Resistance Distance and Kirchhoff Index of a New Join of Two Graphs

Jianhua Li and Qun Liu

***Abstract*—Given graphs G_1 and G_2, let $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ be the edge set of G_1. A new join of two graph G_1 and G_2, denoted by $G_1 \odot G_2$, can be obtained from one copy of G_1 and m_1 copied of G_2 by adding a new vertex corresponding to each edge of G_1, letting the resulting new vertex set be $U = \{u_1, u_2, \ldots, u_{m_1}\}$, and joinning u_i with each vertex of i-th copy of G_2 and with endpoints of e_i for $i = 1, 2, \ldots m_1$. In this paper, the explicit closed formulas of resistance distance and Kirchhoff index of $G_1 \odot G_2$ whenever G_1 is an r_1-regular graph and G_2 is arbitrary graphs are obtianed.**

***Index Terms*—Resistance distance; Kirchhoff index; a new join of two graphs; Generalized inverse**

I. Introduction

In 1993 Klein and Randić [1] first came up with the concept of resistance distance on graphs. The resistance distance r_{uv} between nodes u and v on the network is defined as on every edge of a network one puts a unit resistor and the effective resistance between nodes u and v on the network. A well-known resistance distance-based parameter, called the Kirchhoff index, was given by $Kf(G) = \sum_{\{u,v\} \subseteq V(G)} r_{uv}$. The resistance distance and Kirchhoff index have attracted extensive attention due to their wide applications in physics, chemistry and others. Up till now, many results on the resistance distance and the Kirchhoff index are obtained. See ([2]-[10]) and the references therein to know more.

Let $G = (V(G), E(G))$ be a connected graph with $|V(G)| = n$ and $|E(G)| = m$. Let $D_G = diag(d_1, d_2, \cdots d_{|V(G)|})$ be the diagonal matrix with all vertex degrees of G as its diagonal entries, where d_i be the degree of vertex i in G. For a graph G, the matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where A_G denote the adjacency matrix of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the eigenvalues of L_G.

The hot topic about resistance distance and Kirchhoff index is computation of two indices. However, it is difficult to compute the resistance distance and Kirchhoff index because they are highly sensitive to small perturbations on the network. Thus this has prompted researchers try to find some techniques to compute the resistance distance and Kirchhoff index. In [11], a new join of two graphs of $G_1 \odot G_2$ is investigated. The adjacency spectrum, the Laplacian spectrum and the signless Laplacian spectrum of $G_1 \odot G_2$ are determined. This paper we will considers the explicit formulas of resistance distance and Kirchhoff index of $G_1 \odot G_2$.

Definition 1 [11] Given graphs G_1 and G_2 with n_1, n_2 vertices, respectively. let $E(G_1) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of G_1, the graph $G_1 \odot G_2$ can be obtained from one copy of G_1 and m_1 copies of G_2 as follows. Firstly, we add a new vertex corresponding to each edge of G_1, the resulting new vertex set $U = \{u_1, u_2, \ldots, u_m\}$. Then join u_i with each vertex of i-th copy of G_2 and with the endpoints of e_i, for $i = 1, 2, \ldots, m_1$.

Note that the graph $G_1 \odot G_2$ in Definition 1 contains $n_1 + m_1(n_2+1)$ vertices. As instance, $C_4 \odot P_3$ is as illustrated in Fig.1.

Up until now, many studies have been focused on composite graph of resistance distance and Kirchhoff index, such as subdivision-vertex join and subdivision-edge join [12], R-vertex join and R-edge join [13]. corona and neighborhood corona [14], circulant graph [15], abelian Cayley graph [16], noncommutative groups [17]. Motivated by the results, in this paper, we explore the generalized inverse of $G_1 \odot G_2$ in terms of the generalized inverse of the factor graphs. Thus the resistance distance and Kirchhoff index of $G_1 \odot G_2$ can be derived from the resistance distance and Kirchhoff index of the factor graphs.

II. Preliminaries

The $\{1\}$-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$- inverse [7]. For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$ ([9], [7],). If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$- inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [9].

