Necessary and Sufficient Conditions for Oscillation of Delay Fractional Differential Equations

Xulong Wang, Xiangyu Wang and Anping Liu

***Abstract*—In this work, we obtain necessary and sufficient conditions for the oscillation of all solutions of delay fractional differential equation of the form**

$$T_\alpha(T_\alpha y(t)) + q(t)y^\gamma(\tau(t)) = 0,$$

where $0 < \alpha \leq 1$, $0 < \gamma = \frac{\text{odd integer}}{\text{odd integer}}$. Furthermore, we supplement the theoretical aspects with numerical simulations and illustrations.

***Index Terms*—oscillation, fractional differential equation, necessary and sufficient condition, delay.**

I. Introduction

ARBITRARY order differential and integration generalizations of integer order derivatives and $n-$fold integrals. The history goes back to the derivative of order $\alpha = 1/2$ proposed by Leibniz. Fractional calculus has garnered phenomenal interest because of its various applications in multiple areas of science and engineering ranging from electric circuits, signal and image processing to viscoelasticity, industrial robotics and numerous other branches of both physical and biological sciences [1]–[3]. Furthermore, fractional calculus can also provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a 'memory' term in the model [4]–[7].

The study of oscillation behavior as a part of the qualitative theory of the solutions for various including ordinary and partial differential equations, dynamic equations on time scales, difference equations is an exciting field of research with a broad range of applications. However, to the best of our knowledge, there are few results on the oscillation of fractional differential equations. We refer to [8]–[17] and the references therein.

In 2019, Feng et al. [18] studied the oscillation behavior of the following fractional differential equations

$$T_\alpha^{t_0} x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0,\ t \geq t_0,$$

$$\begin{aligned}&T_\alpha^{t_0}(r(t)(T_\alpha^{t_0} x(t) + p(t)x(\tau(t)))^\beta)\\&+q(t)x^\beta(\sigma(t)) = 0,\ t \geq t_0,\end{aligned}$$

and

$$\begin{aligned}&T_\alpha^{t_0}(r_2 T_\alpha^{t_0}(r_1(T_\alpha^{t_0} y)^\beta))(t) + p(t)(T_\alpha^{t_0} y(t))^\beta\\&+q(t)f(y(g(t))) = 0,\ t \geq t_0,\end{aligned}$$

where T_α denotes the conformable differential operator of order α, $0 < \alpha \leq 1$. The authors generalized the oscillatory criterion of ordinary differential equations to conformable fractional derivative.

In 2020, Zheng and Feng [19] are concerned with oscillation of a class of fractional differential equations with damping term as follows

$$\begin{aligned}&D_t^\alpha(r(t)D_t^\alpha x(t)) + p(t)D_t^\alpha x(t) + q(t)f(x(t)) = 0,\\&t \geq t_0,\ 0 < \alpha < 1,\end{aligned}$$

where $D_t^\alpha(\cdot)$ denotes the conformable fractional derivative with respect to the variable t. Based on certain Riccati transformations, inequality and integration average technique, the authors obtained some sufficient conditions. In [20], [21], the authors investigated oscillation of the following two fractional differential equations

$$D_a^\alpha x(t) + q(t)f(x(t)) = 0$$

and

$$(D_{0+}^{1+\alpha} y)(t) + p(t)(D_{0+}^\alpha y)(t) + q(t)f(y(t)) = g(t)$$

where the fractional derivative is defined in the sense of the Riemann-Liouville derivative.

In this paper, we study the oscillation criteria of conformable fractional differential equation,

$$T_\alpha(T_\alpha y(t)) + q(t)y^\gamma(\tau(t)) = 0. \tag{1}$$

A solution $y(t)$ is called oscillatory if and only if it has arbitrarily large zeros on $[0, \infty)$. An equation is said to be oscillatory if all its solutions of this equation are oscillatory.

The paper is organized as follows. In Sect. 2, we introduce some notation and definitions of conformable fractional integrals. In Sect.3, we present the main theorems on 2α-order equations. Finally, we give some examples and simple numerical simulation to supplement the theoretical analysis.

II. Brief on Conformable Fractional Calculus

Definition 2.1. [22] Given a function $f : [0, \infty) \to R$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} T_\alpha f(t)$ exists, then we define $T_\alpha f(0) = \lim_{t \to 0^+} T_\alpha f(t)$.

Manuscript received April 21, 2022; revised December 15, 2022. This research was partially supported by grants from the National Natural Science Foundation of China, No. 41630643.

Xulong Wang (Corresponding author) is a postgraduate student of School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China. (e-mail: wang_xulong@qq.com).

