# Vertex-Distinguishing E-Total Colorings of Complete Bipartite Graphs with One Part Having Five Vertices

Xiang'en CHEN, Shiling LI

***Abstract*—Suppose $G$ is a simple graph. If $f$ is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$ such that $f(e), f(u), f(v)$ are distinct for each edge $e = uv$ of $G$, then $f$ is called an E-total coloring of $G$ using $k$ colors ($k$-E-total coloring of $G$, in brief). For an E-total coloring $f$ of a graph $G$ and any vertex $x$ of $G$, we denote the set**

$$\{f(x)\} \cup \{f(e)|e \in E(G) \text{ and } e \text{ is incident with } x\}$$

**by $C(x)$ and refer to it as the color set of $x$ under $f$. If $C(u) \neq C(v)$ for any two different vertices $u$ and $v$ of $V(G)$, then we say that $f$ is a vertex-distinguishing E-total coloring of $G$ or a VDET coloring of $G$ for short. Let**

$$\chi^e_{vt}(G) = \{k|G \text{ has a VDET coloring using } k \text{ colors }\}.$$

**Then the positive integer $\chi^e_{vt}(G)$ is called the VDET chromatic number of $G$. The VDET coloring of complete bipartite graph $K_{5,n}$ is discussed in this paper and the VDET chromatic number of $K_{5,n}$ has been obtained.**

***Index Terms*—graph; complete bipartite graphs; E-total coloring; vertex-distinguishing E-total coloring; vertex-distinguishing E-total chromatic number**

## I. Introduction and Preliminaries

COLORING problem in graph theory research has important theoretical significance and applications. In this paper we will discuss a kind of coloring: vertex-distinguishing E-total coloring of graphs. All graphs considered in this paper are simple, finite and undirected.

For a total coloring (proper or not) $f$ of $G$ and a vertex $x$ of $G$, let

$$\{f(x)\} \cup \{f(e)|e \in E(G) \text{ and } e \text{ is incident with } x\}.$$

Note that $C(x)$ is not a multiset. We refer to $C(x)$ as the color set of $x$ under $f$.

For a proper total coloring, if $C(u) \neq C(v)$ for any two distinct vertices $u$ and $v$, then the coloring is called a vertex-distinguishing (proper) total coloring, or a VDT coloring of $G$ for short.

$$\chi_{vt}(G) = \{k|G \text{ has a VDT coloring using } k \text{ colors }\}.$$

Then the positive integer $\chi_{vt}(G)$ is called the VDT chromatic number of $G$. The vertex distinguishing (proper) total colorings of graphs are introduced and studied in [8]. The VDT chromatic number of complete graph, star, complete bipartite graph, wheel, fan, path and cycle are determined in [8] and a conjecture was proposed in [8]: $\chi_{vt}(G) = \mu(G)$ or $\mu(G)+1$, where $\mu(G)$ denote the minimum positive integer $k$ such that $\binom{k}{i+1}$ is not less than $n_i$ $(\delta(G) \leq i \leq \Delta(G))$. We denote the number of vertices of degree $i$ in $G$ by $n_i(G)$ or $n_i$ simply. In [2], the vertex-distinguishing total coloring of $n$-cube were discussed. In [3], the relations of VDT chromatic numbers between a subgraph and its supergraph had been studied. When $p$ is even, $p \geq 4$ and $q \geq 3$, the VDT chromatic numbers of complete $p$-partite graphs with each part of cardinality $q$ had been obtained in [7].

If $f$ is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$ such that $f(e), f(u), f(v)$ are distinct for each edge $e = uv$ of $G$, then $f$ is called an E-total coloring of $G$ using $k$ colors ($k$-E-total coloring of $G$, in brief). For an E-total coloring $f$ of a graph $G$ and any vertex $x$ of $G$, we denote the set

$$\{f(x)\} \cup \{f(e)|e \in E(G) \text{ and } e \text{ is incident with } x\}$$

by $C(x)$ and refer to it as the color set of $x$ under $f$. If $C(u) \neq C(v)$ for any two different vertices $u$ and $v$ of $V(G)$, then we say that $f$ is a vertex-distinguishing E-total coloring of $G$ or a VDET coloring of $G$ for short. Let

$$\chi^e_{vt}(G) = \{k|G \text{ has a VDET coloring using } k \text{ colors }\}.$$

Then the positive integer $\chi^e_{vt}(G)$ is called the VDET chromatic number of $G$.

The VDET colorings of complete graph, complete bipartite graph $K_{2,n}$, star, wheel, fan, path and cycle were discussed in [5]. The VDET chromatic numbers of $mC_3$ and $mC_4$ are obtained in [6]. The VDET coloring of complete bipartite graph $K_{5,n}$ is discussed in this paper and the VDET chromatic number of $K_{5,n}$ has been obtained.

A parameter was introduced in [5]: $\eta(G) = \min\{l : \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{i+1} \geq n_\delta + n_{\delta+1} + \cdots + n_i, 1 \leq \delta \leq i \leq \Delta\}$, $n_i$ denote the number of vertices with degree $i, \delta \leq i \leq \Delta$. At the end of the paper [5], a conjecture was proposed.

**Conjecture 1** ([5]) For a graph $G$ with no isolated vertices and chromatic number at most 5, we have $\chi^e_{vt}(G) = \eta(G)$ or $\eta(G) + 1$.

In this paper, we will consider the VDET coloring of complete bipartite graph $K_{5,n}$ and confirm Conjecture 1 for $K_{5,n}$.

For not necessarily proper total colorings which are adjacent vertex distinguishing, we can see [4]. For other notations and terminologies we can refer to [1].

Let $X = \{u_1, u_2, u_3, u_4, u_5\}$, $Y = \{v_1, v_2, v_3, \cdots, v_n\}$, $V(K_{5,n}) = X \cup Y$ and $E(K_{5,n}) = \{u_iv_j : 1 \leq i \leq 5, 1 \leq j \leq n\}$.

Manuscript received February 20, 2022; revised August 24, 2022.

This work was supported by the National Natural Science Foundation of China (11761064).

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**Global description of the main results:** Let $M_5$, $M_6$, $M_7$, $M_8$ , $M_9$ and $M_k$ denote the integer intervals $[5,11]$, $[12,39]$, $[40,100]$, $[101,220]$ , $[221,437]$ and $[\binom{k-1}{2}+\binom{k-1}{3}+\binom{k-1}{4}+\binom{k-1}{5}+\binom{k-1}{6}-2(k-1), \binom{k}{2}+\binom{k}{3}+\binom{k}{4}+\binom{k}{5}+\binom{k}{6}-2k-1]$, where $k \geq 10$. We will prove the result "If $n \in M_s$ with $s \geq 5$, then $\chi^e_{vt}(K_{5,n}) = s$" by giving six theorems in Section 2.

## II. Basic ideas of the proofs of the theorems in Section III

In order to prove that the VDET chromatic number of a graph $K_{5,n}$ is $l$ ( for $l \in \{5,6,7,8,9,k\}$ ) in each theorems, we have done two jobs as follows:

1. We need to prove that $K_{5,n}$ doesn't have $(l-1)$-VDET coloring by contradiction. Assume that $K_{5,n}$ has a $(l-1)$-VDET coloring using colors $1,2,\cdots,l-1$, we will consider five cases when the number of different colors of five vertices in $X$ is $1,2,3,4$ and $5$ successively. Then we can find the subsets of $\{1,2,\cdots,l-1\}$ which may become the color sets of vertices in $Y$. According to the definition of VDET coloring and the colors of vertices in $X$, we only need to consider some special subsets and finally we can obtain contradictions.