It is well known that resistance distances in a connected graph G can be obtained from any $\{1\}$- inverse of G ([2]). We use $M^{(1)}$ to denote any $\{1\}$- inverse of a matrix M, and let $(M)_{uv}$ denote the (u, v)- entry of M.

Lemma 2.1 ([9]) Let G be a connected graph. Then

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}. \end{aligned}$$

Lemma 2.2 ([12]) For any graph, we have $L_G^{\#} 1 = 0$.

Let 1_n denote the column vector of dimension n with all the entries equal one. We will often use 1 to denote an all-ones column vector if the dimension can be read from the context.

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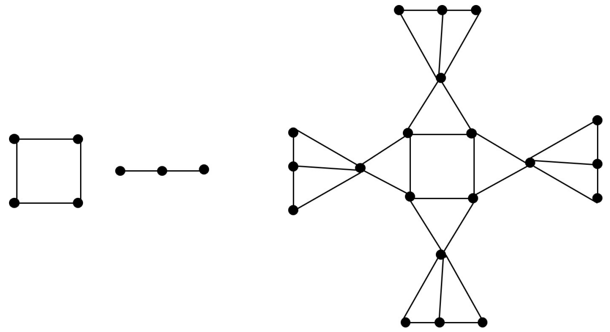

*Fig.*1 *Graph* $C_4 \odot P_3$

Lemma 2.3 ([18]) Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

Lemma 2.4 Let G be a connected graph. For any $i, j \in V(G)$,

$$r_{ij}(G) = d_i^{-1}\left(1 + \sum_{k\in T(i)} r_{kj}(G) - d_i^{-1} \sum_{k,l\in T(i)} r_{kl}(G)\right).$$

For a square matrix M, let $tr(M)$ denote the trace of M.

Lemma 2.5 ([10]) Let G be a connected graph on n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 2.6 ([13]) Let G be a graph of order n. For any $a, b > 0$ satisfying $b \neq a$, we have

$$(L_G + aI_n - \frac{a}{b} j_{n\times n})^{-1} = (L_G + aI_n)^{-1} + \frac{1}{a(b-n)} j_{n\times n}.$$

Lemma 2.7([19]) Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^T H^{\#} & D^{-1} + D^{-1}B^T H^{\#} BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

III. THE RESISTANCE DISTANCE OF $G_1 \odot G_2$

In this section, we focus on determing the resistance distance of graph $G_1 \odot G_2$ whenever G_1 is an r_1-regular graph and G_2 is an arbitrary graph.

Theorem 3.1 Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges, G_2 be an arbitrary graph with n_2 vertices and m_2 edges. Let L_i, R_i be the Laplacian matrix and incidence matrix of G_i, for $i = 1, 2$, respectively. Then $G_1 \odot G_2$ have the resistance distance as follows:

(i) For any $i, j \in V(G_1)$, we have

$$\begin{aligned} r_{ij}(G_1 \odot G_2) &= \frac{2}{3}(L_1^{\#})_{ii} + \frac{2}{3}(L_1^{\#})_{jj} - \frac{4}{3}(L_1^{\#})_{ij} \\ &= \frac{2}{3} r_{ij}(G_1). \end{aligned}$$

(ii) For any $i, j \in V(G_2)$, we have
$r_{ij}(G_1 \odot G_2)$

$$\begin{aligned} &= ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{ii} + \\ &\quad ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{jj} \\ &\quad -2((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{ij}. \end{aligned}$$

(iii) For any $i \in V(G_1)$, $j \in V(G_2)$, we have
$r_{ij}(G_1 \odot G_2)$

$$\begin{aligned} &= \frac{2}{3}(L_1^{\#})_{ii} + ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \\ &\quad \otimes E_{m_1})_{ii} - \frac{2}{3}(L_1^{\#})_{ij}. \end{aligned}$$

(iv) For any $i \in I(G_1)$, $j \in V(G_1) \cup V(G_2)$, Let $u_i v_i \in E(G_1)$ denote the edge corresponding to i, we have
$r_{ij}(G_1 \odot G_2)$

