Optimal Forest Rotation from the Viewpoint of Carbon Sequestration Efficiency based on the Gilpin-Ayala Model

Junwei Chen, Shiyu Hu, Linyi Chen, Jiayi Gu, Fengfan Ge and Sishui Wang

Abstract—**This paper introduces the Gilpin-Ayala equation, gives the explicit expression of the solution, and analyzes the characteristics of the solution. Based on these results, a stock growth model is constructed to study the rotation period of forests. From the viewpoint of carbon sequestration benefits, the paper explicitly expresses the optimal solution of a single rotation period of forests and studies the mathematical law of the solution. By using the monotonicity of the optimal solution, this paper proves that carbon sink unit price and rotation period modulate in the same way. In other words, if carbon sink unit price is high, then rotation period is large. This means that when carbon sink price increases, cutting day will be delayed; when carbon sink price decreases, cutting day will move forward. The impacts of timber unit price and interest rate on the rotation period are as follows: when timber price rises, cutting day will move forward; when timber price falls, cutting day will be delayed; when interest rate rises, cutting day will move forward; when interest rate falls, cutting day will be delayed. In addition, with the help of data instances, this paper demonstrates that for both single- and multi-rotation, carbon sequestration benefits postpone the cutting time. Moreover, the multi-rotation cutting occurs earlier than the single rotation cutting. Therefore, harvesting at the optimal rotation period with carbon sequestration benefits is conducive for the coordinated development of social and ecological benefits.**

Index Terms—**Carbon sequestration efficiency, Gilpin-Ayala model, Forest stock, Optimal rotation.**

I. Introduction

As the world's largest emitter of carbon dioxide, China undergoes pressure from its peers to reduce emissions. China has abundant forest resources and can use forest carbon sink to offset greenhouse gas emissions ([1], [2], [3], [4]), making it easier to reduce emissions. The forest is the main body of the terrestrial ecosystem and the largest carbon pool. It absorbs CO_2 through photosynthesis and stores terrestrial carbon, thereby restraining the increase in CO_2 concentration in the atmosphere, improving the situation of climate warming, adjusting the global carbon balance, and promoting sustainable development ([5]). In 2009, the Chinese government pledged to increase forest carbon sink to achieve double the growth of forestry; in 2020, it proposed to reduce CO_2 emissions, reach a peak by 2030, and achieve carbon neutrality by 2060. In order to increase carbon sink, reduce emissions, maximize forest functions, and balance the relationships between ecological benefits and economic benefits, it is necessary to seek the optimal timing of the forest rotation period with carbon sequestration benefits.

The optimal forest rotation age refers to the deforestation time that can help us derive the best forest benefits. To the best of our knowledge, there is no explicit expression of the optimal forest rotation, the impacts of interest rate, timber price, and carbon sink price on the optimal cutting time are still unknown. The position of the optimal cutting time for forest stock is not clear either. For example, several scholars have only given the optimal implicit forest rotation period with carbon sequestration benefits ([8], [9], [11]-[18]), which is not conducive to estimating the cutting time. The impacts of interest rate, timber price, and carbon sink price on the cutting time have been demonstrated through unrigorous numerical examples. Furthermore, none of the research has proved that the optimal cutting time should be in the forest accumulation growth, after the highest peak and before the saturation period. There has also been no comparison between the carbon and non-carbon harvesting time of the single rotation and multi-rotation models.

The forest stock volume is the basis of the rotation period model, and many scholars have chosen several different models to investigate the forest stock volume. For example, Hui and Zhu ([19], [21]) chose the GM (1,1) model and found that GM (1,1) is less accurate than the logistic model. Some scholars chose the logistic function ([22]-[24]), and some chose the Richards function ([9], [24]-[28]). However, the fitting accuracy of the function is not good enough. There are other research methods; for example, Lin [15] used the ARIMA prediction, but this method does not consider the particularity of biological populations.

Motivated by these, this paper introduces the Gilpin-Ayala equation to fit the forest stock volume. The Gilpin-Ayala equation has one more fitting parameter θ than the logistic function. It not only improves the fitting accuracy but also conforms to the growth characteristics of species and reduces the influences or errors caused by other factors. The characteristics of the Gilpin-Ayala curve are also studied,

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and the maximum value, the inflection point, the initial stage of species growth, the rapid period, the highest peak, and the stagflation saturation period are solved. We also solve the explicit expression of the optimal solution for a single rotation period of the forest from the viewpoint of carbon sequestration benefits. We conduct an experimental study of the monotonicity of the explicit expression to analyze the unit price of carbon sink, timber price and the impact of interest rate on the rotation period. We work out the maximum of LEV too. In addition, we study the multi-rotation mathematical model, and discuss the relationships between the multi-rotation harvesting period and the single rotation harvesting period.

II. Curve Equation of Gilpin-Ayala Model

The logistic model ([23], [29]-[35]) is often used to describe the curve of a single species in which a population receives environmental or resource constraints that slow down its growth rate. A slower growth rate characterizes the curve in the early phase, and then the growth rate accelerates. When the population reaches the maximum value, the growth rate decreases owing to the competitive influence of the resource environment or the population in nature until it finally stagnates (does not grow).

The logistic model can be expressed as follows:

$$X'(t) = bX(t)\left[1 - \frac{X(t)}{k}\right].$$

where $X(t)$ represents the size of the population, $X'(t) = dX(t)/dt$, b is the intrinsic growth rate of the population, k stands for the maximum capacity of the environment.

With the progress in research on life sciences and mathematics, scholars have popularized and expanded the logistic model and found a Gilpin-Ayala model with higher accuracy, a better effect, and a wider application range than the logistic model. The logistic curve is just a special case of the Gilpin-Ayala model ([36]-[45]).

The Gilpin-Ayala model can be expressed as follows:

$$X'(t) = bX(t)\left[1 - \left(\frac{X(t)}{k}\right)^{\theta}\right]. \tag{1}$$

where θ is used to fit the real data.

The analytical solution of the Gilpin-Ayala model can be expressed as follows:

$$X(t) = \frac{k}{(1 + ae^{-\theta bt})^{\frac{1}{\theta}}}, \tag{2}$$

where

$$a = \left(\frac{k}{X(0)}\right)^{\theta} - 1.$$

As a result,

$$\begin{aligned}
&\frac{X'(t)}{X(t)} \\
&= b\left[1 - (\frac{X(t)}{k})^{\theta}\right] \\
&= b\left[1 - \left(\frac{k/(1 + ae^{-\theta bt})^{\frac{1}{\theta}}}{k}\right)^{\theta}\right] \\
&= \frac{abe^{-\theta bt}}{1 + ae^{-\theta bt}} \\
&= \frac{ab}{a + e^{\theta bt}}.
\end{aligned} \tag{3}$$

As indicated above, the Gilpin-Ayala model curve has the following characteristics.

(A1). The curve has two asymptote lines parallel to the t-axis:

$$X = k,\ X = \frac{k}{(1 + a)^{\frac{1}{\theta}}}.$$

Since,

$$\lim_{t \to 0} \frac{k}{(1 + ae^{-\theta bt})^{\frac{1}{\theta}}} = \frac{k}{(1 + a)^{\frac{1}{\theta}}},$$

and

$$\lim_{t \to +\infty} \frac{k}{(1 + ae^{-\theta bt})^{\frac{1}{\theta}}} = k.$$

(A2). The curve is monotonically increasing if $X(0) < K$.