2. We can prove that $K_{5,n}$ has an $l$-VDET coloring. In the 2-subsets, 3-subsets, $\cdots$, $(l-1)$-subsets and $l$-subsets of $\{1,2,\cdots,l\}$, we may select $n+5$ subsets appropriately, and let these $n+5$ subsets correspond to the vertices in $X \cup Y$, such that the different vertices corresponded to different subsets. Then we will find an E-total coloring $f$ of $K_{5,n}$, under this E-total coloring, the color set of every vertex is the subset corresponded to this vertex in advance. So we can obtain that the coloring $f$ is vertex distinguishing. Namely, $f$ is an $l$-VDET coloring of $K_{5,n}$.

Suppose $p_s$ is the maximum number in $M_s$, i.e., $p_s = \max M_s$, $s \geq 5$.

In order to construct required coloring, we can give a 5-VDET coloring $f_{11}$ of $K_{5,11}$ firstly. Then based on the $(s-1)$-VDET coloring $f_{p_{s-1}}$ of $K_{5,p_{s-1}}$ for every $s \in \{6,7,\cdots\}$, we increase a new color $s$, and give required coloring of $p_s - p_{s-1}$ new degree five vertices and their incident edges. So we can obtain an $s$-VDET coloring $f_{p_s}$ of $K_{5,p_s}$.

When we have constructed an $s$-VDET coloring $f_{p_s}$ of $K_{5,p_s}$ for each $s \geq 5$, we delete some vertices in $Y$ and their incident edges gradually, then we can obtain an $s$-VDET coloring of $K_{5,n}$ when $n \in M_s \setminus \{p_s\}$.

This process should be carried out recursively.

## III. Main Results

**Theorem 1**. If $5 \leq n \leq 11$, then $\chi^e_{vt}(K_{5,n}) = 5$.

**Proof**. We only need to prove that $K_{5,n}$ has no 4-VDET coloring, in the same time, we will give a 5-VDET coloring of $K_{5,n}$.

Assume that $K_{5,n}$ has a 4-VDET coloring $f$ using colors 1, 2, 3 and 4. There are three cases to consider.

**Case 1** $u_1,u_2,u_3,u_4$ and $u_5$ receive the same color under $f$. We may suppose that $f(u_i) = 1, i = 1,2,3,4,5$. So none of the $C(v_j)$ include color 1 and each $C(v_j)$ is one of $\{2,3\},\{2,4\},\{3,4\},\{2,3,4\}$. When $5 \leq n \leq 11$, we can obtain a contradiction, since four subsets can not distinguish $n$ vertices in $Y$.

**Case 2** $u_1,u_2,u_3,u_4$ and $u_5$ receive two different colors, say 1 and 2, under $f$. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1,2$. So each $C(v_j)$ is one of $\{3,4\},\{1,2,3\},\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$. We can obtain a contradiction since 6 subsets can not distinguish $n(7 \leq n \leq 11)$ vertices in $Y$.

When $n = 5,6$. From one subset in $\{\{1,2,3\},\{1,2,4\}\}$, say $\{1,2,3\}$, must be the color set of some vertex in $Y$, we can obtain that each $C(u_i)$ contains $\{1,2\}$, and when $n = 5$, each $C(u_i)$ is either $\{1,2\}$ or $\{1,2,4\}$. This is a contradiction. When $n = 6$, each $C(u_i)$ is equal to $\{1,2\}$. This is also a contradiction.

**Case 3** $u_1,u_2,u_3,u_4$ and $u_5$ receive three different colors, say 1, 2 and 3, under $f$. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1,2,3$, and each $C(v_j)$ is not $\{1,2,3\}$. So the color set of each $v_j$ is one of $\{1,2,4\},\{1,3,4\},\{2,3,4\}$ and $\{1,2,3,4\}$. This is a contradiction since 4 subsets can not distinguish $n(5 \leq n \leq 11)$ vertices in $Y$.

Hence $K_{5,n}$ does not have a 4-VDET coloring and $\chi^e_{vt}(K_{5,n}) \geq 5$ when $5 \leq n \leq 11$. We will give a 5-VDET coloring $f_n$ of $K_{5,n}$.

Let $f_n = (a_1b_1c_1d_1e_1g_1, a_2b_2c_2d_2e_2g_2, \cdots,$
$a_nb_nc_nd_ne_ng_n, 2b_1b_2\cdots b_n, 1c_1c_2\cdots c_n, 1d_1d_2\cdots d_n,$
$2e_1e_2\cdots e_n, 2g_1g_2\cdots g_n)$, where "$a_jb_jc_jd_je_jg_j$" (composed with six ordered colors $a_j, b_j, c_j, d_j, e_j$ and $g_j$) represents the colors of the vertex $v_j$ and its incident edges: the color of $v_j$ is $a_j$, the colors of $u_1v_j, u_2v_j, u_3v_j, u_4v_j, u_5v_j$ are $b_j, c_j, d_j, e_j, g_j$, respectively. And the colors of $u_1,u_2,u_3,u_4,u_5$ are 2, 1, 1, 2, 2 respectively.

Next we will give a 5-VDET coloring $f_n$ of $K_{5,n}$.

$f_{11} = (344444, 533333, 533331, 533233, 435355, 544444,$
$544441, 544244, 533231, 544241, 344441, 243333444344,$
$143335444344, 143323442224, 243335444344, 243135414111)$

Based on $K_{5,11}$ and its coloring $f_{11}$, if we delete the vertex whose color set is $\{1,3,4\}$, then we obtain $K_{5,10}$ and its 5-VDET coloring $f_{10}$. Based on $K_{5,10}$ and its coloring $f_{10}$, if we delete $i$ vertices, where $i = 1,2,3,4,5$, whose color sets are in $\{\{1,2,4,5\},\{1,2,3,5\},\{2,4,5\},\{1,4,5\},\{4,5\}\}$, then we obtain $K_{5,10-i}$ and its 5-VDET coloring $f_{10-i}$.

This completes the proof of Theorem 1.

**Theorem 2**. If $12 \leq n \leq 39$, then $\chi^e_{vt}(K_{5,n}) = 6$.

**Proof**. Assume that $K_{5,n}$ has a 5-VDET coloring $f$. There are four cases we need to consider.

**Case 1** $u_1,u_2,u_3,u_4$ and $u_5$ receive the same color under $f$. We may suppose that $f(u_i) = 1, i = 1,2,3,4,5$, so none of the $C(v_j)$ include color 1 and each $C(u_i)$ is in $A = \{\{2,3\},\{2,4\},\{2,5\},\{3,4\}, \{3,5\}, \{4,5\},\{2,3,4\},\{2,3,5\}, \{2,4,5\},\{3,4,5\}, \{2,3,4,5\}\}$. We can obtain a contradiction since 11 subsets can not distinguish $n$ vertices in $Y$ when $12 \leq n \leq 39$.

**Case 2** $u_1,u_2,u_3,u_4$ and $u_5$ receive two different colors under $f$. We may assume that $f(u_i) \in \{1,2\}, i = 1,2,3,4,5$. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1,2$. So the number of the subsets of $\{1,2,3,4,5\}$ which may become the color sets of the vertices in $Y$ is $\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}-7 = 19$. A contradiction may arise since 19 subsets can not distinguish $n$ vertices in $Y$ when $20 \leq n \leq 39$.

Let $B_1 = \{\{3,4\},\{3,5\},\{4,5\}\}$,
$B_2 = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$,

$B_3 = \{\{1,3,4\},\{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}$.

♢ If one set in $B_1$ and one set in $B_2$, say $\{3,4\}$ and $\{1,2,3\}$, are the color sets of vertices in $Y$, we may obtain that $\{1,2,3\} \subseteq C(u_i), i = 1,2,3,4,5$ or $\{1,2,4\} \subseteq C(u_i), i = 1,2,3,4,5$, without loss of generality, we may assume that $\{1,2,3\} \subseteq C(u_i), i = 1,2,3,4,5$. Then each $C(u_i)$ is one of $\{1,2,3\},\{1,2,3,4\},\{1,2,3,5\}$ and $\{1,2,3,4,5\}$. This is a contradiction. So we need to consider the following subcases.