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \odot G_2) + \frac{1}{2} r_{v_i j}(G_1 \odot G_2) \\ &\quad - \frac{1}{4} r_{u_i v_i}(G_1 \odot G_2). \end{aligned}$$

(v) For any $i, j \in I(G_1)$, Let $u_i v_i, u_j v_j \in E(G_1)$ denote the edges corresponding to i, j, we have

$r_{ij}(G_1 \odot G_2)$

$$\begin{aligned} &= 1 + \frac{1}{4}\big(r_{u_i u_j}(G_1 \odot G_2) + r_{u_i v_j}(G_1 \odot G_2) \\ &\quad + r_{v_i u_j}(G_1 \odot G_2) + r_{v_i v_j}(G_1 \odot G_2) \\ &\quad - r_{u_i v_i}(G_1 \odot G_2) - r_{u_j v_j}(G_1 \odot G_2)\big). \end{aligned}$$

Proof With a suitable labeling for vertices of $G_1 \odot G_2$, the Laplacian matrix of $G_1 \odot G_2$ can be written as follows:

$L(G_1 \odot G_2) =$

$$\begin{pmatrix} r_1 E_{n_1} + L_1 & -R_1 & -1_{m_1}^T \otimes 0_{n_1\times n_2} \\ -R_1^T & (2+n_2)E_{m_1} & -1_{n_2}^T \otimes E_{m_1} \\ -1_{m_1} \otimes 0_{n_2\times n_1} & -1_{n_2} \otimes E_{m_1} & (E_{n_2} + L_2) \otimes E_{m_1} \end{pmatrix}.$$

Let $A = r_1 E_{n_1} + L_1$, $B = \begin{pmatrix} -R_1 & -1_{m_1}^T \otimes 0_{n_1\times n_2} \end{pmatrix}$, $B^T = \begin{pmatrix} -R_1^T \\ -1_{m_1} \otimes 0_{n_2\times n_1} \end{pmatrix}$ and

$$D = \begin{pmatrix} (2+n_2)E_{m_1} & -1_{n_2}^T \otimes E_{m_1} \\ -1_{n_2} \otimes E_{m_1} & (E_{n_2} + L_2) \otimes E_{m_1} \end{pmatrix}.$$

First we compute the D^{-1}. By Lemma 2.3, we have

$$\begin{aligned} A_1 - B_1 D_1^{-1} C_1 \\ &= (2+n_2)E_{m_1} - (-1_{n_2}^T \otimes E_{m_1}) \\ &\quad ((L_2+E_{n_2})^{-1} \otimes E_{m_1}))(-1_{n_2} \otimes E_{m_1}) \\ &= (2+n_2)E_{m_1} - n_2 E_{m_1} = 2E_{m_1}, \end{aligned}$$

so $(A_1 - B_1 D_1^{-1} C_1)^{-1} = \frac{1}{2} E_{m_1}$.

By Lemma 2.3, we have

$$\begin{aligned} S &= (D_1 - C_1 A_1^{-1} B_1) \\ &= (L_2+E_{n_2}) \otimes E_{m_1} - (-1_{n_2} \otimes E_{m_1}) \\ &\quad \tfrac{1}{2+n_2}(-1_{n_2}^T \otimes E_{m_1}) \\ &= (L_2+E_{n_2}) \otimes E_{m_1} - \tfrac{1}{n_2+2} j_{n_2} \otimes E_{m_1} \\ &= (L_2+E_{n_2} - \tfrac{1}{n_2+2} j_{n_2 \times n_2}) \otimes E_{m_1}, \end{aligned}$$

so $S^{-1} = (L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1}$.

By Lemma 2.3, we have

$$\begin{aligned} -A_1^{-1} B_1 S^{-1} \\ &= -\tfrac{1}{n_2+2} E_{m_1}(-1_{n_2}^T \otimes E_{m_1}) \\ &\quad (L_2+E_{n_2} - \tfrac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1} \\ &= \tfrac{1}{2} 1_{n_2}^T \otimes E_{m_1}. \end{aligned}$$

Similarly, $-S^{-1} C_1 A_1^{-1} = \frac{1}{2} 1_{n_2} \otimes E_{m_1}$.

