{r,m-2}) & = \frac{s}{2}p + r + 1 \\
 & 1 \le r \le p.
\end{array}$$

The weight of the edges are the following:

$$\begin{array}{ll}
\omega_f(uv) & = 1 + q \\
\omega_f(ux_{r,1}) & = r + 1,\ 1 \le r \le p \\
 & 1 \le r \le p \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 1, \\
 & 1 \le r \le p, 1 \le s \le \frac{m}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 2, \\
 & 1 \le r \le p, \frac{m+2}{2} \le s \le m-4 \\
\omega_f(x_{r,s}x_{r,s+1}) & = q + \frac{s-3}{2}p + 2r + 1, \\
 & 1 \le r \le p,\ s = m-3 \\
\omega_f(x_{r,s}v) & = 2q - n + r, \\
 & 1 \le r \le p,\ s = m-2 \\
\omega_f(x_{r,s}y_{r,s}) & = q + \frac{s-2}{2}p + 2r + 1, \\
 & 1 \le r \le p,\ s = m-2.
\end{array}$$

Case 3 for $m \equiv 1 \ mod\ 4$.
The vertices are labelled as given below:

$$\begin{array}{ll}
f(u) = 1, & f(v) = q \\
f(x_{r,1}) & = r,\ 1 \le r \le p \\
f(x_{r,s}) & = \frac{s}{2}p + 1, \\
 & 2 \le s \le \frac{m-1}{2}, \text{ and } p \text{ even} \\
f(x_{r,s}) & = \frac{s-1}{2}p + r, \\
 & 3 \le s \le \frac{m-3}{2}, \text{ and } s \text{ odd} \\
f(x_{r,s}) & = \frac{s-1}{2}p + r, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil \text{ and } s = \frac{m+1}{2} \\
f(x_{r,s}) & = \frac{s-1}{2}p + r + 1, \\
 & \frac{p+2}{2} \le r \le p \text{ and } s = \frac{m+1}{2} \\
f(x_{r,s}) & = \frac{s}{2}p + r + 1, \\
 & \frac{m+5}{2} \le s \le m-4, \text{ and } s \text{ odd} \\
f(x_{r,s}) & = \frac{s}{2}p + 2, \\
 & \frac{m+3}{2} \le s \le m-3, \text{ and } p \text{ even} \\
f(x_{r,s}) & = q - p + r,\ s = m-2 \\
f(y_{r,m-2}) & = \frac{m-3}{2}p + \frac{p-3}{2}, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil \text{ and } s \text{ odd} \\
f(y_{r,m-2}) & = \frac{m-1}{2}p + \frac{p-3}{2}, \\
 & \lceil \frac{p+2}{2} \rceil \le r \le p, \text{ and } s \text{ odd} \\
f(y_{r,m-2}) & = \frac{m-3}{2}p + \frac{p-4}{2}, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil, \text{ and } p \text{ even} \\
f(y_{r,m-2}) & = \frac{m-1}{2}p + \frac{p-4}{2}, \\
 & \lceil \frac{p+2}{2} \rceil \le r \le p-1, \text{ and } p \text{ even} \\
f(y_{r,m-2}) & = p + 2, \\
 & r = p \text{ and } p \text{ even.}
\end{array}$$

The weight of the edges are:

$$\begin{array}{ll}
\omega_f(ux_{r,1}) & = r + 1,\ 1 \le r \le p \\
\omega_f(uv) & = 1 + q \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 1, \\
 & 1 \le r \le p,\ 1 \le s \le \frac{m-3}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 1, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil,\ s = \frac{m-1}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 2, \\
 & \lceil \frac{p+2}{2} \rceil \le r \le p,\ s = \frac{m-1}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 2, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil,\ s = \frac{m+1}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 3, \\
 & \lceil \frac{p+2}{2} \rceil \le r \le p,\ s = \frac{m+1}{2} \\
\omega_f(x_{r,s}x_{r,s+1}) & = sp + r + 3, \\
 & 1 \le r \le p,\ \frac{m+3}{2} \le s \le m-4 \\
\omega_f(x_{r,s}x_{r,s+1}) & = q + \frac{s-2}{2}p + r, \\
 & 1 \le r \le p,\ s = m-3 \\
\omega_f(x_{r,s}v) & = 2q - n + r, \\
 & 1 \le r \le p,\ s = m-2 \\
\omega_f(x_{r,m-2}y_{r,m-2}) & = q + \frac{m-5}{2}p - \frac{p-3}{2} + r, \\
 & 1 \le r \le \lceil \frac{p}{2} \rceil, \text{ and } p \text{ odd}
\end{array}$$