**2.1** $n = 16$.

♢ If $\{3,4\},\{3,5\}$ and $\{4,5\}$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is equal to $\{1,2\}$. This is a contradiction.

♢ If $\{1,2,3\},\{1,2,4\}$ and $\{1,2,5\}$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5\}$, say 3 and 4. So each $C(u_i)$ is either $\{1,3,4\}$ or $\{2,3,4\}$. This is a contradiction.

**2.2** $n = 15$. There exist four subsets in $B_1 \cup B_2 \cup B_3$ which are not the color sets of vertices in $Y$.

♢ $\{3,4\},\{3,5\}$ and $\{4,5\}$ are not the color sets of vertices in $Y$. From one set in $B_2$ is a color set of some vertex in $Y$, we can know that each $C(u_i)$ contains $\{1,2\}$. So there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ $\{1,2,3\},\{1,2,4\}$ and $\{1,2,5\}$ are not the color sets of vertices in $Y$. From one set in $B_1$ is a color set of some vertex in $Y$, we can know that each $C(u_i)$ contains one common color. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

**2.3** $n = 14$. There exist five subsets in $B_1 \cup B_2 \cup B_3$ which are not the color sets of vertices in $Y$.

♢ Two subsets in $B_1$ and three subsets in $B_2$ are not the color sets of vertices in $Y$. From one set in $B_1$ is a color set of some vertex in $Y$, we can know that each $C(u_i)$ contains one common color. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ Two subsets in $B_2$ and three subsets in $B_1$ are not the color sets of vertices in $Y$. From one set in $B_2$ is a color set of some vertex in $Y$, we can know that each $C(u_i)$ contains $\{1,2\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If $\{3,4\},\{3,5\},\{4,5\}$ and two subsets in $B_3$ are not the color sets of vertices in $Y$, then from $\{1,2,3\}$ is a color set of some vertex in $Y$, we can know that $\{1,2\} \subseteq C(u_i), i = 1,2,3,4,5$, and there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ $\{\{1,2,i\}|i = 3,4,5\}$ and two subsets in $B_3$ are not the sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

**2.4** $n = 13$. There exist six subsets in $B_1 \cup B_2 \cup B_3$ which are not the color sets of vertices in $Y$.

♢ If one subset in $B_1$ , three subsets in $B_2$ and two subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color, say 3. So some sets $C(u_i)$ contain $\{1,3\}$, and others contain $\{2,3\}$. If two subsets, say $\{1,4,5\}$ and $\{2,4,5\}$, are not the color sets of vertices in $Y$, then each $C(u_i)$ is one of $\{1,3\},\{2,3\}$ and $\{1,2,3\}$. This is a contradiction since three subsets are not distinguish 5 vertices in $X$; If two subsets, which contain $\{1,3\}$ or $\{2,3\}$, are not the color sets of vertices in $Y$, then from $\{1,4,5\}$ and $\{2,4,5\}$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ is one of $\{1,3,5\},\{2,3,5\}$ and $\{1,2,3,5\}$ or each $C(u_i)$ is one of $\{1,3,4\},\{2,3,4\}$ and $\{1,2,3,4\}$. This is a contradiction.

♢ There exist exactly one subset in $B_3$ is not a color set of vertex in $Y$, we can know that only one subset in $B_1 \cup B_2$ is a color set of vertex in $Y$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If one subset in $B_2$, three subsets in $B_1$ and two subsets in $B_3$ are not the color sets of vertices in $Y$, then $\{1,2\} \subseteq C(u_i), i = 1,2,3,4,5$, and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ All subsets in $B_1 \cup B_2$ are not the color sets of vertices in $Y$. From $\{1,4,5\}$ and $\{2,4,5\}$ are the color sets of vertices in $Y$, we can know that there exist at least two sets $C(u_i)$ contain $\{1,2\}$, and others contain $\{1,5\},\{1,4\},\{2,5\}$ or $\{2,4\}$. Because not all vertices in $Y$ contain color 4 or 5, so each $u_i$ is not a 2-subset. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is either $\{1,2,4\}$ or $\{1,2,5\}$. This is a contradiction.

♢ If three subsets in $B_1$ and three subsets in $B_3$ are not the color sets of vertices in $Y$, then $\{1,2\} \subseteq C(u_i), i = 1,2,3,4,5$, and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If three subsets in $B_2$ and three subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors, say 3 and 4. So each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$, and there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

**2.5** $n = 12$. There exist seven subsets in $B_1 \cup B_2 \cup B_3$ which are not the color sets of vertices in $Y$.

♢ If one subset in $B_1$, say $\{4,5\}$, three subsets in $B_2$ and three subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color, say 3. So each $C(u_i)$ contains $\{1,3\}$ or $\{2,3\}$. If $\{1,4,5\}$ or $\{2,4,5\}$ is not the color set of vertex in $Y$, then we can obtain a contradiction easily since there exist at most four subsets which may become the color sets of vertices in $X$. So $\{1,4,5\}$ and $\{2,4,5\}$ are the color sets of vertices in $Y$, and we may suppose that each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$. So there exist at most three subsets which may become the color sets of vertices in $X$. This is also a contradiction.

♢ If three subsets in $B_1$, one subset in $B_2$ and three subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$ and each $C(u_i)$ is not $\{1,2\}$ since there exist 5 subsets in $B_3$ do not contain color 1, and 5 subsets in $B_3$ do not contain color 2. So there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ One subset in $B_3$, say $\{1,4,5\}$, and all subsets in $B_1 \cup B_2$ are not the color sets of vertices in $Y$. From $\{1,3,5\}$ and $\{2,3,5\}$ are the color sets of vertices in $Y$, we can know that there exist at least two sets $C(u_i)$ contain $\{1,2\}$, and all sets contain color 3 or 5. By using the same method, each $C(u_i)$ is not a 2-subset. So there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If two subsets in $B_1$, say $\{3,5\}$ and $\{4,5\}$, three subsets in $B_2$ and two subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains one common color, say 3. Since there exist three subsets in $B_3$ which not contain color 3, we can obtain that each $C(u_i)$ is not a 2-subset. So $|C(u_i) \cup C(v_j)| = |B_3 \cup \{\{1,2,3\},\{3,5\},\{4,5\}\}| = 16$. This is a contradiction since 16 subsets can not distinguish $5+n=17$ vertices in $X \cup Y$.

$\diamondsuit$ If three subsets in $B_1$, two subsets in $B_2$, say $\{1,2,3\}$ and $\{1,2,4\}$, and two subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$. Since 5 subsets in $B_3$ do not contain color 2, and 5 subsets in $B_3$ do not contain color 1, we can know that each $C(u_i)$ is not a 2-subset. So $|C(u_i)\cup C(v_j)| = |B_3 \cup \{\{1,2,3\},\{1,2,4\}\}| = 15$. This is a contradiction since 15 subsets can not distinguish $5+n=17$ vertices in $X \cup Y$.

$\diamondsuit$ If three subsets in $B_1$ and four subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$. From the above discussion, each $C(u_i)$ is not a 2-subset and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If three subsets in $B_2$ and four subsets in $B_3$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors, say 3 and 4. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

**Case 3** $u_1, u_2, u_3, u_4$ and $u_5$ receive three different colors under $f$, say 1, 2, and 3. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1,2,3$, and each $C(v_j)$ is not $\{1,2,3\}$. So the number of the subsets of $\{1,2,3,4,5\}$ which may become the color sets of the vertices in $Y$ is $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} - 10 = 16$. A contradiction may arise since 19 subsets can not distinguish $n$ vertices in $Y$ when $17 \le n \le 39$.

Let $C_1 = \{\{4,5\}\}$, $C_2 = \{\{1,2,4\},\{1,2,5\}, \{1,3,4\}, \{1,3,5\},\{2,3,4\},\{2,3,5\}\}$, $C_3 = \{\{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}$.