So D^{-1}

$$= \begin{pmatrix} \frac{1}{2} E_{m_1} & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ \frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1} \end{pmatrix}.$$

Next we begin with the computation of $\{1\}$-inverse of $L_{G_1 \odot G_2}$.

By Lemma 2.7, we have

$$\begin{aligned} H &= r_1 E_{n_1} + L_1 - \begin{pmatrix} -R_1 & 1_{n_2}^T \otimes 0_{n_1\times n_2} \end{pmatrix} \\ &\quad \begin{pmatrix} \frac{1}{2} E_{m_1} & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ \frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2})^{-1} \otimes E_{m_1} \end{pmatrix} \\ &\quad \begin{pmatrix} -R_1^T \\ -1_{n_2} \otimes 0_{n_2\times n_1} \end{pmatrix} \\ &= r_1 E_{n_1} + L_1 - \begin{pmatrix} -\frac{1}{2} R_1 & -\frac{1}{2} R_1 (1_{n_2}^T \otimes E) \end{pmatrix} \\ &\quad \begin{pmatrix} -R_1^T \\ -1_{n_2} \otimes 0_{n_2\times n_1} \end{pmatrix} \\ &= r_1 E + (r_1 E_{n_1} - A(G_1)) - \tfrac{1}{2}(r_1 E + A(G_1)) \\ &= \tfrac{3}{2} L_1, \end{aligned}$$

so $H^{\#} = \frac{2}{3} L_1^{\#}$.
According to Lemma 2.7, we calculate $-H^{\#} B D^{-1}$ and $-D^{-1} B^T H^{\#}$.
$-H^{\#} B D^{-1}$

$$\begin{aligned} &= -\tfrac{2}{3} L_1^{\#} \begin{pmatrix} -R_1 & 1_{m_1}^T \otimes 0_{n_1\times n_2} \end{pmatrix} \\ &\quad \begin{pmatrix} \frac{1}{2} E_{m_1} & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ \frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2})^{-1} \otimes E_{m_1} \end{pmatrix} \\ &= -\tfrac{2}{3} L_1^{\#} \begin{pmatrix} -\frac{1}{2} R_1 & -\frac{1}{2} R_1 (1_{n_2}^T \otimes E_{m_1}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} L_1^{\#} R_1 & \frac{1}{3} L_1^{\#} R_1 (1_{n_2}^T \otimes E_{m_1}) \end{pmatrix} \end{aligned}$$

and
$-D^{-1} B^T H^{\#}$

$$\begin{aligned} &= -2 \begin{pmatrix} \frac{1}{2} E_{m_1} & -\frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ -\frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2})^{-1} \otimes E \end{pmatrix} \\ &\quad \begin{pmatrix} -R_1^T \\ -1_{m_1} \otimes 0_{n_2\times n_1} \end{pmatrix} L_1^{\#} \\ &= -\begin{pmatrix} \frac{1}{3} R_1^T L_1^{\#} \\ \frac{1}{3}(1_{n_2} \otimes E_{m_1}) R_1^T L_1^{\#} \end{pmatrix}. \end{aligned}$$

We are ready to compute the $D^{-1} B^T H^{\#} B D^{-1}$. Let $M = 1_{n_2} \otimes E_{m_1}$, then $D^{-1} B^T H^{\#} B D^{-1}$

$$\begin{aligned} &= \begin{pmatrix} \frac{1}{2} E_{m_1} & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ \frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1} \end{pmatrix} \\ &\quad L_{G_1}^{\#} \begin{pmatrix} -R_1 & 1_{m_1}^T \otimes 0_{m_1\times n_2} \end{pmatrix} \\ &\quad \begin{pmatrix} \frac{1}{2} E_{m_1} & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \\ \frac{1}{2} 1_{n_2} \otimes E_{m_1} & (L_2+E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} R_1^T L_1^{\#} R_1 & \frac{1}{6} R_1^T L_1^{\#} R_1 M^T \\ \frac{1}{6} M R_1^T L_1^{\#} R_1 & \frac{1}{6} M R_1^T L_1^{\#} R_1 M^T \end{pmatrix}. \end{aligned}$$

Let $P = (L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1}$,$Q = \frac{1}{6}(1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})$, $T = \frac{1}{6} R_1^T L_1^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})$ and $M = \frac{1}{3} L_1^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})$.