Xiangyu Wang is a postgraduate student of School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China. (e-mail: 2587301449@qq.com).

Anping Liu is a professor of School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China. (e-mail: wh_apliu@sina.com).

Definition 2.2. [22] The Conformable fractional integral operator of order α, $\alpha \in (0,1)$, of a function f is defined as

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is defined in the sense of the improper Riemann integral.

Lemma 2.1. [22] Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then
(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ for all $a,\ b \in R$.
(2) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in R$.
(3) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(6) If f is differentiable, then $T_\alpha f(t) = t^{1-\alpha}f'(t)$.

Lemma 2.2. [22] (Rolle's Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a,b] \to R$ be a given function that satisfies
(i). f is continuous on $[a,b]$,
(ii). f is α-differentiable for some $\alpha \in (0,1)$,
(iii). $f(a) = f(b)$.
Then, there exists $c \in (a,b)$, such that $T_\alpha f(c) = 0$.

Lemma 2.3. [22] (Mean Value Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a,b] \to R$ be a given function that satisfies
(i). f is continuous on $[a,b]$,
(ii). f is α-differentiable for some $\alpha \in (0,1)$,
Then, there exists $c \in (a,b)$, such that

$$T_\alpha f(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}.$$

Lemma 2.4. [23] Let $a > 0$ and $f : [a,b] \to R$ be a given function that satisfies
(i). f is continuous on $[a,b]$,
(ii). f is α-differentiable for some $\alpha \in (0,1)$,
If $T_\alpha f(t) = 0$ for all $t \in (a,b)$, then f is a constant on $[a,b]$.

Lemma 2.5. [23] Let $a > 0$ and $f : [a,b] \to R$ be a given function that satisfies
(i). f is continuous on $[a,b]$,
(ii). f is α-differentiable for some $\alpha \in (0,1)$,
Then we have the following:
1) If $T_\alpha f(t) > 0$ for all $t \in (a,b)$, then f is increasing on $[a,b]$.
2) If $T_\alpha f(t) < 0$ for all $t \in (a,b)$, then f is decreasing on $[a,b]$.

III. 2α-ORDER CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

In this section, we mainly consider fractional differential equations of the form

$$T_\alpha(T_\alpha y(t)) + q(t)y^\gamma(\tau(t)) = 0,\ t \geq t_0 > 0, \qquad (2)$$

where T_α denotes the conformable differential operator of order $\alpha \in (0,1]$. $q(t)$ and $\tau(t)$ are real-valued and continuous on $[t_0,\infty)$. The constant γ is the ratio of odd integers and satisfies $\gamma > 0$. In this paper, $y(t)$ is differentiable on $[t_0,\infty)$. We make the following assumptions:
(A1) $q(t) \geq 0$ for sufficiently large t, $q(t)$ is not identically zero in any neighborhood of infinity.
(A2) $\tau(t) \leq t,\ \lim_{t\to\infty} \tau(t) = \infty$.

Lemma 3.1. Assume that (A1) and (A2) hold. If $y(t)$ is an eventually positive solution of (2), then there exists a constant $T > t_0$ such that

$$y(t) > 0,\ T_\alpha y(t) > 0 \text{ and } T_\alpha(T_\alpha y(t)) \leq 0,\ t \geq T.$$

Proof. Since $y(t)$ is an eventually positive solution of (2), we can obtain that there exists a T, $T > t_0$, such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq T$. From (2), it follows that

$$T_\alpha(T_\alpha y(t)) = -q(t)y^\gamma(\tau(t)) \leq 0 \text{ for } t \geq T.$$

Therefore, by Lemma 2.5, $T_\alpha y(t)$ is nonincreasing. We claim that $T_\alpha y(t) > 0$ for $t \geq T$. We argue by contradiction. Assume that $T_\alpha y(t) \leq 0$ for $t \geq T$ and consider the following scenarios:

- If $T_\alpha y(t) = 0$ for $t \geq T$ we can obtain $T_\alpha(T_\alpha y(t)) = 0$ for $t \geq T$. This is obviously impossible.
- If $T_\alpha y(t) < 0$ for $t \geq T$, then we can find $t^* \geq T$ and $c_1 > 0$ such that $T_\alpha y(t) \leq -c_1$ for all $t \geq t^*$. Integrating the inequality $y'(t) \leq -c_1 t^{\alpha-1}$, from t^* to $t(t > t^*)$, we obtain
$$y(t) \leq y(t^*) + \frac{c_1}{\alpha}(t^{*\alpha} - t^\alpha) \to -\infty,$$
as $t \to \infty$. This contradicts $y(t)$ being a positive solution.