$\diamondsuit$ If one subset in $\{\{1,2,4\},\{1,2,5\}\}$, one subset in $\{\{1,3,4\},\{1,3,5\}\}$ and one subset in $\{\{2,3,4\}, \{2,3,5\}\}$ are the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2,3\}$. So each $C(u_i)$ is one of $\{1,2,3\}$, $\{1,2,3,4\}$, $\{1,2,3,5\}$ and $\{1,2,3,4,5\}$. This is a contradiction since 4 subsets can not distinguish 5 vertices in $X$.

**3.1** $n = 14$**.** There exist two subsets in $C_1 \cup C_2 \cup C_3$ which are not the color sets of vertices in $Y$.

$\diamondsuit$ If $\{4,5\}$ and one subset in $C_2 \cup C_3$ (or one subset in $C_3$) are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph. If two sets in $C_2$ are not the color sets of vertices in $Y$, say $\{1,2,4\}$ and $\{1,2,5\}$, then from $\{4,5\}$ is a color set of vertex in $Y$, we can obtain that each $C(u_i)$ contains $\{3,5\}$ or each $C(u_i)$ contains $\{3,4\}$, we may suppose that each $C(u_i)$ contains $\{3,4\}$. So the color sets of $\{u_1,u_2,u_3,u_4,u_5\}$ contain $\{3,4\}$ or $\{1,3,4\}$ or $\{2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can know that each $C(u_i)$ is $\{3,4\}$ or $\{1,3,4\}$ or $\{2,3,4\}$. This is also a contradiction.

**3.2** $n = 13$**.** There exist three subsets in $C_1 \cup C_2 \cup C_3$ which are not the color sets of vertices in $Y$.

$\diamondsuit$ Three subsets in $C_2$, say $\{1,2,4\},\{1,2,5\}$ and $\{1,3,4\}$, are not the color sets of vertices in $Y$. Since $\{4,5\},\{1,3,5\}$ and $\{2,3,4\}$ are the color sets of vertices in $Y$, and we may assume that some $v_j$ has color 5, we can know that the color sets of $\{u_1,u_2,u_3,u_4,u_5\}$ contain $\{1,3,4\}$ or $\{2,3,4\}$ or $\{1,2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain a contradiction.

$\diamondsuit$ $\{4,5\}$ and two subsets in $C_2$, say $\{1,2,4\}$ and $\{1,2,5\}$, are not the color sets of vertices in $Y$. From the above discussion, we can obtain that each $C(u_i)$ contains color 3. In order to distinguish each $u_i$ with vertices in $Y$, we can know that each $C(u_i)$ is one of $\{1,3\},\{2,3\}$ and $\{1,2,3\}$. This is a contradiction.

$\diamondsuit$ One set in $C_3$ and two subsets in $C_2$, say $\{1,2,4\}$ and $\{1,2,5\}$, are not the color sets of vertices in $Y$. From $\{4,5\},\{1,3,4\}$ and $\{2,3,4\}$ are the color sets of vertices in $Y$, and we may suppose that some $v_j$ has color 5. We can know that each $C(u_i)$ contains $\{3,4\}$, from the above discussion, we can also obtain a contradiction.

**3.3** $n = 12$**.** There exist four subsets in $C_1 \cup C_2 \cup C_3$ which are not the color sets of vertices in $Y$.

$\diamondsuit$ Four subsets in $C_2$, say $\{1,2,4\},\{1,2,5\},\{1,3,4\}$ and $\{1,3,5\}$, are not the color sets of vertices in $Y$. From $\{4,5\}$ and $\{2,3,4\}$ are the color sets of vertices in $Y$, and we may suppose that some $v_j$ has color 5. We can know that at least two sets $C(u_i)$ contain $\{2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can know that there exist at least two sets $C(u_i)$ which are equal to the color set of some vertex $v_j$. This is a contradiction.

$\diamondsuit$ $\{4,5\}$ and three subsets in $C_2$, say $\{1,2,4\},\{1,2,5\}$ and $\{1,3,4\}$, are not the color sets of vertices in $Y$. From $\{1,3,5\}$ and $\{2,3,4\}$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains $\{1,3\}$ or $\{2,3\}$ or $\{1,2,3\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is one of $\{1,3\},\{2,3\},\{1,2,3\}$ and $\{1,3,4\}$. This is a contradiction.

$\diamondsuit$ One subset in $C_3$ and three subsets in $C_2$, say $\{1,2,4\},\{1,2,5\}$ and $\{1,3,4\}$, are not the color sets of vertices in $Y$. From the above discussion, we can obtain that there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ Two subsets in $C_2$, say $\{1,2,4\}$ and $\{1,2,5\}$, and two subsets in $C_3$ are not the color sets of vertices in $Y$. From $\{4,5\},\{1,3,4\}$ and $\{2,3,4\}$ are the color sets of vertices in $Y$, and we may suppose that some $v_j$ has color 5, we can know that each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$ or $\{1,2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ $\{4,5\}$, one subset in $C_3$ and two subsets in $C_2$, say $\{1,2,4\}$ and $\{1,2,5\}$ are not the color sets of vertices in $Y$. By using the same method, we can obtain that there exist at most three subsets which may become the color sets of

TABLE I
COLORING METHODS

| the color set of $v_j$ | the colors of $v_j$ and $u_iv_j(i = 1, 2, 3, 4, 5)$ |
|---|---|
| $\{a, k\}$ | $a; k, k, k, k, k$ |
| $\{1, 2, k\}$ | $k; 1, 2, 2, 1, 1$ |
| $\{a, b, k\}$ | $a; k, b, k, k, k$ |
| $\{1, b, k\}$ | $b; k, k, k, 1, 1$ |
| $\{2, b, k\}$ | $b; k, 2, k, k, k$ |
| $\{a, b, c, k\}$ | $a; k, b, c, k, k$ |
| $\{1, b, c, k\}$ | $b; k, c, k, 1, 1$ |
| $\{1, 2, c, k\}$ | $c; 1, 2, 2, 1, k$ |
| $\{2, b, c, k\}$ | $b; c, 2, k, k, k$ |
| $\{a, b, c, d, k\}$ | $a; k, b, c, d, k$ |
| $\{1, b, c, d, k\}$ | $b; k, c, d, 1, k$ |
| $\{2, b, c, d, k\}$ | $b; c, 2, d, k, k$ |
| $\{1, 2, c, d, k\}$ | $c; 1, 2, d, 1, k$ |
| $\{a, b, c, d, e, k\}$ | $a; k, b, c, d, e$ |
| $\{1, b, c, d, e, k\}$ | $b; k, c, d, 1, e$ |
| $\{2, b, c, d, e, k\}$ | $b; c, 2, d, e, k$ |
| $\{1, 2, c, d, e, k\}$ | $c; 1, d, 2, e, k$ |

vertices in $X$. This is a contradiction.

**Case 4** $u_1, u_2, u_3, u_4$ and $u_5$ receive four different colors under $f$, say 1, 2, 3 and 4. Then the color set $C(v_j)$ is not a 2-subset, and each $C(v_j)$ is not $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ or $\{1, 2, 3, 4\}$. So the number of the subsets in $\{1, 2, 3, 4, 5\}$ which may become the color sets of the vertices in $Y$ is $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}-5 = 11$. A contradiction may arise since 11 subsets can not distinguish $n$ vertices in $Y$ when $12 \le n \le 39$.

Hence $K_{5,n}$ does not have a 5-VDET coloring and $\chi^e_{vt}(K_{5,n}) \ge 6$ when $12 \le n \le 39$.