Based on Lemma 2.3 and 2.7, the following matrix
$N =$

$$\begin{pmatrix} \frac{2}{3} L_1^{\#} & \frac{1}{3} L_1^{\#} R_1 & M \\ \frac{1}{3} R_1^T L_1^{\#} & \frac{1}{2} E_{m_1} + \frac{1}{6} R_1^T L_1^{\#} R_1 & \frac{1}{2} 1_{n_2}^T \otimes E_{m_1} + T \\ M^T & \frac{1}{2} 1_{n_2} \otimes E_{m_1} + T^T & P + Q \end{pmatrix} \qquad (1)$$

is a symmetric $\{1\}$-inverse of $L_{G_1 \odot G_2}$.

For any $i, j \in V(G_1)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}(G_1 \odot G_2) &= \frac{2}{3}(L_1^{\#})_{ii} + \frac{2}{3}(L_1^{\#})_{jj} - \frac{4}{3}(L_1^{\#})_{ij} \\ &= \frac{2}{3} r_{ij}(G_1), \end{aligned}$$

as stated in (i).

For any $i, j \in V(G_2)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}&(G_1 \odot G_2) \\ &= ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{ii} + \\ &\quad ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{jj} \\ &\quad -2((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \otimes E_{m_1})_{ij}, \end{aligned}$$

as stated in (ii).

For any $i \in V(G_1)$, $j \in V(G_2)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}&(G_1 \odot G_2) \\ &= \frac{2}{3}(L_1^{\#})_{ii} + ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} \\ &\quad \otimes E_{m_1})_{ii} - \frac{2}{3}(L_1^{\#})_{ij}, \end{aligned}$$

as stated in (iii).

For any $i \in I(G_1)$, $j \in V(G_1) \cup V(G_2)$, Let $u_i v_i \in E(G_1)$ denote the edge corresponding to i, By Lemma 2.4, we have $r_{ij}(G_1 \odot G_2)$

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \odot G_2) + \frac{1}{2} r_{v_i j}(G_1 \odot G_2) \\ &\quad -\frac{1}{4} r_{u_i v_i}(G_1 \odot G_2), \end{aligned}$$

as stated in (iv).

For any $i, j \in I(G_1)$, Let $u_i v_i, u_j v_j \in E(G_1)$ denote the edges corresponding to i, j, By Lemma 2.4, we have

$$\begin{aligned}
r_{ij}(G_1 \odot G_2) &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \odot G_2) + \frac{1}{2} r_{v_i j}(G_1 \odot G_2) \\
&\quad - \frac{1}{4} r_{u_i v_i}(G_1 \odot G_2) \\
&= 1 + \frac{1}{4} \big(r_{u_i u_j}(G_1 \odot G_2) + r_{u_i v_j}(G_1 \odot G_2) \\
&\quad + r_{v_i u_j}(G_1 \odot G_2) + r_{v_i v_j}(G_1 \odot G_2) \\
&\quad - r_{u_i v_i}(G_1 \odot G_2) - r_{u_j v_j}(G_1 \odot G_2) \big),
\end{aligned}$$

as stated in (v).

IV. The Kirchhoff index of $G_1 \odot G_2$

In this section, we focus on determing the Kirchhoff index of graphs of $G_1 \odot G_2$ whenever G_1 is an r_1-regular graph and G_2 is an arbitrary graph.