Therefore we obtain $T_\alpha y(t) > 0$ for $t \geq T$ and conclude the proof.

Lemma 3.2. Let (A2) holds and assume $y(t) \in C^{2\alpha}[T,\infty)$ satisfies

$$y(t) > 0,\ T_\alpha y(t) > 0 \text{ and } T_\alpha(T_\alpha y(t)) \leq 0,\ t \geq T.$$

Then for each $k_1 \in (0,1)$ there is a $T_{k_1} \geq T$ such that

$$y(\tau(t)) \geq \frac{k_1(\tau(t))^\alpha}{t^\alpha}y(t),\ t \geq T_{k_1}.$$

Proof. It suffices to consider only those t for which $\tau(t) < t$. Then we have for $t > \tau(t) \geq T$, $y(t) - y(\tau(t)) \leq T_\alpha y(\tau(t))(\frac{1}{\alpha}t^\alpha - \frac{1}{\alpha}(\tau(t))^\alpha)$ by Lemma 2.3 and the monotonicity of $T_\alpha y(t)$. Hence,

$$\frac{y(t)}{y(\tau(t))} \leq 1 + \frac{T_\alpha y(\tau(t))}{y(\tau(t))}(\frac{1}{\alpha}t^\alpha - \frac{1}{\alpha}(\tau(t))^\alpha),\ t > \tau(t) \geq T.$$

Moreover, we can obtain by using $y(\tau(t)) - y(T) \geq T_\alpha y(\tau(t))(\frac{1}{\alpha}(\tau(t))^\alpha - \frac{1}{\alpha}T^\alpha)$ that, for any $k_1 \in (0,1)$, there is a $T_{k_1} \geq T$ such that

$$\frac{y(\tau(t))}{T_\alpha y(\tau(t))} \geq k_1 \frac{1}{\alpha}(\tau(t))^\alpha,\ t \geq T_{k_1}.$$

Hence, from the two inequalities above, we obtain

$$\frac{y(t)}{y(\tau(t))} \leq \frac{t^\alpha}{k_1(\tau(t))^\alpha},\ t \geq T_{k_1}.$$

This completes the proof.

Lemma 3.3. Let $y(t) \in C^{2\alpha}[T,\infty)$, satisfies

$$y(t) > 0,\ T_\alpha y(t) > 0 \text{ and } T_\alpha(T_\alpha y(t)) \leq 0,\ t \geq T.$$

Then for each $k_2 \in (0,1)$ there is a $T_{k_2} \geq T$ such that

$$y(t) \geq k_2 t^\alpha T_\alpha y(t),\ t \geq T_{k_2}.$$

Proof. Suppose $t > T$. Then by Lemma 2.3 we have

$$y(t) - y(T) = T_\alpha y(c)(\frac{1}{\alpha}t^\alpha - \frac{1}{\alpha}T^\alpha),\ \text{for some } c \in (T, t).$$

From this we obtain

$$y(t) \geq T_\alpha y(t)(\frac{1}{\alpha}t^\alpha - \frac{1}{\alpha}T^\alpha).$$

Now for any $k_2 \in (0, 1)$, there is a $T_{k_2} = \frac{T}{(1-k\alpha)^{\frac{1}{\alpha}}} > T$ such that $t \geq T_{k_2}$,

$$y(t) \geq k_2 t^\alpha T_\alpha y(t).$$

The proof is complete.

Theorem 3.1. Let $0 < \gamma < 1$. Equation (2) is oscillatory if and only if

$$\int_T^{+\infty} \frac{q(t)\tau^{\alpha\gamma}(t)}{t^{1-\alpha}}dt = +\infty,\ \text{for all } T > t_0. \tag{3}$$

Proof. We first show that condition (3) is sufficient. Assume that there exists a solution $y(t)$ to equation (2) which is not zero for large t. Since $-y(t)$ is also a solution, we can assume that $y(t) > 0$ for large t. By Lemma 3.1, there exists a $T > t_0$ such that $y(t) > 0$, $T_\alpha y(t) > 0$ and $T_\alpha(T_\alpha y(t)) \leq 0$ for $t \geq T$, and meanwhile, we assume that $q(t) \geq 0$ for $t \geq T$. Multiply both sides of equation (2) by $(T_\alpha y(t))^{-\gamma}$ and use Lemma 3.2 and Lemma 3.3, we obtain

$$\frac{T_\alpha(T_\alpha y(t))}{(T_\alpha y(t))^\gamma} + (k_1k_2)^\gamma q(t)\tau^{\alpha\gamma}(t) \leq 0,\ t \geq t_1 \geq T. \tag{4}$$