Based on $K_{5,11}$ and its coloring $f_{11}$, we can give a 6-VDET coloring $f_n$ of $K_{5,n}$ $(12 \le n \le 39)$. In order to distinguish each $u_i$ with vertices in $Y$, subsets $\{2, 3, 4, 6\}$, $\{1, 3, 4, 5, 6\}, \{1, 2, 3, 4, 6\}, \{2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$ are not the color sets of any vertices in $Y$. So $f_n = f_{11}+(366666, 344244, 345451, 344241, 345245, 345241, \cdots)$. We can by coloring other vertices $v_j$ and its incident edges $(18 \le j \le 39)$ according to the method given in Table I (the second column in Table I shows that the colors of $v_j$; $u_1v_j, u_2v_j, u_3v_j, u_4v_j, u_5v_j$), in the same time, the colors of $u_1, u_2, u_3, u_4$ and $u_5$ are 2, 1, 1, 2 and 2 respectively. Finally we can obtain the 6-VDET coloring $f_n$ $(12 \le n \le 39)$ of $K_{5,n}$.

The proof of Theorem 2 is completed.

**Theorem 3**. If $40 \le n \le 100$, then $\chi^e_{vt}(K_{5,n}) = 7$.

**Proof**. Assume that $K_{5,n}$ has a 6-VDET coloring $f$. There are five cases we need to consider.

**Case 1** $u_1, u_2, u_3, u_4$ and $u_5$ receive the same color under $f$. We may suppose that $f(u_i) = 1, i = 1, 2, 3, 4, 5$, so none of the $C(v_j)$ include color 1 and the number of the subsets in $\{1, 2, 3, 4, 5, 6\}$ which may become the color sets of the vertices in $Y$ is $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 26$. A contradiction may arise since 26 subsets can not distinguish $n$ vertices in $Y$ when $40 \le n \le 100$.

**Case 2** $u_1, u_2, u_3, u_4$ and $u_5$ receive two different colors under $f$. We may assume that $f(u_i) \in \{1, 2\}, i = 1, 2, 3, 4, 5$. Then the color sets $C(v_j)$ do not include color $i$ when $|C(v_j)| = 2, i = 1, 2$. So the number of the subsets of $\{1, 2, 3, 4, 5, 6\}$ which may become the color sets of the vertices in $Y$ is $\binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} - 9 = 48$. A contradiction may arise since 48 subsets can not distinguish $n$ vertices in $Y$ when $49 \le n \le 100$.

If four subsets in $\{\{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}$ and one subset in $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}\}$ are the color sets of vertices in $Y$, then each $C(u_i)$ contains at least four common colors, say 1, 2, 3 and 4. So each $C(u_i)$ is one of $\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$. This is a contradiction. So we only need to consider the following subcases.

Let $A = \{\{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}\}$. We denoted the set, which contains all 48 subsets except $A$, as $S$.

**2.1** $n = 45$.

If three 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1, 2, 3, 4, 5, 6\}$, say 1, 2 and 3. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is equal to $\{1, 2, 3\}$. This is a contradiction.

**2.2** $n = 44$.

♢ Four 3-subsets in $A$ are not the color sets of vertices in $Y$. From all 2-subsets in $A$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains at least three common colors in $\{3, 4, 5, 6\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1, 3, 4, 5\}$ or $\{2, 3, 4, 5\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is either $\{1, 3, 4, 5\}$ or $\{2, 3, 4, 5\}$. This is a contradiction.

♢ If four 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3, 4, 5, 6\}$, say 3. So $\{1, 2, 3\} \subset C(u_i), i = 1, 2, 3, 4, 5$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is $\{1, 2, 3\}$. This is a contradiction.

♢ If one 3-subset and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

♢ If one subset in $S$ and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3, 4, 5, 6\}$, say 3. So $\{1, 2, 3\} \subset C(u_i), i = 1, 2, 3, 4, 5$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that there exist at most one subset which may become the color set of vertices in $X$. This is a contradiction.

**2.3** $n = 43$.

♢ If five 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3, 4, 5, 6\}$, say 3. So $\{1, 2, 3\} \subset C(u_i), i = 1, 2, 3, 4, 5$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is $\{1, 2, 3\}$. This is a contradiction.

♢ Two 3-subset and three 2-subsets in $A$ are not the color sets of vertices in $Y$. This discussion is similar to the last paragraph.

♢ If one 3-subset and four 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can also obtain a contradiction similar to the last paragraph.

♢ One subset in $S$ and four 2-subsets in $A$ are not the color sets of vertices in $Y$. From the above discussion, we may assume that each $C(u_i)$ contains $\{1, 2, 3\}$. Then there exist at most one subset which may become the color set of vertices in $X$. This is a contradiction.

♢ If one subset in $S$ and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{3,4,5,6\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1,3,4,5\}$ or $\{2,3,4,5\}$, and there exist at most one subset which may become the color set of vertices in $X$. This is a contradiction.

♢ One 2-subset and four 3-subsets in $A$ are not the color sets of vertices in $Y$. From the above discussion, we may assume that each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$. This is also a contradiction.

♢ If two subsets in $S$ and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. So there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If one subset in $S$, one 3-subset and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

**2.4** $n = 42$**.**

♢ If six 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is equal to $\{1,2\}$. This is a contradiction.

♢ If one 3-subset and five 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is equal to $\{1,2,3\}$. This is a contradiction.

♢ If one subset in $S$ and five 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

♢ If one subset in $S$, one 2-subset and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5,6\}$, say 3 and 4. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that there exist at most one subset which may become the color set of vertex in $X$. This is a contradiction.

♢ If one subset in $S$, one 3-subset and four 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. So there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If one subset in $S$, two 3-subsets and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

♢ Two subsets in $S$, one 3-subset and three 2-subsets in $A$ are not the color sets of vertices in $Y$. From the above discussion, we can know that each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. So there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If three 3-subsets and three 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is equal to $\{1,2,3\}$. This is a contradiction.

♢ If two 3-subsets and four 2-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

♢ Three subsets in $S$ and three 2-subsets in $A$ are not the color sets of vertices in $Y$. From the above discussion, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If two 2-subsets and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5,6\}$, say 3 and 4. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is either $\{1,3,4\}$ or $\{2,3,4\}$. This is a contradiction.

♢ If two subsets in $S$ and four 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If two subsets in $S$ and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{3,4,5,6\}$, say 3, 4 and 5. So each $C(u_i)$ is either $\{1,3,4,5\}$ or $\{2,3,4,5\}$. This is a contradiction.

♢ If one 2-subset and one 3-subset in $A$ are the color sets of vertices in $Y$, and there exist at most three subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

**2.5** $n = 41$**.**

♢ One subset in $S$ and six 2-subsets in $A$ are not the color sets of vertices in $Y$. From $\{1,2,3\}$ is a color set of vertex in $Y$, we can know that each $C(u_i)$ contains $\{1,2\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If one 3-subset, say $\{1,2,3\}$, and six 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$, and each $C(u_i)$ is either $\{1,2\}$ or $\{1,2,3\}$. This is a contradiction.

♢ If two subsets in $S$, one 2-subset and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5,6\}$, say 3 and 4. So each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If one subset in $S$, two 2-subsets and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

♢ If three 2-subsets and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3,4,5,6\}$, say 3. So each $C(u_i)$ contains $\{1,3\}$ or $\{2,3\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If three 2-subsets and four subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{1,2,3,4,5,6\}$, say 1, 2 and 3. So there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

♢ If three subsets in $S$ and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{3,4,5,6\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1,3,4,5\}$ or $\{2,3,4,5\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

**2.6** $n=40$**.**

$\diamondsuit$ If one subset in $S$, one 3-subset and six 2-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains $\{1,2\}$ and there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If one 2-subset and four 3-subsets in $A$ and three subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5,6\}$, say 3 and 4. So each $C(u_i)$ contains $\{1,3,4\}$ or $\{2,3,4\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If two 2-subsets and four 3-subsets in $A$ and two subsets in $S$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

$\diamondsuit$ If three 2-subsets and one 3-subset in $A$ and four subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3,4,5,6\}$, say 3. So each $C(u_i)$ contains $\{1,2,3\}$. From $\{1,4,5\}$ and $\{2,4,5\}$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains $\{1,2,3,4\}$ or $\{1,2,3,5\}$. So there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If three 2-subsets in $A$ and five subsets in $S$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

$\diamondsuit$ If one subset in $S$, three 2-subset and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3,4,5,6\}$, say 3. So each $C(u_i)$ contains $\{1,3\}$ or $\{2,3\}$. But there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If four 2-subsets in $S$ and four subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3,4,5,6\}$, say 3. So each $C(u_i)$ contains $\{1,2,3\}$ and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If four 3-subsets in $S$ and four subsets in $S$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{3,4,5,6\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1,3,4,5\}$ or $\{2,3,4,5\}$ and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ If four 2-subsets and four 3-subsets in $A$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least one common color in $\{3,4,5,6\}$, say 3. So each $C(u_i)$ contains $\{1,3\}$ or $\{2,3\}$. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is one of $\{1,3\},\{2,3\}$ and $\{1,2,3\}$. This is a contradiction.