Theorem 4.1 Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges, G_2 be an arbitrary graph with n_2 vertices and m_2 edges. Let L_i, R_i be the Laplacian matrix and incidence matrix of G_i, for $i = 1, 2$, respectively. Then $G_1 \odot G_2$ have the Kirchhoff index as follows:

$$\begin{aligned}
&Kf(L_{G_1 \odot G_2}) \\
&= (n_1 + m_1(n_2+1)) \Bigg(\frac{2}{3n_1} Kf(G_1) \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2)+1} \\
&\quad + tr(\frac{1}{6}(1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1}) \\
&\quad + \frac{3m_1 - n_1 + 1}{6} \Bigg) - \frac{m_1 + n_2^2(n_2+2)}{2}.
\end{aligned}$$

Proof Let $m = n_1 + m_1(n_2+1)$. By Lemma 2.5, we have

$$\begin{aligned}
&Kf(L_{G_1 \odot G_2}) \\
&= m \Bigg(\frac{2}{3} tr(L_{G_1}^{\#}) + tr(\frac{1}{2} E_{m_1} + \frac{1}{6} R_1^T L_{G_1}^{\#} R_1) + \\
&\quad tr((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1}) + \\
&\quad tr(\frac{1}{6}(1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1}) \Bigg) \\
&\quad - 1^T N 1 \\
&= m \Bigg(\frac{2}{3n_1} Kf(G_1) + \frac{m_1}{2} + \frac{1}{6} \sum_{i<j, i,j \in E(G_1)} [(2L_{G_1}^{\#})_{ii} \\
&\quad + (2L_{G_1}^{\#})_{jj} - r_{ij}(G_1)] + tr((L_2 + E_{n_2} - \\
&\quad \frac{1}{n_2+2} j_{n_2})^{-1} \otimes E_{m_1}) + tr(\frac{1}{6}(1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} \\
&\quad R_1 (1_{n_2}^T \otimes E_{m_1})) - 1^T N 1 \\
&= m \Bigg(\frac{2}{3n_1} Kf(G_1) + \frac{m_1}{2} + \frac{1}{6} tr(D_{G_1} L_{G_1}^{\#}) - \frac{n_1 - 1}{6} \\
&\quad + tr((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1}) + \\
&\quad tr(\frac{1}{6}(1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1}) \Bigg) - 1^T N 1.
\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}
&\left((L_2 + E_{n_2} - \tfrac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1} \right)^{-1} \\
&= (L_{G_2} + E_{n_2})^{-1} + \frac{1}{2} j_{n_3 \times n_3}.
\end{aligned}$$

Note that the eigenvalues of $(L_{G_2} + I_{n_2})$ are $\mu_1(G_2) + 1, \mu_2(G_2) + 1, ..., \mu_{n_2}(G_2) + 1$. Then

$$\begin{aligned}
&tr((L_2 + E_{n_2} - \tfrac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1}) \\
&= m_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2)+1}. \qquad (2)
\end{aligned}$$

Next, we calculate the $1^T(L_G^{(1)})1$. By Lemma 2.2, $L_G^{\#} 1 = 0$, then

$$\begin{aligned}
&1^T N 1 \\
&= \frac{m_1}{2} + \frac{1}{6} 1^T \left(R_1^T(G_1) L_{G_1}^{\#} R(G_1) \right) 1 + 1^T \Bigg(\frac{1}{6} R_1^T \\
&\quad L_{G_1}^{\#} (1_{n_2}^T \otimes E_{m_1}) \Bigg) 1 + 1^T \left(\frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \right) 1 + \frac{1}{6} \\
&\quad 1^T ((1_{n_2} \otimes E_{m_1}) R_1^T L_G^{\#} R_1) 1 + \frac{1}{2} 1^T (1_{n_2} \otimes E_{m_1}) 1 \\
&\quad + 1^T ((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1}) 1 \\
&\quad + \frac{1}{6} 1^T ((1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})) 1.
\end{aligned}$$