(A3). Curve convexity and inflection point.

The first and second derivatives of the curve can be solved separately:

$$\begin{aligned}
X'(t) &= \frac{d}{dt}\left[\frac{k}{(1 + ae^{-\theta bt})^{\frac{1}{\theta}}}\right] \\
&= k \cdot \frac{d}{dt}\left[\frac{1}{(1 + ae^{-\theta bt})^{\frac{1}{\theta}}}\right] \\
&= k \cdot (-\frac{1}{\theta})(1 + ae^{-\theta bt})^{-\frac{1}{\theta}-1}\frac{d}{dt}\left[1 + ae^{-\theta bt}\right] \\
&= \frac{-k \cdot (1 + ae^{-\theta bt})^{-\frac{1}{\theta}-1} a\frac{d}{dt}[e^{-\theta bt}]}{\theta} \\
&= \frac{-k \cdot (1 + ae^{-\theta bt})^{-\frac{1}{\theta}-1} ae^{-\theta bt}\frac{d}{dt}[-\theta bt]}{\theta} \\
&= \frac{-k \cdot (1 + ae^{-\theta bt})^{-\frac{1}{\theta}-1} ae^{-\theta bt}(-\theta b)}{\theta} \\
&= k \cdot ab(1 + ae^{-\theta bt})^{-\frac{1}{\theta}-1}e^{-\theta bt} \\
&= \frac{k \cdot abe^{bt}}{(a + e^{\theta bt})^{1+\frac{1}{\theta}}}.
\end{aligned} \tag{4}$$

As a result, one can obtain $X'(t) > 0$. In addition, one can see that

$$\begin{aligned}
&X''(t) = \frac{d}{dt}\left[\frac{abke^{bt}}{(a + e^{\theta bt})^{1+\frac{1}{\theta}}}\right] \\
&= abk \cdot \frac{d}{dt}\left[\frac{e^{bt}}{(a + e^{\theta bt})^{1+\frac{1}{\theta}}}\right] \\
&= \frac{abk}{(a + e^{\theta bt})^{2(1+\frac{1}{\theta})}} \\
&\cdot\left\{(a + e^{\theta bt})^{1+\frac{1}{\theta}}\frac{d}{dt}[e^{bt}] - e^{bt}\frac{d}{dt}[(a + e^{\theta bt})^{1+\frac{1}{\theta}}])\right\} \\
&= \frac{abk}{(a + e^{\theta bt})^{2(1+\frac{1}{\theta})}} \\
&\cdot[be^{bt}(a + e^{\theta bt})^{1+\frac{1}{\theta}} - (a + e^{\theta bt})^{\frac{1}{\theta}}(\frac{\theta + 1}{\theta})e^{bt}\frac{d}{dt}(e^{\theta bt})] \\
&= \frac{abk}{(a + e^{\theta bt})^{2(1+\frac{1}{\theta})}} \\
&\cdot[be^{bt}(a + e^{\theta bt})^{1+\frac{1}{\theta}} - (a + e^{\theta bt})^{\frac{1}{\theta}}(\frac{\theta + 1}{\theta})\theta be^{\theta bt}e^{bt}] \\
&= \frac{abk}{(a + e^{\theta bt})^{2(1+\frac{1}{\theta})}} \\
&\cdot\left[be^{bt}(a + e^{\theta bt})^{1+\frac{1}{\theta}} - (a + e^{\theta bt})^{\frac{1}{\theta}}(\theta + 1)be^{bt+\theta bt}\right] \\
&= -ab^2ke^{bt}(\theta e^{\theta bt} - a)/(a + e^{\theta bt})^{2+\frac{1}{\theta}}.
\end{aligned}$$

Letting $X''(t) = 0$ gives

$$\theta e^{\theta bt} - a = 0.$$

That is to say,

$$e^{\theta bt} = \frac{a}{\theta}.$$

In other words,

$$t = \frac{\ln a - \ln\theta}{b\theta}.$$

By a simple calculation, one can see that:
if $t > \frac{\ln(a)-\ln(\theta)}{b\theta}$, then $X''(t) > 0$. The curve is concave.
if $t < \frac{\ln(a)-\ln(\theta)}{b\theta}$, then $X''(t) < 0$. The curve is convex.
One then obtain the inflection point of the curve:

$$\left(\frac{\ln a - \ln\theta}{b\theta}, \frac{k}{\left(1 + a(\theta/a)^{r/b}\right)^{1/\theta}}\right).$$

Thereby, the Gilpin-Ayala curve is a growing S-shaped curve similar to the logistic curve. The slope of the curve appears small at first, then it turns faster and then finally returns slow.

When $t = \frac{\ln a - \ln\theta}{b\theta}$, the growth rate reaches its peak.

In addition, one can see that

$$\begin{aligned}
X'''(t) &= \frac{d}{dt}\left[\frac{-ab^2ke^{bt}(\theta e^{\theta bt} - a)}{(a + e^{\theta bt})^{2+1/\theta}}\right] \\
&= \frac{-ab^2k}{(a + e^{\theta bt})^{2(2+1/\theta)}}\Bigg\{(a + e^{\theta bt})^{2+1/\theta}\frac{d}{dt}\left[e^{bt}(\theta e^{\theta bt} - a)\right] \\
&\quad -e^{bt}(\theta e^{\theta bt} - a)\frac{d}{dt}\left[(a + e^{\theta bt})^{2+1/\theta}\right]\Bigg\} \\
&= \frac{-ab^2k}{(a + e^{\theta bt})^{2(2+1/\theta)}}\Bigg\{(a + e^{\theta bt})^{2+1/\theta}\Big[be^{bt}(\theta e^{\theta bt} - a) \\
&\quad +e^{bt}\theta\theta be^{\theta bt}\Big] - e^{bt}(\theta e^{\theta bt} - a)(2 + 1/\theta) \\
&\quad (a + e^{\theta bt})^{1+1/\theta}b\theta e^{\theta bt}\Bigg\} \\
&= \frac{-ab^2k}{(a + e^{\theta bt})^{2(2+1/\theta)}}\Bigg\{(a + e^{\theta bt})^{2+\frac{1}{\theta}}\Big[be^{bt}(\theta e^{\theta bt} - a) \\
&\quad +e^{bt}\theta\theta be^{\theta bt}\Big] - b(\theta e^{\theta bt} - a)(2\theta + 1) \\
&\quad (a + e^{\theta bt})^{1+1/\theta}e^{\theta bt+bt}\Bigg\} \\
&= ab^3k\frac{\left[\theta^2e^{2bt\theta} - a\theta(\theta + 3)e^{bt\theta} + a^2\right]e^{bt}}{(a + e^{\theta bt})^{3+1/\theta}}.
\end{aligned}$$

Letting $X'''(t) = 0$ yields

$$\theta^2e^{2bt\theta} - a\theta(\theta + 3)e^{bt\theta} + a^2 = 0.$$

Let $z = e^{bt\theta}$, then

$$\theta^2z^2 - a\theta(\theta + 3)z + a^2 = 0.$$

Solving the equation leads to

$$z = \pm\frac{a\sqrt{\theta^2 + 6\theta + 5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2}.$$

In view of

$$z = e^{bt\theta} = \pm\frac{a\sqrt{\theta^2 + 6\theta + 5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2},$$

thus,

$$bt\theta = \ln\left(\pm\frac{a\sqrt{\theta^2 + 6\theta + 5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2}\right).$$

As a result, we get the solution of the equation:

$$t_1 = \frac{\ln(-\frac{a\sqrt{\theta^2+6\theta+5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2})}{b\theta}.$$

$$t_2 = \frac{\ln(\frac{a\sqrt{\theta^2+6\theta+5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2})}{b\theta}.$$

It can be seen that there are two inflection points in the velocity function, so the process of the growth trend of the curve is divided into three parts:

- Initial period: $t \in (-\infty, t_1)$, period of slow growth.
- Fast period: $t \in (t_1, t_2)$, period of rapid growth.
- Stagnation period: $t \in (t_2, +\infty)$, quantities tend to saturate.