$$
\begin{aligned}
\omega_f(x_{r,m-2}y_{r,m-2}) &= q+\tfrac{m-3}{2}p-\tfrac{p-3}{2}+r,\\
&\quad \lceil\tfrac{p+2}{2}\rceil \le r\le p \ \text{ and } p \text{ odd}\\
\omega_f(x_{r,m-2}y_{r,m-2}) &= q+\tfrac{m-5}{2}p-\tfrac{p-4}{2}+r,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil \ \text{ and } p \text{ even}\\
\omega_f(x_{r,m-2}y_{r,m-2}) &= q+\tfrac{m-3}{2}p-\tfrac{p-4}{2}+r,\\
&\quad \lceil\tfrac{p+2}{2}\rceil \le r\le p-1 \ \text{ and } p \text{ even}\\
\omega_f(x_{r,m-2}y_{r,m-2}) &= 2q+r-n,\\
&\quad i=n \text{ and } p \text{ even.}
\end{aligned}
$$

Case 4 for $m \equiv 3 \ mod \ 4$.
The vertices are labelled as follows:

$$
\begin{aligned}
f(u)=1,\quad & f(v)=q\\
f(x_{r,1}) &= r,\ 1\le r\le p\\
f(x_{r,s}) &= \tfrac{s}{2}p+1,\\
&\quad 2\le s\le \tfrac{m-3}{2} \text{ and } s \text{ even}\\
f(x_{r,s}) &= \tfrac{s-1}{2}p+r,\\
&\quad 3\le s\le \tfrac{m-1}{2} \text{ and } s \text{ odd}\\
f(x_{r,s}) &= \tfrac{s}{2}p+1,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil \text{ and } s=\tfrac{m+1}{2}\\
f(x_{r,s}) &= \tfrac{s}{2}p+2,\\
&\quad \tfrac{p+2}{2}\le r\le p \text{ and } s=\tfrac{m+1}{2}\\
f(x_{r,s}) &= \tfrac{s}{2}p+r+1,\\
&\quad \tfrac{m+3}{2}\le s\le m-4 \text{ and } s \text{ odd}\\
f(x_{r,s}) &= \tfrac{s}{2}p+2,\\
&\quad \tfrac{m+3}{2}\le s\le m-3 \text{ and } s \text{ even}\\
f(x_{r,s}) &= q-p+r,\\
&\quad s=m-2\\
f(y_{r,m-2}) &= \tfrac{m-3}{2}p+\tfrac{p-3}{2},\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil \text{ and } p \text{ odd}\\
f(y_{r,m-2}) &= \tfrac{m-1}{2}p+\tfrac{p-3}{2},\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p \text{ and } p \text{ odd}\\
f(y_{r,m-2}) &= \tfrac{m-3}{2}p+\tfrac{p-4}{2},\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil \text{ and } p \text{ even}\\
f(y_{r,m-2}) &= \tfrac{m-1}{2}p+\tfrac{p-4}{2},\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p-1 \text{ and } p \text{ even}\\
f(y_{r,m-2}) &= p+2,\\
&\quad r=p \text{ and } p \text{ even.}
\end{aligned}
$$