Integrating both sides of inequality (4), we see that

$$I_\alpha^{t_1}(\frac{T_\alpha(T_\alpha y(x))}{(T_\alpha y(x))^\gamma}) + (k_1k_2)^\gamma I_\alpha^{t_1}(q(x)\tau^{\alpha\gamma}(x)) \leq 0. \tag{5}$$

From (5) it follows that

$$\begin{aligned} &\frac{1}{1-\gamma}[(T_\alpha y(t))^{1-\gamma} - (T_\alpha y(t_1))^{1-\gamma}] + \\ &(k_1k_2)^\gamma \int_{t_1}^t \frac{q(x)\tau^{\alpha\gamma}(x)}{x^{1-\alpha}}dx \leq 0. \end{aligned} \tag{6}$$

Consequently the left hand side term of (6) is bounded by terms which are either constants or positive. But since condition (3) holds we eventually have a contradiction and hence equation (2) is oscillatory.

In order to prove necessity we only need to construct a non-oscillatory solution on some half-line $[t_1, \infty)$, $t_1 > t_0$. Suppose that $\int^{+\infty} \frac{q(t)\tau^{\alpha\gamma}(t)}{t^{1-\alpha}}dt < +\infty$. Choose t_1 so large that

$$\int_{t_1}^{+\infty} \frac{q(t)\tau^{\alpha\gamma}(t)}{t^{1-\alpha}}dt < \frac{\alpha^\gamma}{2}. \tag{7}$$

Consider the solution $y(t)$ which is defined by the initial data

$$T_\alpha y(t_1) = 1,\ y(t) = 0,\ t \leq t_1. \tag{8}$$

We claim that this solution dose not vanish on $[t_1, \infty)$ and argue by contradiction. If, on the contrary, $y(t_2) = 0$ for some $t_2 > t_1$ (assume that there is no other zero point between t_1 and t_2) then, by Lemma 2.2, there must be some point $c \in (t_1, t_2)$ for which $T_\alpha y(c) = 0$. However, this will be in contradiction to the following fact: the function $T_\alpha y(t)$ can never vanish on $[t_1, t_2)$. According to the initial data and (A2) we obtain $T_\alpha(T_\alpha y(t)) \leq 0$ on (t_1, t_2), by integrating this term twice we have

$$y(t) \leq \frac{1}{\alpha}(t^\alpha - {t_1}^\alpha) \leq \frac{1}{\alpha}t^\alpha,\ t_1 \leq t \leq t_2. \tag{9}$$

From equation (2),

$$\begin{aligned} T_\alpha y(t) &= 1 - \int_{t_1}^t \frac{q(x)y^\gamma(\tau(x))}{x^{1-\alpha}}dx \\ &\geq 1 - \frac{1}{\alpha^\gamma}\int_{t_1}^\infty \frac{q(x)\tau^{\alpha\gamma}(x)}{x^{1-\alpha}}dx \geq \frac{1}{2}. \end{aligned} \tag{10}$$

Hence $T_\alpha y(t)$ never vanishes and the proof of this Theorem is complete.

Theorem 3.2. Let $\gamma = 1$. Equation (2) is oscillatory if

$$\lim_{t\to+\infty} \sup t^\alpha \int_t^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds > 1. \tag{11}$$

Proof. Suppose that (11) does not hold, then there exists a non-oscillatory solution $y(t)$ of (2). Without loss of generality we may assume that $y(t)$ is eventually positive. By Lemma 3.1, there exists a constant $T > 0$ such that $y(t) > 0$, $T_\alpha y(t) > 0$ and $T_\alpha(T_\alpha y(t)) \leq 0$ for $t \geq T$. Multiply both terms of equation (2) by $t^{\alpha-1}$ and integrate from t to $+\infty$, $t \geq T$, we have

$$T_\alpha y(t) \geq \int_t^{+\infty} \frac{q(s)y(\tau(s))}{s^{1-\alpha}}ds. \tag{12}$$

According to Lemma 3.3 there exists a number $t_2 \geq T$ and $k_2 \in (0, 1)$ such that $y(t) \geq k_2 t^\alpha T_\alpha y(t)$ for $t \geq t_2$. Then the inequality (12) yields

$$y(t) \geq k_2 t^\alpha \int_t^{+\infty} \frac{q(s)y(\tau(s))}{s^{1-\alpha}}ds,\ t \geq t_2. \tag{13}$$