Hence, there are three important points on the curve:

$$t_1 = \frac{\ln(-\frac{a\sqrt{\theta^2+6\theta+5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2})}{b\theta}. \tag{5}$$

$$t = \frac{\ln a - \ln\theta}{b\theta}. \tag{6}$$

$$t_2 = \frac{\ln(\frac{a\sqrt{\theta^2+6\theta+5}}{2\theta} + \frac{3a}{2\theta} + \frac{a}{2})}{b\theta}. \tag{7}$$

III. Mathematical analysis on a single forest rotation model

Now let us study the optimal solution of a single forest rotation period based on the improved Faustmann-Hartman model ([9], [13], [16]) and the Gilpin-Ayala growth model.

The objective function of the land expectation value of a single rotational forest is:

$$\begin{aligned}
Y(T) = (p - c)\delta X(T)e^{-rT} - R + \int_0^T \alpha qX'(t)e^{-rt}dt \\
-q\alpha(1 - \beta)X(T)e^{-rT}.
\end{aligned} \tag{8}$$

where

- T represents the length of the rotation period;
- $X(t)$ stands for the amount of forest stock volume at the time of felling in year T;
- $Y(T)$ is the land expectation value of the forest;
- p and q measure the unit price of timber and the unit price of carbon sink, respectively;
- c is the cost of harvesting and transporting per cubic metre of timber;
- r characterizes the real interest rate;
- R represents the present value of afforestation costs and maintenance costs over the years and logging costs.
- $\alpha > 0$ is the carbon conversion coefficient;
- $\delta \in (0, 1)$ stands for the yield of forest trees;
- $\beta \in (0, 1)$ represents the proportion of uncorrupted and fixed wood in the total;
- $(p - c)\delta X(T)e^{-rT} - R$ characterizes the present value of the net income of timber;
- $\int_0^T \alpha qX'(t)e^{-rt}dt - q\alpha(1-\beta)X(T)e^{-rT}$ measures the net income of carbon sink.

A. Solution of the optimal rotation period

Our aim is to maximize the land expectation, and now we are in the position to look for the time T that maximizes the objective function as the optimal rotation period. It can be seen that

$$\begin{aligned}\frac{dY(T)}{dt} &= (p-c)\delta[(X'(T)e^{-rT} - rX(T)e^{-rT})] \\ &\quad +q\alpha X'(T)e^{-rT} \\ &\quad -q\alpha(1-\beta)[X'(T)e^{-rT} - rX(T)e^{-rT}] \\ &= [X'(T)e^{-rT} - rX(T)e^{-rT}][(p-c)\delta \\ &\quad -q\alpha(1-\beta)] + q\alpha X'(T)e^{-rT} \\ &= e^{-rT}\{[X'(T) - rX(T)][(p-c)\delta \\ &\quad -q\alpha(1-\beta)] + q\alpha X'(T)\}.\end{aligned}$$

Letting $\frac{dY(T)}{dt} = 0$ yields

$$[X'(T) - rX(T)][(p-c)\delta - q\alpha(1-\beta)] = -q\alpha X'(T).$$

Hence

$$\begin{aligned}&X'(T)[(p-c)\delta - q\alpha(1-\beta) + q\alpha] \\ &= rX(T)[(p-c)\delta - q\alpha(1-\beta)].\end{aligned}$$

In other words,

$$\frac{X'(T)}{X(T)} = \frac{r[(p-c)\delta - q\alpha(1-\beta)]}{(p-c)\delta - q\alpha(1-\beta) + q\alpha}.$$

It then follows from (3) that

$$\frac{X'(T)}{X(T)} = \frac{r[(p-c)\delta - q\alpha(1-\beta)]}{(p-c)\delta - q\alpha(1-\beta) + q\alpha} = \frac{ab}{a + e^{\theta bT}}.$$

Therefore,

$$a + e^{\theta bT} = \frac{ab[(p-c)\delta - q\alpha(1-\beta) + q\alpha]}{r[(p-c)\delta - q\alpha(1-\beta)]},$$

Consequently,

$$\begin{aligned}e^{\theta bT} &= \frac{ab[(p-c)\delta - q\alpha(1-\beta) + q\alpha]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a, \\ &= \frac{ab[(p-c)\delta + q\alpha\beta)]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a.\end{aligned}$$

Then,

$$\theta bT = \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta)]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a\right\}.$$

Let T_C be the rotation period with carbon sequestration benefits and T_0 be the rotation period that does not contain carbon sequestration benefits, respectively. One can observe that

$$\begin{aligned}T_c &= \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a\right\}/(\theta b) \\ &= \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta]}{r[(p-c)\delta + q\alpha\beta - q\alpha]} - a\right\}/(\theta b).\end{aligned} \tag{9}$$

By (9), it can be seen that, if $q = 0$, the objective function is the optimal time for forest deforestation when the economic benefits of the forest are considered but the benefits of carbon sequestration are not considered; that is to say,

$$T_0 = \frac{\ln[\frac{ab}{r} - a]}{\theta b}. \tag{10}$$

In addition, it can be seen that

$$\begin{aligned}\frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta - q\alpha(1-\beta)} &= \frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta + q\alpha\beta - q\alpha} \\ &> \frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta + q\alpha\beta} = 1.\end{aligned}$$

As a result,

$$\frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta - q\alpha(1-\beta)} > 1,$$

hence,

$$\frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta - q\alpha(1-\beta)}\frac{ab}{r} > \frac{ab}{r},$$

therefore,

$$\frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta - q\alpha(1-\beta)}\frac{ab}{r} - a > \frac{ab}{r} - a.$$

Notice that in real-world problems, b represents the internal growth rate, r represents the real interest rate; thus $b > r > 0$. As a result,

$$\frac{b}{r} > 1,$$

then,

$$(\frac{b}{r} - 1)a > 0,$$

hence,

$$\frac{ab}{r} - a > 0.$$

Consequently,

$$\ln\left(\frac{(p-c)\delta + q\alpha\beta}{(p-c)\delta - q\alpha(1-\beta)}\frac{ab}{r} - a\right) > \ln\left(\frac{ab}{r} - a\right).$$

According to (9), one can find that

$$\begin{aligned}T_c &= \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a\right\}/(\theta b) \\ &> \ln\left(\frac{ab}{r} - a\right)/(\theta b) = T_0.\end{aligned}$$

As a result, one can obtain that

$$T_c > T_0.$$

In other words, with the increasing of carbon sequestration efficiency, the optimal time for deforestation is postponed.

B. Analysis of rotational period formulas

We are now in the position to study the effects of timber unit price, carbon sink unit price, and interest rate on optimal rotation period with carbon sequestration benefits.