**Case 3** $u_1,u_2,u_3,u_4$ and $u_5$ receive three different colors under $f$, say 1, 2 and 3. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)|=2, i=1,2,3$ and each $C(v_j)$ is not $\{1,2,3\}$. So the number of the subsets of $\{1,2,3,4,5,6\}$ which may become the color sets of the vertices in $Y$ is $\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}-13=44$. A contradiction may arise since 44 subsets can not distinguish $n$ vertices in $Y$ when $45\le n\le 100$.

$\diamondsuit$ If one subset in $\{\{1,2,4\},\{1,2,5\},\{1,2,6\}\}$, one subset in $\{\{1,3,4\},\{1,3,5\},\{1,3,6\}\}$, one subset in $\{\{2,3,4\},\{2,3,5\},\{2,3,6\}\}$ and one subset in $\{\{4,5\},\{4,6\},\{5,6\}\}$ are the color sets of vertices in $Y$, then each $C(u_i)$ contains at least four common colors in $\{1,2,3,4,5,6\}$, say 1, 2, 3 and 4. So each $C(u_i)$ is one of $\{1,2,3,4\},\{1,2,3,4,5\},\{1,2,3,4,6\}$ and $\{1,2,3,4,5,6\}$. This is a contradiction.

When $n=41,40$.

$\diamondsuit$ If $\{4,5\},\{4,6\}$ and $\{5,6\}$ are not the color sets of vertices in $Y$, then we may assume that each $C(u_i)$ contains $\{1,2,3\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

$\diamondsuit$ $\{1,2,4\},\{1,2,5\}$ and $\{1,2,6\}$ are not the color sets of vertices in $Y$. From one subset in $\{\{1,3,4\}, \{1,3,5\}, \{1,3,6\}\}$, one subset in $\{\{2,3,4\},\{2,3,5\},\{2,3,6\}\}$ and one subset in $\{\{4,5\}, \{4,6\},\{5,6\}\}$ are the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3,4,5,6\}$, say 3 and 4. So each $C(u_i)$ is one of $\{1,3,4\},\{2,3,4\}$ and $\{1,2,3,4\}$. This is a contradiction.

$\diamondsuit$ If $\{1,3,4\},\{1,3,5\}$ and $\{1,3,6\}$ (or $\{2,3,4\},\{2,3,5\}$ and $\{2,3,6\}$ ) are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to the last paragraph.

**Case 4** $u_1,u_2,u_3,u_4$ and $u_5$ receive four different colors under $f$, say 1, 2, 3 and 4. Then the color sets $C(v_j)$ do not include color $i$ when $|C(v_j)|=2, i=1,2,3,4$ and each $C(v_j)$ is not $\{1,2,3\},\{1,2,4\},\{1,3,4\}, \{2,3,4\}$ or $\{1,2,3,4\}$. So the number of the subsets in $\{1,2,3,4,5,6\}$ which may become the color sets of the vertices in $Y$ is $\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}-19=38$. A contradiction may arise since 38 subsets can not distinguish $n$ vertices in $Y$ when $40\le n\le 100$.

**Case 5** $u_1,u_2,u_3,u_4$ and $u_5$ receive five different colors under $f$, say 1, 2, 3, 4 and 5. Then each color set $C(v_j)$ is not a 2-subset and the number of the subsets in $\{1,2,3,4,5,6\}$ which may become the color sets of the vertices in $Y$ is $\binom{6-1}{2}+\binom{6-1}{3}+\binom{6-1}{4}+\binom{6-1}{5}=26$. A contradiction may arise since 26 subsets can not distinguish $n$ vertices in $Y$ when $40\le n\le 100$.

Hence $K_{5,n}$ does not have a 6-VDET coloring and $\chi^e_{vt}(K_{5,n})\ge 7$ when $40\le n\le 100$.

Based on $K_{5,39}$ and its coloring $f_{39}$, we can give a 7-VDET coloring $f_n$ of $K_{5,n}$ $(40\le n\le 100)$. In order to distinguish each $u_i$ with vertices in $Y$, subsets $\{1,2,4,5,6,7\},\{1,2,3,4,6,7\},\{1,2,5,6,7\}$ and $\{1,2,3,5,6,7\}$ are not the color sets of any vertices in $Y$. So $f_n=f_{39}+(377777, 542116, 532116, 522116, 432116, 542631,\cdots)$. We can by coloring other vertices $v_j$ and its incident edges $(46\le j\le 100)$ according to the method given in Table 1 (in which we let $k=7$). Finally we can obtain the 7-VDET coloring $f_n$ $(40\le n\le 100)$ of $K_{5,n}$.

The proof of Theorem 3 is completed.

**Theorem 4**. If $101\le n\le 220$, then $\chi^e_{vt}(K_{5,n})=8$.

**Proof**. Assume that $K_{5,n}$ has a 7-VDET coloring $f$. There are five cases we need to consider.

**Case 1** $u_1,u_2,u_3,u_4$ and $u_5$ receive the same color under

$f$. We may suppose that $f(u_i) = 1, i = 1, 2, 3, 4, 5$, so none of the $C(v_j)$ include color 1 and the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7\}$ which may become the color sets of the vertices in $Y$ is $\binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 57$. A contradiction may arise since 57 subsets can not distinguish $n$ vertices in $Y$ when $101 \le n \le 220$.

**Case 2** $u_1, u_2, u_3, u_4$ and $u_5$ receive two different colors under $f$. We may assume that $f(u_i) \in \{1, 2\}, i = 1, 2, 3, 4, 5$. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2$. So the number of the subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ which may become the color sets of the vertices in $Y$ is $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} - 11 = 108$. A contradiction may arise since 108 subsets can not distinguish $n$ vertices in $Y$ when $109 \le n \le 220$.

We denoted the set, which contains the 108 subsets except all 2-subsets in $\{3, 4, 5, 6, 7\}$ and $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$, as $D$.

♢ There exist at least eight 2-subsets in $\{3, 4, 5, 6, 7\}$ and one subset in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are the color sets of vertices in $Y$. We can obtain that each $C(u_i)$ contains at least three common colors in $\{3, 4, 5, 6, 7\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1, 2, 3, 4, 5\}$ and each $C(u_i)$ is one of $\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 7\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$. This is a contradiction.

♢ If one 2-subset in $\{3, 4, 5, 6, 7\}$ and and one subset in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are the color sets of vertices in $Y$, and there exist at most three subsets in $D$ are not the color sets of vertices in $Y$, then we can obtain that each $C(u_i)$ contains at least one common color in $\{3, 4, 5, 6, 7\}$, say 3. So each $C(u_i)$ contains $\{1, 2, 3\}$ and there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

Next, we only need to consider the following subcases:

When $n = 103$ and all the subsets in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are not the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains at least four common colors in $\{3, 4, 5, 6, 7\}$, say 3, 4, 5 and 6. In order to distinguish each $u_i$ with vertices in $Y$, we can obtain that each $C(u_i)$ is either $\{1, 3, 4, 5, 6\}$ or $\{2, 3, 4, 5, 6\}$. This is a contradiction.