Using Lemma 3.2 and (13), we obtain

$$y(t) \geq k^2 t^\alpha \int_t^{+\infty} \frac{q(s)\tau^\alpha(s)y(s)}{s}ds,\ t \geq t_3 \geq t_2, \tag{14}$$

where $k = \min\{k_1, k_2\}$. Since the function $y(t)$ is positive and increasing, it follows from the above inequality (14) that

$$1 \geq k^2 t^\alpha \int_t^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds. \tag{15}$$

From (15) it follows that

$$\lim_{t\to+\infty} \sup t^\alpha \int_t^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds < +\infty. \tag{16}$$

If we suppose

$$\lim_{t\to+\infty} \sup t^\alpha \int_t^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds = l \tag{17}$$

and suppose that (11) holds, then there exists a sequence of points $\{b_p\}$ such that $\lim_{p\to+\infty} b_p = +\infty$ and

$$\lim_{p\to+\infty} \sup b_p^\alpha \int_{b_p}^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds = l > 1. \tag{18}$$

So for $\varepsilon = \frac{l-1}{2} > 0$ there exists a number P such that for every $p > P$ we have

$$b_p^\alpha \int_{b_p}^{+\infty} \frac{q(s)\tau^\alpha(s)}{s}ds > \frac{l+1}{2}. \tag{19}$$

Now if we choose $p > P$ so that $b_p \geq t_3$ and moreover, numbers $k_1,\ k_2 \in (0,1)$ such that $\sqrt{\frac{2}{l+1}} < k < 1$, then (19) implies

$$\begin{aligned} k^2 b_p^\alpha \int_{b_p}^{+\infty} \frac{q(s)\tau^\alpha(s)}{s} ds &> \frac{l+1}{2} k^2 \\ &> \frac{l+1}{2} \cdot \frac{2}{l+1} = 1, \end{aligned} \tag{20}$$

which contradicts (15). This completes the proof.

Theorem 3.3. Let $\gamma > 1$, assume $\tau(t) = t - \theta(t)$, $0 < \theta(t) \leq M$(M is a constant) and $\theta(t)$ is continuous function on $[t_0, \infty)$. Equation (2) is oscillatory if and only if

$$\int_T^{+\infty} \frac{q(s)}{s^{1-2\alpha}} ds = +\infty, \text{ for all } T > t_0. \tag{21}$$

Proof. To prove sufficiency, assume that $y(t)$ is a nonoscillatory solution of (2). Without loss of generality we may assume that $y(t)$ is eventually positive. By Lemma 3.1, there exists a constants $T > M$ such that $y(t) > 0$, $T_\alpha y(t) > 0$ and $T_\alpha(T_\alpha y(t)) \leq 0$ for $t \geq T$. We multiply both terms of (2) by $t^\alpha y^{-\gamma}(\tau(t))$ and integrate,

$$I_\alpha^c \left[t^\alpha \frac{T_\alpha[T_\alpha y(t)]}{y^\gamma(\tau(t))} \right] + I_\alpha^c(q(t)t^\alpha) = 0, \tag{22}$$

where c is fixed and $c \gg T$. Since $y(t)$ is increasing we see that $y(t-M) < y(\tau(t))$ and hence

$$I_\alpha^c \left[t^\alpha \frac{T_\alpha[T_\alpha y(t)]}{y^\gamma(t-M)} \right] + I_\alpha^c(q(t)t^\alpha) \leq 0. \tag{23}$$

We can then obtain from integration by parts,

$$\begin{aligned} & t^\alpha \frac{T_\alpha y(t)}{y^\gamma(t-M)} \bigg|_c^t - I_\alpha^c(T_\alpha y(t) T_\alpha(\frac{t^\alpha}{y^\gamma(t-M)})) \\ & \qquad + I_\alpha^c(q(t)t^\alpha) \leq 0 \\ & t^\alpha \frac{T_\alpha y(t)}{y^\gamma(t-M)} \bigg|_c^t - I_\alpha^c(\alpha y^{-\gamma}(t-M) T_\alpha y(t)) \\ & \quad + I_\alpha^c(\gamma y^{-\gamma-1}(t-M) y'(t-M) t T_\alpha y(t)) \\ & \qquad + I_\alpha^c(q(t)t^\alpha) \leq 0. \end{aligned} \tag{24}$$