(i) The impact of timber unit price on the rotation period.

Assume that r, q are unchanged. Consider the per unit price of timber minus the cost of timber as the value of timber, i.e. $p - c$. It is easy to see that when $p - c$ increases,

$$\ln\left\{\left(\frac{q\alpha}{(p-c)\delta + q\alpha\beta - q\alpha} + 1\right)\frac{ab}{r} - a\right\}/(\theta b)$$

decreases. As a result,

$$\begin{aligned}T_c &= \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta)]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a\right\}/(\theta b) \\ &= \ln\left\{\left(\frac{q\alpha}{(p-c)\delta + q\alpha\beta - q\alpha} + 1\right)\frac{ab}{r} - a\right\}/(\theta b)\end{aligned}$$

decreases. Consequently, when the unit price of timber rises, then the rotation period T will be shortened, and the harvesting time will move forward.

(ii) The impact of carbon sink unit price on the rotation period.

Assume that r, p, c are unchanged. It is easy to see that when q increases,

$$\ln\left\{\frac{ab[(p-c)\delta+q\alpha\beta)]}{r[(p-c)\delta-q\alpha(1-\beta)]}-a\right\}/(\theta b)$$

increases. As a result,

$$T_c=\ln\left\{\frac{ab[(p-c)\delta+q\alpha\beta)]}{r[(p-c)\delta-q\alpha(1-\beta)]}-a\right\}/(\theta b)$$

increases. Consequently, when the unit price of carbon sink increases, the rotation period T will increase and the harvesting of forests will be delayed.

(iii) The impact of interest rate on the rotation period.

Assume that q, p, c are unchanged. When r increases,

$$r[(p-c)\delta-q\alpha(1-\beta)]$$

increases,

$$\frac{1}{r[(p-c)\delta-q\alpha(1-\beta)]}$$

decreases. Notice that

$$ab[(p-c)\delta+q\alpha\beta)]$$

doesn't change, hence

$$\frac{ab[(p-c)\delta+q\alpha\beta)]}{r[(p-c)\delta-q\alpha(1-\beta)]}$$

decreases. As a result,

$$\ln\left\{\frac{ab[(p-c)\delta+q\alpha\beta)]}{r[(p-c)\delta-q\alpha(1-\beta)]}-a\right\}/(\theta b)$$

decreases. Thereby,

$$T_c=\ln\left\{\frac{ab[(p-c)\delta+q\alpha\beta)]}{r[(p-c)\delta-q\alpha(1-\beta)]}-a\right\}/(\theta b)$$

decreases. Consequently, when interest rate rises, the rotation period will decrease and the harvesting time will move forward.

In sum, when carbon sequestration benefits increase, the best time for forest felling will be delayed, and the unit price of carbon sink modulates in the same way with the rotation period. Regarding changes in the opposite direction to the rotation period, timber will be harvested in advance as the price of it increases and the harvest (of wood) will be delayed as the price of wood decreases.

Without considering the effect of carbon sequestration efficiency on the optimal rotation period (11), the rotation period will be shortened if the timber price and interest rate increase, resulting in early logging. It can be seen that regardless of the benefit of carbon sink, once the interest rate rises or the price of timber rises, the harvesting will be advanced, and the forest benefits are the greatest. The following is an empirical analysis of a single rotation period, taking the forest data of the state-owned forest farm in Sanming City, Fujian Province ([9]) as an example.

TABLE I: Fitting parameters

θ	k	a	b	R^2	residual
1	402.304	19.927	0.212	0.979	16258.257
0.5	405.341	5.065	0.358	0.985	11427.279
0.1	408.588	0.604	1.533	0.990	7446.276
0.25	407.251	1.828	0.651	0.988	8932.227

IV. An Empirical Analysis on Single Rotation Period of Forest

Now, let us present an application by using the forest data from the state-owned forest farms in Sanming, Fujian Province, P.R. China ([9]). According to (2), we should fit the parameters k, a, b, and θ. By using the data in [30] and SPSS, we show several typical fitting parameters in Table I below.

From Table I, it can be seen that

- when $\theta = 0.1$, $b = 1.533 > 1$, the intrinsic growth rate represented by b is not realistic;
- When $\theta = 1$, it is the logistic population curve. However, $R^2 = 0.979$, residual=16258.257.
- When $\theta = 0.25$, R^2 takes the maximum value, and the residual is minimal too.

As a result, we choose $\theta = 0.25$, $k = 407.251$, $a = 1.828$ and $b = 0.651$. In this case, the Gilpin-Ayala model is

$$X(t)=\frac{407.25}{(1+1.828e^{-0.25*0.651*t})^{\frac{1}{0.25}}},$$

which fits the reality better than the logistic model (see Figure 1 below).

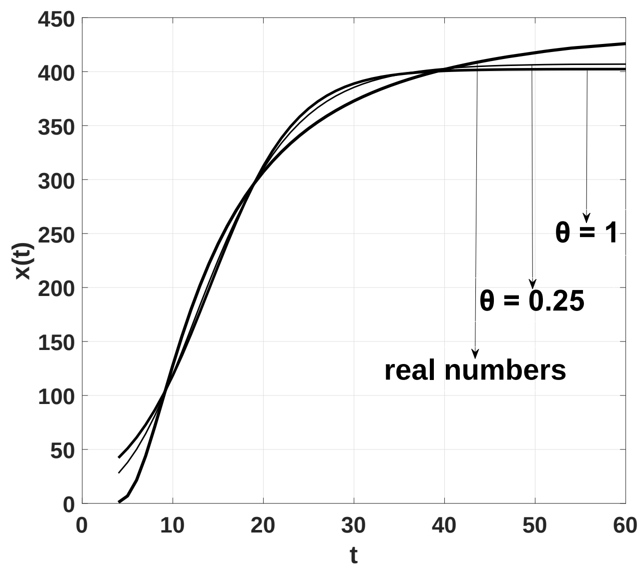

Fig. 1: Comparison between the Gilpin-Ayala model ($\theta = 0.25$) and the Logistic model ($\theta = 1$)

It then follows from (5)-(7) that

$$t_1 = 5.6699502,\ t = 12.22437,\ t_2 = 18.77879733.$$

- $(-\infty, t_1)$ is the initial period, during which the forest stock volume grows slowly;
- (t_1, t_2) is the fast period, during which the forest stock volume grows fast. The highest peak $t = 12.22437$ of forest stock volume is just in this range;

- $(t_2, +\infty)$ is the slowdown period, during which the forest stock volume tends to saturate.

The following figure is the image of forest volume $X(t)$ with $\theta = 0.25$. t_1 and t_2 are special points. For $t \in (t_1, t_2)$, the curve is steep. And then it tends to be flat.

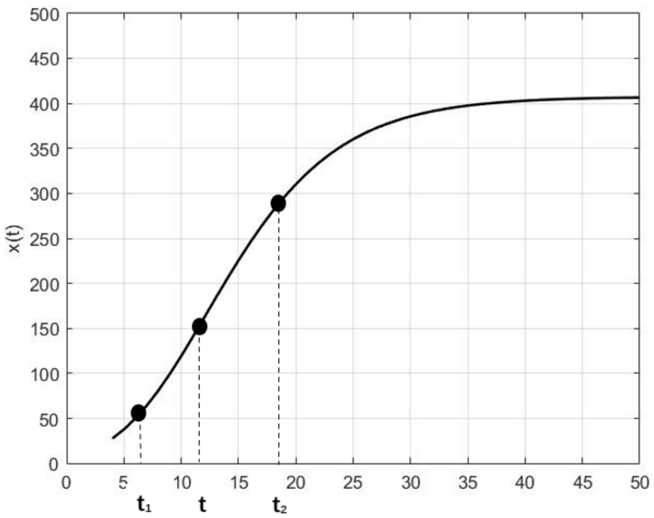

Fig. 2: The curve of $X(t)$ with θ = 0.25.