When $n = 102$, we have the following two subcases.

♢ If one 2-subset in $\{3, 4, 5, 6, 7\}$ and all the subsets in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least three common colors in $\{3, 4, 5, 6, 7\}$, say 3, 4 and 5. From the above discussion, we can obtain that each $C(u_i)$ is either $\{1, 3, 4, 5\}$ or $\{2, 3, 4, 5\}$. This is a contradiction.

♢ All subsets in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ and one subset in $D$ are not the color sets of vertices in $Y$. From the above discussion, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

When $n = 101$, we have the following three subcases.

♢ One subset in $D$, one 2-subset in $\{3, 4, 5, 6, 7\}$ and five subsets in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are not the color sets of vertices in $Y$. Then each $C(u_i)$ contains at least three common colors in $\{3, 4, 5, 6, 7\}$, say 3, 4 and 5. From the above discussion, we can obtain that each $C(u_i)$ is either $\{1, 3, 4, 5\}$ or $\{2, 3, 4, 5\}$. This is a contradiction.

♢ If two 2-subsets in $\{3, 4, 5, 6, 7\}$ and five subsets in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$ are not the color sets of vertices in $Y$, then we can obtain a contradiction similar to last paragraph.

♢ If three 2-subsets in $\{3, 4, 5, 6, 7\}$ and four subsets in $D$ are not the color sets of vertices in $Y$, then each $C(u_i)$ contains at least two common colors in $\{3, 4, 5, 6, 7\}$, say 3 and 4. So each $C(u_i)$ contains $\{1, 3, 4\}$ or $\{2, 3, 4\}$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most four subsets which may become the color sets of vertices in $X$. This is a contradiction.

**Case 3** $u_1, u_2, u_3, u_4$ and $u_5$ receive three different colors under $f$, say 1, 2 and 3. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3$ and each $C(v_j)$ is not $\{1, 2, 3\}$. So the number of the subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ which may become the color sets of the vertices in $Y$ is $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} - 16 = 103$. A contradiction may arise since 103 subsets can not distinguish $n$ vertices in $Y$ when $104 \le n \le 220$.

Since one subset in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7\}$, one subset in $\{\{1, 3, i\} : i = 4, 5, 6, 7\}$ and one subset in $\{\{2, 3, i\} : i = 4, 5, 6, 7\}$ are the color sets of vertices in $Y$, we can know that $\{1, 2, 3\} \subset C(u_i), i = 1, 2, 3, 4, 5$. But there exist at most two subsets which may become the color sets of vertices in $X$. This is a contradiction.

**Case 4** $u_1, u_2, u_3, u_4$ and $u_5$ receive four different colors under $f$, say 1, 2, 3 and 4. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4$ and each $C(v_j)$ is not $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ or $\{1, 2, 3, 4\}$. So the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7\}$ which may become the color sets of the vertices in $Y$ is $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} - 23 = 96$. A contradiction may arise since 96 subsets can not distinguish $n$ vertices in $Y$ when $101 \le n \le 220$.

**Case 5** $u_1, u_2, u_3, u_4$ and $u_5$ receive five different colors under $f$, say 1, 2, 3, 4 and 5. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4, 5$ and each $C(v_j)$ is not 3-subset, 4-subset and 5-subset in $\{1, 2, 3, 4, 5\}$. So the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7\}$ which may become the color sets of the vertices in $Y$ is $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} - 36 = 83$. A contradiction may arise since 83 subsets can not distinguish $n$ vertices in $Y$ when $101 \le n \le 220$.

Hence $K_{5,n}$ does not have a 7-VDET coloring and $\chi^e_{vt}(K_{5,n}) \ge 8$ when $101 \le n \le 220$.

Based on $K_{5,100}$ and its coloring $f_{100}$, we can give a 8-VDET coloring $f_n$ of $K_{5,n}$ $(101 \le n \le 220)$. In order to distinguish each $u_i$ with vertices in $Y$, $\{1, 2, 5, 6, 7, 8\}$ is not the color set of any vertex in $Y$. So $f_n = f_{100} + (388888, 542671, 532671, 562711, 432671, \cdots)$. We can by coloring other vertices $v_j$ and its incident edges $(106 \le j \le 220)$ according to the method given in Table 1 (in which we let $k = 8$). Finally we can obtain the 8-VDET coloring $f_n$ $(101 \le n \le 220)$ of $K_{5,n}$.

The proof of Theorem 4 is completed.

**Theorem 5**. If $221 \le n \le 437$, then $\chi^e_{vt}(K_{5,n}) = 9$.

**Proof**. Assume that $K_{5,n}$ has a 8-VDET coloring $f$. There are five cases we need to consider.

**Case 1** $u_1, u_2, u_3, u_4$ and $u_5$ receive the same color under $f$. We may suppose that $f(u_i) = 1, i = 1, 2, 3, 4, 5$, so none of the $C(v_j)$ include color 1 and the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which may become the color sets of the vertices in $Y$ is $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} = 119$. A contradiction may arise since 119 subsets can not distinguish

$n$ vertices in $Y$ when $221 \le n \le 437$.

**Case 2** $u_1, u_2, u_3, u_4$ and $u_5$ receive two different colors under $f$. We may assume that $f(u_i) \in \{1, 2\}, i = 1, 2, 3, 4, 5$. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2$. So the number of the subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which may become the color sets of the vertices in $Y$ is $\binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} - 13 = 225$. A contradiction may arise since 225 subsets can not distinguish $n$ vertices in $Y$ when $226 \le n \le 437$.

Since there exist at least 10 2-subsets in $\{3, 4, 5, 6, 7, 8\}$ and one subset in $\{\{1, 2, i\} : i = 3, 4, 5, 6, 7, 8\}$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains at least three common colors in $\{3, 4, 5, 6, 7, 8\}$, say 3, 4 and 5. So each $C(u_i)$ contains $\{1, 2, 3, 4, 5\}$. When $n = 225, 224, 223, 222, 221$, we can obtain a contradiction since there exist at most four subsets which may become the color sets of vertices in $X$.

**Case 3** $u_1, u_2, u_3, u_4$ and $u_5$ receive three different colors under $f$, say 1, 2 and 3. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3$ and each $C(v_j)$ is not $\{1, 2, 3\}$. So the number of the subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which may become the color sets of the vertices in $Y$ is $\binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} - 19 = 219$. A contradiction may arise since 219 subsets can not distinguish $n$ vertices in $Y$ when $221 \le n \le 437$.

**Case 4** $u_1, u_2, u_3, u_4$ and $u_5$ receive four different colors under $f$, say 1, 2, 3 and 4. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4$ and each $C(v_j)$ is not $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$, $\{2, 3, 4\}$ or $\{1, 2, 3, 4\}$. So the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which may become the color sets of the vertices in $Y$ is $\binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} - 27 = 211$. A contradiction may arise since 211 subsets can not distinguish $n$ vertices in $Y$ when $221 \le n \le 437$.

**Case 5** $u_1, u_2, u_3, u_4$ and $u_5$ receive five different colors under $f$, say 1, 2, 3, 4 and 5. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4, 5$ and each $C(v_j)$ is not 3-subset, 4-subset or 5-subset in $\{1, 2, 3, 4, 5\}$. So the number of the subsets in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which may become the color sets of the vertices in $Y$ is $\binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} - 41 = 197$. A contradiction may arise since 197 subsets can not distinguish $n$ vertices in $Y$ when $221 \le n \le 437$.

Hence $K_{5,n}$ does not have a 8-VDET coloring and $\chi^e_{vt}(K_{5,n}) \ge 9$ when $221 \le n \le 437$.