From (24),

$$\begin{aligned} & I_\alpha^c(\alpha y^{-\gamma}(t-M) T_\alpha y(t)) \\ & \quad = \int_c^t s^{\alpha-1} \alpha y^{-\gamma}(s-M)(T_\alpha y)(s) ds \\ & \quad \leq \int_c^t s^{\alpha-1} \alpha y^{-\gamma}(s-M)(T_\alpha y)(s-M) ds \\ & \quad \leq \int_c^t \alpha y^{-\gamma}(s-M) y'(s-M) ds \\ & \quad = \frac{\alpha}{1-\gamma} y^{-\gamma+1}(s-M) \bigg|_c^t, \end{aligned}$$

then we obtain

$$\begin{aligned} & t^\alpha \frac{T_\alpha y(t)}{y^\gamma(t-M)} \bigg|_c^t + \frac{\alpha}{\gamma-1} y^{-\gamma+1}(s-M) \bigg|_c^t + \\ & \quad I_\alpha^c(\gamma y^{-\gamma-1}(t-M) y'(t-M) t T_\alpha y(t)) \\ & \qquad + I_\alpha^c(q(t)t^\alpha) \leq 0. \end{aligned} \tag{25}$$

Consequently the left side of (25) are either constant or positive. But since condition (21) holds we eventually have a contradiction as $t \to +\infty$ and equation (2) is oscillatory.

Next, we show that (21) is necessary. Assume that (21) does not hold; so there exists $T > t_0$ such that

$$\int_T^{+\infty} \frac{q(s)}{s^{1-2\alpha}} ds < \frac{\alpha}{2\gamma}. \tag{26}$$

Define

$$S = \left\{ y \in C([t_0, \infty), R) : 1 - \tfrac{1}{2\gamma} \leq y(t) \leq 1, t \geq t_0 \right\},$$
$$\|y(t)\| = \sup\{|y(t)| : t \geq t_0\}.$$

Let $\Phi : S \to S$ be such that

$$(\Phi y)(t) = \begin{cases} (\Phi y)(T), & t \in [t_0, T], \\ 1 - \dfrac{1}{\alpha} \displaystyle\int_t^{+\infty} (s^\alpha - t^\alpha) \frac{q(s) y^\gamma(\tau(s))}{s^{1-\alpha}} ds, & t \geq T. \end{cases}$$

For every $y \in S$, $(\Phi y)(t) \leq 1$ and $(\Phi y)(t) \geq 1 - \frac{1}{2\gamma}$ implies that $(\Phi y)(t) \in S$. Now for $y_1,\ y_2 \in S$, we have

$$\begin{aligned} & |(\Phi y_1) - (\Phi y_2)| \\ & = \left| \int_t^{+\infty} \frac{1}{\alpha}(s^\alpha - t^\alpha) \frac{q(s)}{s^{1-\alpha}} [{y_1}^\gamma(\tau(s)) - {y_2}^\gamma(\tau(s))] ds \right| \\ & \leq \int_t^{+\infty} \frac{1}{\alpha}(s^\alpha - t^\alpha) \frac{q(s)}{s^{1-\alpha}} ds \cdot \|y_1(t) - y_2(t)\| \gamma \\ & \leq \int_t^{+\infty} \frac{1}{\alpha} \frac{q(s)}{s^{1-2\alpha}} ds \cdot \|y_1(t) - y_2(t)\| \gamma \\ & \leq \frac{1}{2} \|y_1 - y_2\|. \end{aligned}$$

Therefore, $|(\Phi y_1) - (\Phi y_2)| \leq \frac{1}{2} \|y_1 - y_2\|$ implies that Φ is a contraction. It follows, by Banach's contraction mapping principle, that Φ has a unique fixed point $y(t) \in S$ which is a nonoscillatory solution. Hence, (21) is the necessary condition for oscillation. This completes the proof of the theorem 3.3.

IV. Example

Example 4.1. Consider the following fractional differential equation

$$T_{\frac{7}{9}}[T_{\frac{7}{9}} y(t)] + (1 + \sin t) y^{\frac{1}{3}}(\frac{99t}{100}) = 0 \tag{27}$$

and

$$T_{\frac{1}{3}}[T_{\frac{1}{3}} y(t)] + \frac{1}{t} y^{\frac{1}{3}}(t) = 0. \tag{28}$$

By Theorem 3.1, for $\forall T > t_0$ we have

$$\begin{aligned} & \int_T^{+\infty} \frac{(1 + \sin(t)) \left(\frac{99t}{100}\right)^{\frac{7}{27}}}{t^{\frac{2}{9}}} dt \\ & = \left(\frac{99}{100}\right)^{\frac{7}{27}} \int_T^{+\infty} (1 + \sin(t)) t^{\frac{1}{27}} dt = +\infty, \\ & \int_T^{+\infty} \frac{t^{-1} t^{\frac{1}{9}}}{t^{\frac{2}{3}}} dt = \frac{9}{5} T^{-\frac{5}{9}} < +\infty, \end{aligned}$$

then equation (27) is oscillatory and (28) is not. Taking the initial condition $\varphi(t) = t$ for $t \leq 1$ and $t_0 = 1$. The oscillatory behavior of (27) and (28) is illustrated in Fig.1 and Fig.2.