From the viewpoint of forest stock volume, one should try to cut down near the inflection point. At this time, the forest is almost no longer growing; cutting down is more conducive to rationally using the resources, and can avoid the waste of time, space, and manpower.

Now let us find the optimal solution of the forest rotation period. We choose

$$a = 1.828,\ b = 0.651,\ \theta = 0.25,$$

and

$$p = 900 \text{ rmb}/m^3,\ q = 66 \text{ rmb}/t,\ c = 256 \text{ rmb}/m^3,$$

$$\alpha = 0.225,\ \beta = 0.05,\ \delta = 0.7,\ r = 0.08.$$

According to (9), we can get

$$T_c = \ln\left\{\frac{ab[(p-c)\delta + q\alpha\beta]}{r[(p-c)\delta - q\alpha(1-\beta)]} - a\right\}/(\theta b) = 16.0161,$$

(The above formula is the best rotation period with carbon sequestration benefits.)

In addition, it follows from (10) that

$$T_0 = \ln\left(\frac{ab}{r} - a\right)/(\theta b) = 15.7824,$$

(The above formula is the best rotation period without carbon sequestration benefits.)

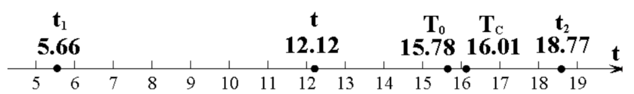

Fig. 3: The forest rotation period with $a = 1.828$, $b = 0.651$, $\theta = 0.25$, and $p = 900$ rmb$/m^3$, $q = 66$ rmb$/t$, $c = 256$ rmb$/m^3$, $\alpha = 0.225$, $\beta = 0.05$, $\delta = 0.7$, $r = 0.08$.

From Fig. 3, it can be seen that the optimal rotation periods for the land expectation value with carbon and without carbon are after the peak of forest stock volume growth and near the saturation point, and the finding is consistent with the theoretical analysis of the Gilpin-Ayala model. In addition, the carbon-containing rotation period is prolonged, and the cutting day is delayed.

Now let us discuss the influences of $p - c, q, r$ on the rotation time T_c (T_0) by numerical simulations.

TABLE II: Without the impacts of carbon sink

$q = 0$	$p - c = 644$	r	0.06	0.065	0.07	0.075
		T_0	17.76	17.22	16.71	16.23

The above table does not consider the impact of carbon sequestration. It can be seen that when the interest rate r increases, T_0 decreases, that is, the rotation period is advanced.

TABLE III: With the impacts of carbon sink

$q = 66$	$p - c = 644$	r	0.06	0.065	0.07	0.075	
		T_c	17.99	17.45	16.94	16.46	$\downarrow$
$q = 66$	r=0.08	$p - c$	500	600	700	800	
		T_c	16.08	16.03	16.00	15.97	$\downarrow$
p-c=644	r=0.08	q	55	60	65	70	
		T_c	15.98	15.99	16.01	16.03	$\uparrow$

The effects of each factor on the value of T_c are given in Table III, in which $\uparrow$ means that the value goes up, $\downarrow$ means that the value goes down.

From Table III, it can be seen that the unit price of carbon sink changes in the same direction as the rotation; that is, cutting day is advanced if the unit price of carbon sink decreases, and cutting day is delayed if the unit price of carbon sink increases (Fig. 6). The unit price of timber changes in the opposite direction of the rotation; that is, cutting day is advanced if the unit price of timber increases, and cutting day is delayed if the unit price of timber decreases (Fig. 4). The interest rate changes in the opposite direction of the rotation; that is, cutting day is advanced if interest rate increases, and cutting day is delayed if interest rate decreases (Fig. 5).

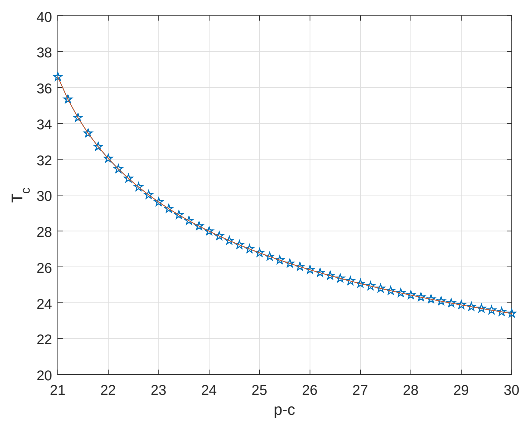

Fig. 4: The relations between $p-c$ (unit price of timber) and T_c.

The unit price of carbon sink changes in the same direction as the rotation period; that is, the increase in the price of carbon sink delays harvesting, and the decrease in the price of carbon sink advances harvesting. The unit price of timber changes in the opposite direction of the rotation period; that

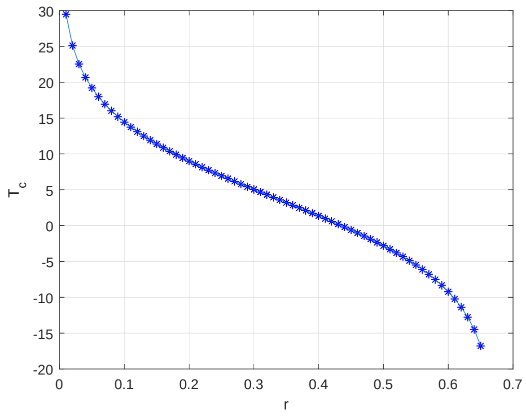

Fig. 5: The relations between r (interest rate) and T_c.

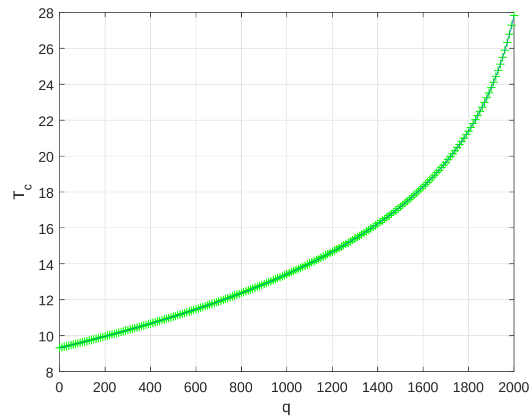

Fig. 6: The relations between q (unit price of carbon sink) and T_c.

is, the harvesting day is advanced as the price of timber increases, and the harvesting day is delayed as the price of timber decreases. The interest rate changes in the opposite direction to the rotation period; that is, as the interest rate rises, the harvesting day is advanced, and as the interest rate falls, the harvesting day is delayed.

V. Analysis of models for multiple rotation period of forests

In the previous sections, we have examined the case of a single rotation period. Now let us pay attention to the case of multiple rotation period. In fact, if the price of timber and the cost of harvesting and transporting do not change after a single rotation period, rational forest managers will consider the production of the next rotation period and continue from that time, thus resulting in a multi-rotation operation model (Faustmann model) ([7]). According to Faustmann's rotation period model ([7]), the timber incomes obtained at the end of all rotation periods are the same.