The weight of the edges are:

$$
\begin{aligned}
\omega_f(ux_{r,1}) &= r+1,\\
&\quad 1\le r\le p\\
\omega_f(uv) &= 1+q\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+1,\\
&\quad 1\le r\le p,\ 1\le s\le \tfrac{m-3}{2}\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+1,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil,\ s=\tfrac{m-1}{2}\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+2,\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p,\ s=\tfrac{m-1}{2}\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+2,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil,\ s=\tfrac{m+1}{2}\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+3,\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p,\ s=\tfrac{m+1}{2}\\
\omega_f(x_{r,s}x_{r,s+1}) &= sp+r+3,\\
&\quad 1\le r\le p,\ \tfrac{m+3}{2}\le s\le m-4\\
\omega_f(x_{r,s}x_{r,s+1}) &= q+\tfrac{s-2}{2}p+r,\\
&\quad 1\le r\le p,\ s=m-3\\
\omega_f(x_{r,s}v) &= 2q-p+r,\\
&\quad 1\le r\le p,\ s=m-2\\
\omega_f(x_{r,s}y_{r,s}) &= q+\tfrac{m-5}{2}p-\tfrac{p-3}{2}+r,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil,\ s=m-2,\ p \text{ odd}\\
\omega_f(x_{r,s}y_{r,s}) &= q+\tfrac{m-3}{2}p-\tfrac{p-3}{2}+r,\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p,\ s=m-2,\ p \text{ odd}\\
\omega_f(x_{r,s}y_{r,s}) &= q+\tfrac{m-5}{2}p-\tfrac{p-4}{2}+r,\\
&\quad 1\le r\le \lceil\tfrac{p}{2}\rceil,\ s=m-2,\ p \text{ even}\\
\omega_f(x_{r,s}y_{r,s}) &= q+\tfrac{m-3}{2}p-\tfrac{p-4}{2}+r,\\
&\quad \lceil\tfrac{p+2}{2}\rceil\le r\le p-1,\ s=m-2,\ p \text{ even}\\
\omega_f(x_{r,s}y_{r,s}) &= 2q+r-p,\\
&\quad r=p,\ s=m-2,\ p \text{ even.}
\end{aligned}
$$

For vertex labelling f, it was observed that the edge weights of the graph $B_p(C_mpK_1)$ are different and the vertex labels are less than m and form a sequence of integers from 2 to $2k$ with $q=\lceil\frac{mp+2}{2}\rceil$. The weight of the edges of $B_p(C_mpK_1)$ respect to labelling f form the set $\{2,3,4,...,(m-1)p+2\}$. ■

Proposition 2.5 proves that there is an edge irregular q-labelling of book graph with the additional p pendant edges, $B_p(C_mpK_1)$, with $q=\lceil\frac{mp+2}{2}\rceil$, and thus we have that the edge irregularity strength of $B_p(C_mpK_1)$ is $es(B_p(C_mpK_1))=\lceil\frac{mp+2}{2}\rceil$.

In [3], the lower bound of the edge irregularity strength of graph B given by Ahmad et al., the form is $es(B)\ge max\left\{\lceil\frac{|E(B)|+1}{2}\rceil,\Delta(B)\right\}$, and $\Delta(B)$ is the maximum degree of B. Based on Definition 2.1, we have the number of edges and maximum degree of book graph $B_p(C_m)$ are $|E(B_p(C_m)|=(m-1)p+1$ and $\Delta(B)=p+1$, and therefore $es(B_p(C_m))\ge\lceil\frac{(m-1)p+2}{2}\rceil$.

Furthermore, if we remove the p pendant edges of the graph $B_p(C_mpK_1)$ labelled by the vertex labelling as defined in Proposition 2.5, we have a book graph $B_p(C_m)$, which edges weights are all different. From this we conclude that the edge irregularity strength of book graph $B_p(C_m)$ is in the interval $\lceil\frac{(m-1)p+2}{2}\rceil\le es(B_p(C_m))\le\lceil\frac{mp+2}{2}\rceil$.

Example 2.3 Figure 4 shows an edge irregularity q-labeling of $B_2(C_62K_1)$ with $q=7$. We can see that the weight of edges are different.