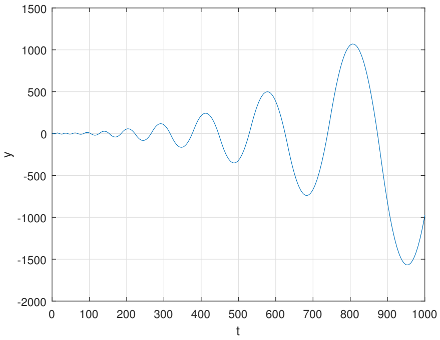

Fig. 1. The Oscillatory Behavior of Solutions for Equation (27).

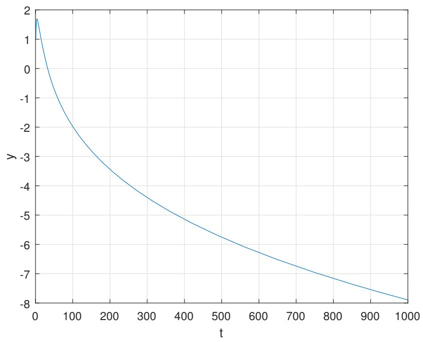

Fig. 2. The Oscillatory Behavior of Solutions for Equation (28).

Example 4.2. Consider the following fractional differential equation

$$T_{\frac{1}{3}}[T_{\frac{1}{3}}y(t)] + t^{\frac{1}{3}}y^3(t-0.2) = 0 \tag{29}$$

and

$$T_{\frac{1}{3}}[T_{\frac{1}{3}}y(t)] + t^{-2}y^3(t-0.2) = 0. \tag{30}$$

By Theorem 3.3, $\forall T > t_0$, we have

$$\int_T^{+\infty} \frac{s^{\frac{1}{3}}}{s^{\frac{1}{3}}} ds = +\infty,$$
$$\int_T^{+\infty} \frac{s^{-2}}{s^{\frac{1}{3}}} ds = \frac{3}{4}T^{-\frac{4}{3}} < +\infty,$$

then equation (29) is oscillatory while (30) is not. Taking the initial condition $\varphi(t) = -1$ for $t \leq 5$ and $t_0 = 5$. The oscillatory behavior of (29) and (30) is illustrated in Fig.3 and Fig.4.

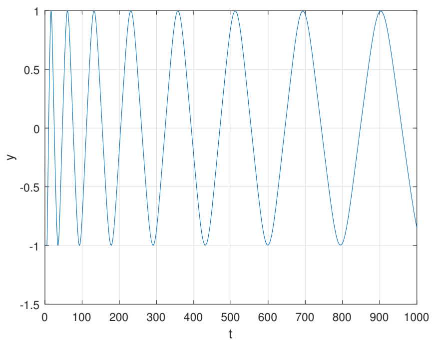

Fig. 3. The Oscillatory Behavior of Solutions of Equation (29).

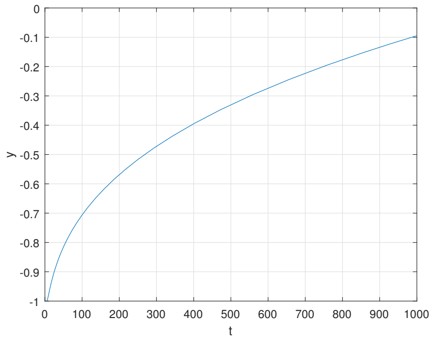

Fig. 4. The Oscillatory Behavior of Solutions of Equation (30).

Example 4.3. Consider the following fractional differential equation

$$T_1[T_1y(t)] + \frac{3}{t^2}y(t) = 0. \tag{31}$$

By Theorem 3.2, we have

$$\lim_{t\to+\infty} \sup t\int_t^{+\infty} \frac{3}{s^2}ds = 3 > 1,$$

then equation (31) is oscillatory. Taking the initial condition $y(1) = 0, y'(1) = 1$ and $t_0 = 1$. The equation can be solved by Matlab as $y = \frac{2}{\sqrt{11}}\sqrt{t}\sin(\frac{\sqrt{11}\ln t}{2})$, which satisfies the conditions of Theorem 3.2 and is therefore oscillatory. The oscillatory behavior of (31) is illustrated in Fig.5.