The net present value ([8], [9], [13], [16]) of the discounted timber value is

$$Z_p(T) = [(p-c)\delta X(T)e^{-rT} - R]/(1-e^{-rT}).$$

Similarly, the carbon sequestration benefit is

$$Z_q(T) = \frac{\int_0^T \alpha q X'(t)e^{-rt}\mathrm{d}t - q\alpha(1-\beta)X(T)e^{-rT}}{(1-e^{-rT})}.$$

Thus, the objective function of the present land expectation value with multiple rotation period considering carbon sequestration benefits is ([8], [13]):

$$\begin{aligned} Z(T) &= Z_p(T) + Z_q(T) \\ &= \Big\{[(p-c)\delta - q\alpha(1-\beta)]X(T)e^{-rT} - R \\ &\quad + \int_0^T \alpha q X'(t)e^{-rt}\mathrm{d}t\Big\}/(1-e^{-rT}). \end{aligned} \tag{11}$$

Letting $Z'(T) = 0$ gives

$$\begin{aligned} &\Big\{[(p-c)\delta - q\alpha(1-\beta)][X'(T)e^{-rT} - rX(T)e^{-rT}] \\ &+\alpha q X'(T)e^{-rT}\Big\}(1-e^{-rT}) \\ &= \Big\{[(p-c)\delta - q\alpha(1-\beta)]X(T)e^{-rT} \\ &\quad -R + \int_0^T \alpha q X'(t)e^{-rt}\mathrm{d}t\Big\}re^{-rT}. \end{aligned} \tag{12}$$

Dividing both sides by e^{-rT} leads to

$$\begin{aligned} &\Big\{[(p-c)\delta - q\alpha(1-\beta)][X'(T) - rX(T)] \\ &+\alpha q X'(T)\Big\}(1-e^{-rT}) \\ &= \Big\{[(p-c)\delta - q\alpha(1-\beta)]X(T)e^{-rT} \\ &\quad -R + \int_0^T \alpha q X'(t)e^{-rt}\mathrm{d}t\Big\}r. \end{aligned} \tag{13}$$

Hence,

$$\begin{aligned} &rR + \Big[(p-c)\delta - q\alpha(1-\beta) + \alpha q\Big]X'(T) \\ &-r\Big[(p-c)\delta - q\alpha(1-\beta)\Big]X(T) \\ &-\Big[(p-c)\delta - q\alpha(1-\beta) + \alpha q\Big]X'(T)e^{-rT} \\ &= rq\alpha \int_0^T X'(t)e^{-rt}dt. \end{aligned} \tag{14}$$

As a result,

$$\begin{aligned} &\Big[(p-c)\delta + q\alpha\beta\Big]X'(T) - r[(p-c)\delta - q\alpha(1-\beta)]X(T) \\ &+rR - \Big[(p-c)\delta + q\alpha\beta\Big]X'(T)e^{-rT} \\ &= rq\alpha \int_0^T X'(t)e^{-rt}dt. \end{aligned} \tag{15}$$

It follows that

$$\begin{aligned} &\Big[(p-c)\delta + q\alpha\beta\Big]X''(T) - r[(p-c)\delta - q\alpha(1-\beta)\Big]X'(T) \\ &-\Big[(p-c)\delta + q\alpha\beta\Big]\Big[X''(T)e^{-rT} - rX'(T)e^{-rT}\Big] \\ &= q\alpha rX'(T)e^{-rT}. \end{aligned} \tag{16}$$

We then obtain

$$X'(T)\Big\{e^{-rT}r\Big[(p-c)\delta+q\alpha\beta\Big]-r\Big[(p-c)\delta-q\alpha(1-\beta)\Big] -q\alpha re^{-rT}\Big\}+X''(T)\Big[(p-c)\delta+q\alpha\beta\Big](1-e^{-rT})=0. \quad (17)$$

That is to say

$$X''(T)\Big[(p-c)\delta+q\alpha\beta\Big]=X'(T)r\Big[(p-c)\delta-q\alpha(1-\beta)\Big].$$

In other words,

$$\frac{X''(T)}{X'(T)}=\frac{[(p-c)\delta+q\alpha\beta]}{r[(p-c)\delta-q\alpha(1-\beta)]}. \quad (18)$$

It then follows from (2) that

$$X'(t)=bX(t)\left(1-\left(\frac{X(t)}{k}\right)^{\theta}\right) =bX(t)-bX^{\theta+1}(t)\left(\frac{1}{k}\right)^{\theta}.$$

Consequently,

$$\begin{aligned}X''(t)&=bX'(t)-bX^{\theta}(t)\left(\frac{1}{k}\right)^{\theta}\\&=b\left[bX(t)-bX^{\theta+1}(t)\left(\frac{1}{k}\right)^{\theta}\right]-bX^{\theta}(t)\left(\frac{1}{k}\right)^{\theta}\\&=b^2X(t)-\frac{b^2}{k^{\theta}}X^{\theta+1}(t).\end{aligned}$$

That is to say,

$$\begin{aligned}\frac{X''(t)}{X'(t)}&=\frac{bX'(t)-b(\theta+1)X^{\theta}(t)/k^{\theta}}{X'(t)}\\&=b-b(\theta+1)\left(\frac{X(t)}{k}\right)^{\theta}\frac{1}{bX(t)[1-X^{\theta}(t)/k^{\theta}]}\\&=\frac{(\theta+1)X^{\theta}(t)}{X(t)[k^{\theta}-X^{\theta}(t)]}.\end{aligned} \quad (19)$$

According to (18) and (19), we have

$$\frac{X''(T)}{X'(T)}=\frac{[(p-c)\delta+q\alpha\beta]}{r[(p-c)\delta-q\alpha(1-\beta)]} =\frac{(\theta+1)X^{\theta}(T)}{X(T)[k^{\theta}-X^{\theta}(T)]}. \quad (20)$$

Making us of (2) gives,

$$X(t)=\frac{k}{(1+ae^{-\theta bt})^{\frac{1}{\theta}}}.$$

Thus,

$$\left(\frac{k}{X(T)}\right)^{\theta}=1+ae^{-\theta bT}.$$

Set $u=e^{-rT}$, then, $e^{-T}=u^{\frac{1}{r}}$. As a result,

$$\left(\frac{k}{X(T)}\right)^{\theta}=1+au^{\frac{b\theta}{r}}. \quad (21)$$

Substituting (21) into (20) yields

$$\begin{aligned}&\frac{[(p-c)\delta+q\alpha\beta]}{(\theta+1)r[(p-c)\delta-q\alpha(1-\beta)]}\\&=\frac{X^{\theta}(T)}{X(T)[k^{\theta}-X^{\theta}(T)]}\\&=\frac{1}{X(T)[k^{\theta}/X^{\theta}(T)-1]}\\&=\frac{(au^{\frac{b\theta}{r}}+1)^{1/\theta}}{kau^{\frac{b\theta}{r}}}.\end{aligned}$$

It follows that

$$\frac{k[(p-c)\delta+q\alpha\beta]}{(\theta+1)r[(p-c)\delta-q\alpha(1-\beta)]}=\frac{(au^{\frac{b\theta}{r}}+1)^{1/\theta}}{au^{\frac{b\theta}{r}}}.$$

Notice that $u=e^{-rT}$, hence

$$\frac{k[(p-c)\delta+q\alpha\beta]}{(\theta+1)r[(p-c)\delta-q\alpha(1-\beta)]}=\frac{(ae^{-b\theta T}+1)^{1/\theta}}{ae^{-b\theta T}}, \quad (22)$$

which is a function of multiple rotation period with carbon sequestration benefit q.