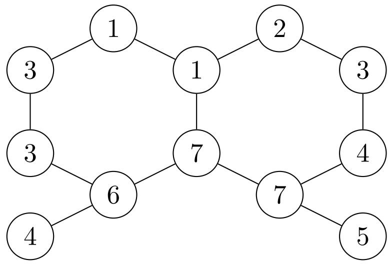

Fig. 4. Edge irregularity 7-labeling of book graph $B_2(C_62K_1)$

By removing all pendant edges from graph $B_2(C_62K_1)$, we will get a book graph $B_2(C_6)$ with different weights on all edges as shown in Figure 5.

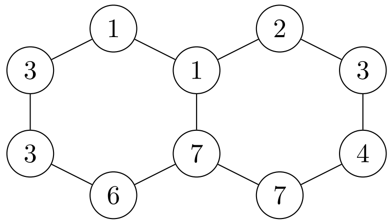

Fig. 5. Edge irregular q-labeling of book graf $B_2(C_6)$ with k=7

III. Conclusion

Based on the main result, we can constructed the edge irregular q-labelling and used for determining the exact value of the edge irregularity strength of triangular, rectangular, pentagonal book graphs, and book graph with additional $(m-2)p$ and p pendant edges for $m \geq 6$. We have $es(B_p(C_3)) = p+2$, $es(B_p(C_4)) = \lceil \frac{3p+2}{2} \rceil$, $es(B_p(C_5)) = \lceil \frac{4p+2}{2} \rceil$, and $es(B_p(C_m(m-2)pK_1)) = \lceil \frac{(2m-3)p+2}{2} \rceil$ and $es(B_p(C_m pK_1)) = \lceil \frac{pm+2}{2} \rceil$ for $m \geq 6$.
We also obtain that the edge irregularity strength of book graph $B_p(C_m)$ for $m \geq 6$ is in the interval $\lceil \frac{(m-1)p+2}{2} \rceil \leq es(B_p(C_m)) \leq \lceil \frac{mp+2}{2} \rceil$. We can continue this research to obtain the exact value of the irregularity strength of book graph $B_p(C_m)$.

References

[1] G. Chartrand , M. Jacobson, J. Lehel, O. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr Numer* 64, 187-192, 1988.

[2] M. Bača , S. Jendrol, M. Miller and J. Ryan, On irregular total labeling, *Discrete Math* 307, 1378-1388, 2007.

[3] Ahmad, A., l-Mushayt, O.B.S., & Bača, M. On edge irregularity strength of graphs. *Applied Mathematics and Computation* 243, 607-610, 2014.

[4] Ahmad, A., Bača, M., & Nadeem, M.F., On edge irregularity strength of toeplitz graphs. *U.P.B. Sci Bull. Series A* 78(4), 155-162, 2016.

[5] Tarawneh, I., Hasni, R., & Ahmad, A. (2016). On the edge irregularity strength of corona product of cycle with isolated vertices, *AKCE International Journal of Graphs and Combinatorics* 13, 213-217.

[6] Tarawneh, I., Hasni, R., & Ahmad, A. (2018). On the edge irregularity strength of grid graphs, *AKCE International Journal of Graphs and Combinatorics* 17, 414-418.

[7] Tarawneh, I., Hasni, R., Ahmad, A., & Asim, M.A. (2021). On the edge irregularity strength for some classes of plane graphs, *AIMS Mathematics* 6(3), 2724-2731.

[8] Imran, M., Aslam, A., Zafar, S., & Nazeer, W. (2017). Further result on edge irregularity strength of graphs, *Indonesian Journal of Combinatorics* 1(2), 82-91.

[9] Ahmad, A., Gupta, A., & Simanjuntak, R. (2018). Computing the edge irregularity strength of chain graphs and the join of two graphs, *Electronic Journal of Graph Theory and Applications* 6(1) (2018), 201-207.

[10] Suparta, I.N., & Suharta, I.G.P. (2020). A note on the edge irregularity strength of some graphs, *Indonesian Journal of Combinatorics* 4(1), 10-20.

[11] Alrawajfeh, A., Al-Hasanat, B.N., Alhasanat, H., & Al Faqih, F.M., (2021). On the edge irregularity strength of bipartite graph and corona product of two graphs, *Internatioanal Journal of Mathematics and Computer Science* 16(2), 639-645.