Based on $K_{5,220}$ and its coloring $f_{220}$, we can give a 9-VDET coloring $f_n$ of $K_{5,n}$ $(221 \le n \le 437)$. $f_n = f_{220} + (399999, 526781, \cdots)$. We can by coloring other vertices $v_j$ and its incident edges $(223 \le j \le 437)$ according to the method given in Table 1 (in which we let $k = 9$). Finally we can obtain the 9-VDET coloring $f_n$ $(221 \le n \le 437)$ of $K_{5,n}$.

The proof of Theorem 5 is completed.

**Theorem 6**. If $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \le n \le \binom{k}{2} + \binom{k}{3} + \binom{k}{4} + \binom{k}{5} + \binom{k}{6} - 2k - 1, k \ge 10$, then $\chi^e_{vt}(K_{4,n}) = k$.

**Proof** . Firstly, we prove that $K_{5,n}$ does not have a $(k-1)-$VDET coloring. Assume that $K_{5,n}$ has a $(k-1)-$VDET coloring $f$. There are five cases to consider.

**Case 1** $u_1, u_2, u_3, u_4$ and $u_5$ receive the same color under $f$. We may suppose that $f(u_i) = 1, i = 1, 2, 3, 4, 5$, so none of the $C(v_j)$ include color 1 and the number of the subsets in $\{1, 2, \cdots, k-1\}$ which may become the color sets of the vertices in $Y$ is $\binom{k-2}{2} + \binom{k-2}{3} + \binom{k-2}{4} + \binom{k-2}{5} + \binom{k-2}{6} < \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \le n$, since the $\binom{k-2}{2} + \binom{k-2}{3} + \binom{k-2}{4} + \binom{k-2}{5} + \binom{k-2}{6}$ subsets can not distinguish $n$ vertices of degree 5. This is a contradiction.

**Case 2** $u_1, u_2, u_3, u_4$ and $u_5$ receive two different colors under $f$. We may assume that $f(u_i) \in \{1, 2\}, i = 1, 2, 3, 4, 5$. Each $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2$. The total number of the subsets of $\{1, 2, 3, \cdots, k-1\}$ which may become the color sets of the vertices in $Y$ is $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2k + 5$. When $n \ge \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} - 2k + 6$, we can obtain a contradiction since $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2k + 5$ sets can not distinguish $n$ vertices in $Y$.

Since there exist at least $\sum_{i=1}^{k-4} i - 5$ 2-subsets in $\{3, 4, \cdots, k-1\}$ and one subset in $\{\{1, 2, i\} : i = 3, 4, \cdots, k-1\}$ are the color sets of vertices in $Y$, we can know that each $C(u_i)$ contains at least $k-6$ common colors in $\{3, 4, \cdots, k-1\}$, say $3, 4, \cdots, k-5$ and $k-4$. So $\{1, 2, 3, \cdots, k-4\} \subset C(u_i), i = 1, 2, 3, 4, 5$. In order to distinguish each $u_i$ with vertices in $Y$, there exist at most three subsets which may become the color sets of vertices in $X$. This is a contradiction.

**Case 3** $u_1, u_2, u_3, u_4$ and $u_5$ receive three different colors under $f$, say 1, 2 and 3. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3$ and each $C(v_j)$ is not $\{1, 2, 3\}$. The total number of the subsets of $\{1, 2, 3, \cdots, k-1\}$ which may become the color sets of the vertices in $Y$ is $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 3k + 8$. Note that $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 3k + 8 < \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \le n$ when $k \ge 10$. So we can obtain a contradiction since $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 3k + 8$ sets can not distinguish $n$ vertices of degree 5.

**Case 4** $u_1, u_2, u_3, u_4$ and $u_5$ receive four different colors under $f$, say 1, 2, 3 and 4. Then each color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4$ and each $C(v_j)$ is not $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$, $\{2, 3, 4\}$ or $\{1, 2, 3, 4\}$. So the number of the subsets in $\{1, 2, \cdots, k-1\}$ which may become the color sets of the vertices in $Y$ is $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 4k + 9$. As $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 4k + 9 < \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \le n$. Thus the $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 4k + 9$ sets can not distinguish $n$ vertices of degree 5. This is a contradiction.

**Case 5** $u_1, u_2, u_3, u_4$ and $u_5$ receive five different colors under $f$, say 1, 2, 3, 4 and 5. Then the color set $C(v_j)$ does not include color $i$ when $|C(v_j)| = 2, i = 1, 2, 3, 4, 5$ and each $C(v_j)$ is not 3-subset, 4-subset or 5-subset in $\{1, 2, 3, 4, 5\}$. So the number of the subsets in $\{1, 2, \cdots, k-1\}$ which may become the color sets of the vertices in $Y$ is is $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 5k + 4$. As $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 5k + 4 < \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \le n$. Thus the $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 5k + 4$ sets can not distinguish $n$ vertices of degree 5. This is a contradiction.

Next, we will give a $k-$ VDET coloring of $K_{5,n}$ recursively.

In the following, we let $s = \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1), t = \binom{k}{2} + \binom{k}{3} + \binom{k}{4} + \binom{k}{5} + \binom{k}{6} - 2k - 1$. Note that $s$ and $t$ depend on $k$.

When $k \geq 10$, suppose $(k-1)-$ VDET coloring $f_{s-1}$ of $K_{5,s-1}$ has been constructed. Based on $K_{5,s-1}$ and its $(k-1)-$ VDET coloring $f_{s-1}$, we will give $K_{5,t}$ and its $k-$ VDET coloring $f_t$ by coloring each vertex $v_j$ and its incident edges $(s \leq j \leq t)$. We arrange all 2-subsets, 3-subsets, 4-subsets, 5-subsets and 6-subsets of $\{1, 2, \cdots, k\}$ which contain $k$, except for $\{1, k\}$ and $\{2, k\}$, into a sequence $P_k$. Then $P_k$ has $\binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} - 2$ terms. Let the terms in $P_k$ be corresponded to vertices $v_s, v_{s+1}, \cdots, v_t$. Then based on $f_{s-1}$ using colors $1, 2, \cdots, k-1$, we can obtain $k-$ VDET coloring $f_t$ by coloring vertex $v_j$ and its incident edges $(s \leq j \leq t)$ according to the method given in Table 1. Of course, we can obtain $K_{5,n}$ and its $k-$ VDET coloring $f_n$ when $s \leq n < t$.

Thus $\chi^e_{vt}(K_{5,n}) = k$ when $\binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4} + \binom{k-1}{5} + \binom{k-1}{6} - 2(k-1) \leq n \leq \binom{k}{2} + \binom{k}{3} + \binom{k}{4} + \binom{k}{5} + \binom{k}{6} - 2k - 1, k \geq 10$.

The proof of Theorem 6 is completed.

## IV. Conclusion

By simple computation, we have

$\eta(K_{5,5}) = \eta(K_{5,6}) = 4$;

$\eta(K_{5,n}) = l, \ \ 2^{l-1} - l - 4 \leq n \leq 2^l - (l+1) - 5, l \geq 5.$

From the six Theorems in Section 2, we know that

1. If $n = 5, 6$, or $11 \leq n \leq 21$, or $40 \leq n \leq 52$, or $101 \leq n \leq 115$, or $221 \leq n \leq 242$, or $\binom{l-1}{2} + \binom{l-1}{3} + \binom{l-1}{4} + \binom{l-1}{5} + \binom{l-1}{6} - 2(l-1) \leq n \leq 2^{l-1} - l - 5, l \geq 10$, then $\chi^e_{vt}(K_{5,n}) = \eta(K_{5,n}) + 1$.

2. If $7 \leq n \leq 10$, or $22 \leq n \leq 39$, or $53 \leq n \leq 100$, or $116 \leq n \leq 220$, or $2^{l-1} - l - 4 \leq n \leq \binom{l}{2} + \binom{l}{3} + \binom{l}{4} + \binom{l}{5} + \binom{l}{6} - 2l - 1, l \geq 10$, then $\chi^e_{vt}(K_{5,n}) = \eta(K_{5,n})$.

Thus Conjecture 1 is right for $K_{5,n}$.

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