When $q=0$, (22) becomes

$$\frac{k(p-c)\delta}{(\theta+1)r(p-c)\delta}=\frac{(ae^{-b\theta T}+1)^{1/\theta}}{ae^{-b\theta T}}, \quad (23)$$

which is a function of multiple rotation period without carbon sequestration benefit q.

According to the data in Section IV, $a=1.828$, $b=0.651$, $\theta=0.25$, $k=407.251$, $p=900\ rmb/m^3$, $c=256\ rmb/m^3$, $p-c=644\ rmb/m^3$. According to [9], [13], [16], we choose $\alpha=0.225$, $\beta=0.05$, $\delta=0.7$, $r=0.08$.

(1). If $q=66\ rmb/t$, that is to say, the carbon sequestration benefits are taken into account, by (22), we have $T_c=13.7731$.

(2). If $q=0\ rmb/t$, that is to say, the carbon sequestration benefits are not taken into account, by (23), we get $T_0=13.6041$.

We can see that, $T_c>T_0$, in other words, the carbon sink benefits can extend the rotation period ([43]-[48]).

Now let us discuss the influences of $p-c,q,r$ on the rotation time T_c (T_0) by numerical simulations.

TABLE IV: Without the impacts of carbon sink

$q=0$	$p-c=644$	r	0.06	0.065	0.07	0.075
		T_0	14.714	14.423	14.1423	13.869

From Table IV, it can be seen that as the increasing of the interest rate r, T_0 decreases. That is to say, when the interest rate increases, the rotation time will move forward.

TABLE V: With the impacts of carbon sink

$q=66$	p-c=644	r	0.06	0.065	0.07	0.075
		T_c	14.86	14.58	14.30	14.03
$q=66$	r=0.08	$p-c$	500	600	700	800
		T_c	14.12	13.86	13.68	13.53
p-c=644	r=0.08	q	55	60	65	70
		T_c	13.75	13.79	13.77	13.78

Table V shows that as the increasing of the interest rate r, T_c decreases. In other words, when the interest rate increases, the rotation period time will move forward. As the increasing of the timber price, T_c decreases. In other words, as the timber price increases, the rotation period will move forward. As the increasing of the carbon sink unit price, T_c increases. That is, when the carbon sink unit price increases, the rotation time will be postponed.

In summary, for multiple rotation period, with the impacts of carbon sink, the rotation period is delayed, and the higher the carbon sink price, the longer the rotation period. Without the impacts of carbon sink, the increasing of interest rate or timber price leads to earlier rotation period. These results are in agreement with the actual situation, where the increasing interest rate or timber price leads to higher holding costs for forest owners, so the forest should be cut ahead of time.

VI. Discussions

A. The relationships between single rotation and multi-rotation period

In this section, we discuss the relationships between single rotation and multi-rotation period. In multiple rotation period, let T_{01} and T_{c1} represent the best cutting time without and with carbon respectively. In single rotation, let T_0 and T_c stand for the best cutting time without and with carbon respectively. Now let us plot T_{01}, T_{c1}, T_0, T_c and the growth time point of the stock volume in the following Figure 7.

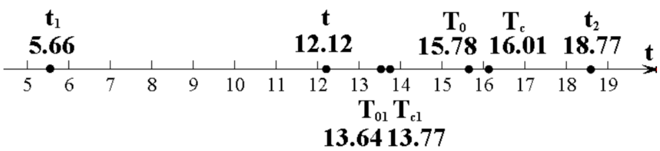

Fig. 7

From Figure 7, it can be seen that

- the carbon sequestration benefit has delayed the cutting time (i.e., $T_{01} < T_{c1}$, $T_0 < T_c$) in both single and multiple rotation cutting period, and the finding is consistent with the conclusion in Section 3.1;
- The optimal rotation period is between the peak of tree stock t and stagflation period t_2;
- No matter whether carbon sequestration benefits are taken into account or not, multi-rotation harvesting is earlier than single rotation harvesting (i.e., $T_{01} < T_{c1} < T_0 < T_c$). This conclusion is consistent with the actual situation.

As a result, using the Gilpin-Ayala equation as a forest rotation model is appropriate.

Now let us see Table VI which contains Table III and Table V. From Table VI, one can find out when the benefits of carbon sequestration are taken into account, if r, q, and $p-c$ are the same, the multi-rotation period is earlier than the single rotation period. Now let us fix two of them (i.e., r, q, $p-c$), and enlarge the remanent one; then we plot the changes of T in single rotation and multi-rotation period under the benefit of carbon sequestration in the following figures.

TABLE VI

$q=66$	p-c=644	r	0.06	0.065	0.07	0.075
		T_{c1}	14.86	14.57	14.3	14
		T_c	17.99	17.45	16.94	16.46
$q=66$	r=0.08	$p-c$	500	600	700	800
		T_{c1}	14.12	13.86	13.68	13.53
		T_c	16.08	16.03	16.00	15.97
p-c=644	r=0.08	q	55	60	65	70
		T_{c1}	13.74	13.76	13.77	13.78
		T_c	15.98	15.99	16.01	16.03

From Fig.8-Fig.10, one can observe that $T_{c1} < T_c$ when $p-c$ or r do not change, and $T_c < T_{c1}$ when q does not change. Regardless of single rotation or multi-rotation period, the unit price of carbon sink has the same effects

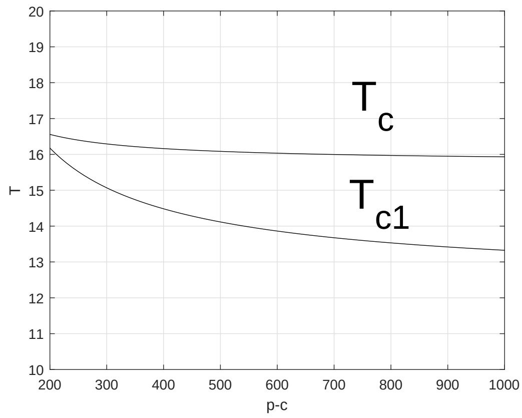

Fig. 8: The relations between $p-c$ and T_c.

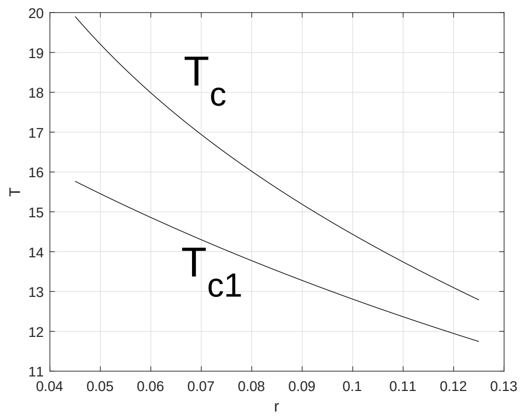

Fig. 9: The relations between r and T_c.

on the rotation period. That is to say, the harvesting day is delayed if the carbon sink price increases, and the harvesting day moves forward if the carbon sink price decreases. The impact of the unit price of timber and the interest rate on the rotation period is reversed; that is, the harvesting day moves forward if the timber price increases, and the harvesting day is delayed if the timber price decreases; the harvesting day moves forward if the interest rate increases, and the harvesting day is delayed if the interest rate decreases.This is consistent with the conclusion of Section 3.2 and [9, 13, 16].