Note that $R1 = \pi$, where $\pi = (d_1, d_2, ..., d_n)$, then

$$\begin{aligned}
&1^T N 1 \\
&= \frac{m_1}{2} + \frac{1}{6} \pi^T L_{G_1}^{\#} \pi + \frac{1}{6} 1^T \left(R_1^T L_{G_1}^{\#} (1_{n_2}^T \otimes E_{m_1}) \right) 1 \\
&\quad + 1^T \left(\frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \right) 1 + \frac{1}{6} 1^T ((1_{n_2} \otimes E_{m_1}) R_1^T \\
&\quad L_G^{\#} R_1) 1 + \frac{1}{2} 1^T (1_{n_2} \otimes E_{m_1}) 1 + 1^T ((L_2 + E_{n_2} \\
&\quad - \frac{1}{n_2+2} j_{n_2})^{-1} \otimes E_{m_1}) 1 + \frac{1}{6} 1^T ((1_{n_2} \otimes E_{m_1}) R_1^T \\
&\quad L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})) 1.
\end{aligned}$$

Since $1^T R^T = R1 = r_1 \cdot 1$, then

$$\begin{aligned}
&1^T ((1_{n_2} \otimes E_{m_1}) R_1^T L_{G_1}^{\#} R_1 (1_{n_2}^T \otimes E_{m_1})) 1 \\
&= \begin{pmatrix} 1_{m_1}^T & 1_{m_1}^T & \cdots & 1_{m_1}^T \end{pmatrix} \begin{pmatrix} E_{m_1} \\ E_{m_1} \\ \cdots \\ E_{m_1} \end{pmatrix} \\
&\quad R_1^T L_{G_1}^{\#} R_1 \begin{pmatrix} E_{m_1} & E_{m_1} & \cdots & E_{m_1} \end{pmatrix} \\
&\quad \begin{pmatrix} 1_{m_1} \\ 1_{m_1} \\ \cdots \\ 1_{m_1} \end{pmatrix} = m_1^2 1_{m_1}^T R_1^T L_{G_1}^{\#} R_1 1_{m_1} = 0.
\end{aligned}$$

$$\begin{aligned}
&1^T R_1^T L_{G_1}^{\#} (1_{n_2}^T \otimes E_{m_1}) 1 \\
&= 1^T R_1^T L_{G_1}^{\#} \begin{pmatrix} E_{m_1} & E_{m_1} & \cdots & E_{m_1} \end{pmatrix} \\
&\quad \begin{pmatrix} 1_{m_1} \\ 1_{m_1} \\ \cdots \\ 1_{m_1} \end{pmatrix} \\
&= 1^T R_1^T L_{G_1}^{\#} 1_{m_1} = 0. \qquad (3)
\end{aligned}$$

Similarly, $1^T \left((1_{n_2} \otimes E_{m_1}) R_1^T L_G^{\#} R_1 \right) 1 = 0$, $1^T \left(\frac{1}{2} 1_{n_2}^T \otimes E_{m_1} \right) 1 1^T \left(\frac{1}{2} 1_{n_2} \otimes E_{m_1} \right) 1 = 0$.
Let $P = \left((L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \otimes E_{m_1} \right) \otimes I_n$, then

$$\begin{aligned}1^TP^{-1}1 &= \begin{pmatrix} 1_{n_2}^T & 1_{n_2}^T & \cdots & 1_{n_2}^T \end{pmatrix} \\ &\quad \begin{pmatrix} P^{-1} & 0 & 0 & ... & 0 \\ 0 & P^{-1} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & P^{-1} \end{pmatrix} \begin{pmatrix} 1_{n_2} \\ 1_{n_2} \\ \cdots \\ 1_{n_2} \end{pmatrix} \\ &= n_2 1_{n_2}^T (L_2 + E_{n_2} - \frac{1}{n_2+2} j_{n_2\times n_2})^{-1} 1_{n_2} \\ &= n_2^2 \frac{n_2+2}{2}. \qquad (4)\end{aligned}$$

Plugging $(2), (3)$ and (4) into $Kf(L_{G_1 \odot G_2})$, we obtain the required result in $vi)$.

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