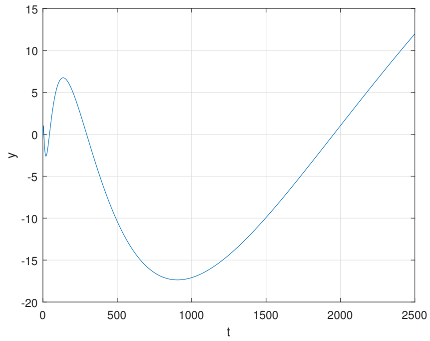

Fig. 5. The Oscillatory Behavior of Solutions of Equation (31).

References

[1] M. S. Osman, "New analytical study of water waves described by coupled fractional variant boussinesq equation in fluid dynamics," *Pramana - Journal of Physics*, vol. 93, no. 2, p. 26, 2019.

[2] B. Ghanbari and A. Atangana, "A new application of fractional atangana–baleanu derivatives: Designing abc-fractional masks in image processing," *Physica A: Statistical Mechanics and its Applications*, vol. 542, p. 123516, 2020.

[3] D. Xue, *Fractional-Order Control Systems: Fundamentals and Numerical Implementations*, Berlin, Boston: De Gruyter, 2017.

[4] A. D. Freed and K. Diethelm, "Fractional calculus in biomechanics: A 3d viscoelastic model using regularized fractional derivative kernels with application to the human calcaneal fat pad," *Biomechanics and Modeling in Mechanobiology*, vol. 5, no. 4, pp. 203–215, 2006.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*, Singapore: World Scientific, 2000.

[6] W. G. Glöckle and T. F. Nonnenmacher, "A fractional calculus approach to self-similar protein dynamics," *Biophysical Journal*, vol. 68, no. 1, pp. 46–53, 1995.

[7] R. T. Baillie, "Long memory processes and fractional integration in econometrics," *Journal of Econometrics*, vol. 73, no. 1, pp. 5–59, 1996.

[8] Q. Feng and A. P. Liu, "Oscillatory behavior of a class impulsive fractional partial differential equation," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp. 452–457, 2020.

[9] Q. Feng and A. P. Liu, "Oscillation for a class of fractional differential equation," *Journal of Applied Mathematics and Physics*, vol. 7, no. 7, pp. 1429–1439, 2019.

[10] Q. X. Ma and A. P. Liu, "Oscillation criteria of neutral type impulsive hyperbolic equations," *Acta Mathematica Scientia*, vol. 34, no. 6, pp. 1845–1853, 2014.

[11] L. H. Erbe, Q. K. Kong, and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, New York: Routledge, 1995.

[12] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*. Dordrecht: Kluwer Academic, 1992.

[13] S. Y. Zhu, H. Li, and A. P. Liu, "Forced oscillation of nonlinear fractional delay differential equations with damping term," *In Proceedings of 2018 2nd International Conference on Applied Mathematics, Modeling and Simulation*, vol. 305, pp. 53–61, 2018.

[14] S. R. Grace, R. P. Agarwal, P. J. Y. Wong, and A. Zafer, "On the oscillation of fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 2, pp. 222–231, 2012.

[15] D. X. Chen, "Oscillation criteria of fractional differential equations," *Advances in Difference Equations*, vol. 2012, no. 1, p. 33, 2012.

[16] B. Abdalla and T. Abdeljawad, "On the oscillation of caputo fractional differential equations with mittag–leffler nonsingular kernel," *Chaos, Solitons & Fractals*, vol. 127, pp. 173–177, 2019.

[17] A. Raheem and M. D. Maqbul, "Oscillation criteria for impulsive partial fractional differential equations," *Computers & Mathematics with Applications*, vol. 73, no. 8, pp. 1781–1788, 2017.

[18] L. M. Feng and S. R. Sun, "Oscillation theorems for three classes of conformable fractional differential equations," *Advances in Difference Equations*, vol. 2019, no. 1, p. 313, 2019.

[19] B. Zheng and Q. Feng, "New oscillatory criteria for a class of fractional differential equations," *Engineering Letters*, vol. 28, no. 3, pp. 970–977, 2020.

[20] Y. Z. Wang, Z. L. Han, P. Zhao, and S. R. Sun, "On the oscillation and asymptotic behavior for a kind of fractional differential equations," *Advances in Difference Equations*, vol. 2014, no. 1, p. 50, 2014.

[21] J. C. Yang, A. P. Liu, and T. Liu, "Forced oscillation of nonlinear fractional differential equations with damping term," *Advances in Difference Equations*, vol. 2015, no. 1, p. 1–7, 2015.

[22] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[23] O. Iyiola and E. Nwaeze, "Some new results on the new conformable fractional calculus with application using d'alambert aproach," *Progress in Fractional Differentiation and Applications*, vol. 2, no. 2, pp. 115–122, 2016.