According to the discussions given above, we choose $k=407.25$; $a=1.828$, $b=0.651$, $\theta=0.25$, $\alpha=0.225$, $\beta=0.05$, $\delta=0.7$, $r=0.08$, $p-c=644$ and $R=9000$.

(I) The LEV with a single rotation period is:

$$Y(T) = (p-c)\delta X(T)e^{-rT} - R + \int_0^T \alpha q X'(t)e^{-rt}dt - q\alpha(1-\beta)X(T)e^{-rT}$$

(i) When the carbon sequestration benefits are taken into account, set $q=66$, one can obtain that when $T=16.02$, LEV gets its maximum value 22464.8.

(ii) When the carbon sequestration benefits are taken into account, set $q=0$, one can obtain that when $T=15.78$, LEV gets its maximum value 21741.8.

(II) The LEV with multiple rotation period is:

$$Z(T) = \{[(p-c)\delta - q\alpha(1-\beta)]X(T)e^{-rT} - R + \int_0^T \alpha q X'(t)e^{-rt}dt\}/(1-e^{-rT})$$

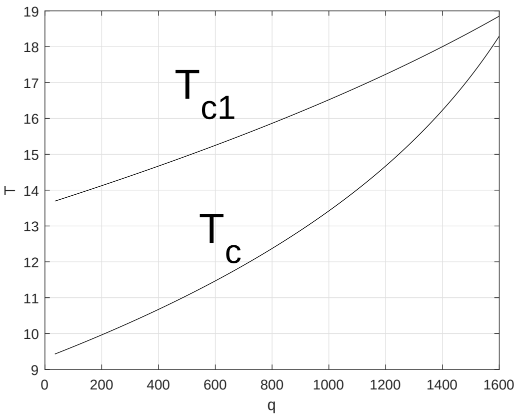

Fig. 10: The relations between q and T_c.

(i') When the carbon sequestration benefits are taken into account, set $q = 66$, one can observe that when $T = 13.77$, LEV gets its maximum value 36725.8.

(ii') When the carbon sequestration benefits are not taken into account, set $q = 0$, one can find out that when $T = 13.60$, LEV gets its maximum value 35980.1.

These results are summarized in the following table.

TABLE VII

	With carbon or not	T	LEV
Single rotation	No	15.78	21741.84
	Yes	16.02	22464.80
Multiple rotation	No	13.64	35980.15
	Yes	13.77	36725.85

From Table VII, we can obtain that for both single and multi-rotation, when the carbon sequestration benefits are taken into account, the optimal harvesting time will be delayed and LEV will increase.

B. Average annual land expectation value

We have discussed the single- and multi-rotation periods that maximize land expectation. Now let us discuss the single- and multi-rotation periods that maximize the annual average land expectation. For the sake of convenience, we introduce several notations.

- T represents the length of the rotation period;
- V_1 stands for timber benefits;
- V_2 stands for carbon sequestration benefits;
- V stands for overall benefits;
- V_1/T stands for average annual benefits from timber;
- V_2/T stands for average annual benefits from carbon sequestration;
- V/T stands for overall average annual benefits.

The following table is the data of the maximized land expectation value under a single rotation.

TABLE VIII

T	V_1	V_2	V	V_1/T	V_2/T	V/T
16.02	21732.35	732.45	22464.8	1356.91	45.73	1402.64

The average annual land expectation value function under a single rotation is:

$$\begin{aligned} Y_1(T) &= Y(T)/T \\ &= [(p-c)\delta X(T)e^{-rT} - R + \textstyle\int_0^T \alpha q X'(t)e^{-rt}dt \\ &\quad -q\alpha(1-\beta)X(T)e^{-rT}]/T. \end{aligned}$$

The following table is the data of the maximized average annual land expectation value under a single rotation.

TABLE IX

T	V_1	V_2	V	V_1/T	V_2/T	V/T
11.93				1579.63	34.07	1613.69

When the average annual land expectancy is considered, single rotation periods are shortened, the average annual benefit of timber as well as the total average annual benefit increases, the average annual carbon sequestration benefit decreases, and the forest circulates faster. The data related to multiple rotation periods with maximized land expectancy are as follows.

TABLE X

T	V_1	V_2	V	V_1/T	V_2/T	V/T
13.77	20982.33	550.82	32247.62	1523.43	39.99	2341.34

The average annual land expectation function under multiple rotation periods is

$$\begin{aligned} Z_1(T) &= Z(T)/T \\ &= \left\{ \left[(p-c)\delta - q\alpha(1-\beta) \right] X(T)e^{-rT} - R \right. \\ &\quad \left. + \int_0^T \alpha q X'(t)e^{-rt}dt \right\} / \left[(1-e^{-rT})T \right] \end{aligned}$$

The data for the multiple rotation period when the annual average land expectation is maximized are as follows.

TABLE XI

T	V_1	V_2	V	V_1/T	V_2/T	V/T
9.28				2777.7	45.3	2823

When the average annual land expectancy is considered, multiple rotation periods are shortened, the average annual benefit of timber as well as the total average annual benefit increases, the average annual carbon sequestration benefit decreases, and the forest circulates faster.

VII. CONCLUSIONS

In this paper, we have analyzed the characteristics of the Gilpin-Ayala model and given the explicit expression of the solution. By fitting the real data of forest stocks, we have developed the Gilpin-Ayala accumulation model. Furthermore, the objective function of the expected land value has been derived to obtain an explicit expression of the optimal solution for a single rotation period. According to the results and monotonicity of the expression, we have analyzed the effects of carbon sink, timber price, and interest rate on the optimal rotation period. Regardless of single rotation or multi-rotation period, the unit price of carbon sink has the same effect on the rotation period; that is, the harvesting day will be delayed if the carbon sink price increases, and the harvesting day will move forward if the carbon sink price decreases. The impact of the unit price of timber and the interest rate on the rotation period is reversed; that is, the harvesting day will move forward if the timber price increases, and the harvesting day will be delayed if the timber price decreases. We have also shown that for both single- and multi-rotation, the optimal harvesting time is always delayed when carbon sink benefits are taken into account; and multi-rotation is always earlier than single-rotation.

Based on the Gilpin-Ayala model, we can see that harvesting at the optimal rotation period with carbon sequestration benefits is conducive for coordinating social and ecological benefits. This paper has helped reveal the ecological benefits brought by forest carbon sequestration. The rotation period with carbon sequestration benefits is delayed compared with the rotation period without carbon sequestration benefits, and the cutting time is delayed. Following the suggestion of the paper not only obtains the greatest economic benefits but also prolongs the time for forests ([49]) to absorb fixed carbon dioxide and produce oxygen, benefiting both forest managers and society, and providing theoretical support for the sustainable development of forests. As an emerging economy, China emits more than 10 billion tons of carbon dioxide into the atmosphere every year ([1]), ranking first in the world. Thus, it is particularly important to seek the best balance between carbon emissions and economic growth. This paper has provided the basic argument for the Chinese government to formulate a reasonable ecological compensation mechanism.

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