Adjacent Vertex Reducible Edge Coloring of Several Types of Joint Graphs

Jing-Wen Li*, Zhe Ding, Rong Luo

***Abstract*—For graph G (p, q), if there is a mapping $f:E(G)\to\{1,2,\cdots,k\}$, make any adjacent same-degree vertex u and v, where $S(u)=\{f(uv)\mid uv\in E(G)\}$, satisfies $S(u)=S(v)$, then it is called Adjacent Vertex Reducible Edge Coloring. A new algorithm based on stepwise optimization is designed to color graphs within 15 vertices. We found the coloring regulation of four types of joint graphs, and we proved the related theorems.**

***Index Terms*—adjacent vertex reducible edge coloring; algorithm; stepwise optimization; joint graphs**

I. Introduction

The four-color theorem is the source of the coloring problem, which has historically been a major component of graph theory[1-2] . More scholars are starting to examine vertex coloring and edge coloring as a result of the four-color conjecture problem. Thus, other conditional coloring were developed, including Total Coloring[3]. In 2006, Professor Zhong-Fu Zhang and coworkers proposed D(β)-vertex distinguishing edge coloring, which is distinguishable edge coloring with the distance between any two vertices no more than β[4]. With a distance of 1, the adjacent strong edge coloring [5-6] can be considered the vertex distinguishing edge coloring. In 2009, Professor Zhong-Fu Zhang proposed a series of new coloring concepts[7], such as adjacent vertex reducible total coloring, vertex reducible total coloring, vertex reducible edge coloring and adjacent vertex reducible edge coloring [8] - [11]. In real life, many objects are classified according to certain rules, which has a significant practical impact, such as transport networks[12] - [15]. The network can be abstracted as a random graph, then the nodes can be classified according to their importance in real life, and a reasonable scheduling plan can be designed to effectively alleviate congestion.

As a result of this study, we can solve the problem using the reducible coloring of graphs. Thus, Xiao-Hui Li[17]-[20] investigated the reducibility algorithm for random graphs in 2015 by coloring the edges (or vertices and edges) with k different colors, ensuring that all vertex with the same degree in the graph have the same set of colors, and ensuring that k is the maximum value satisfying the coloring condition, and a reasonable scheduling scheme can be designed based on the value of k. According to The Three Degrees of Influence Rule, the connection is strongest within three degrees. Therefore, we will focus on the case of a distance of one-degree distance.

With reference to the above-mentioned coloring concept, an algorithms for adjacent vertex reducible edge coloring is proposed in literature, and its advantages are verified by the experimental results. This paper summarizes and proves some theorems about the coloring regulation of several types of joint graphs.

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II. Preliminary knowledge

This paper is primarily concerned with the Adjacent Vertex Reducible Edge Coloring of joint graphs. Graphs G (V, E) are simple undirected-joint graphs with p vertices and q edges, where d(x) represents the degree of x.

Definition 1: Suppose that $G(V,E)$ is a simple graph. If there is a positive integer $k(1\le k\le |E|)$ and a mapping $f:E(G)\to\{1,2,....,k\}$. For any two adjacent vertices are $u,v\in V(G)$, when $d(u)=d(v)$, there are $S(u)=S(v)$, where $S(u)=\cup_{uw\in E(G)}\{f(uw)\}$, and $d(u)$ represents the degree of u. Then f is called the Adjacent Vertex Reducible Edge coloring, referred to as AVREC and $\chi'_{avrec}(G)=\max\{k\mid k-\text{AVREC of } G\}$ is called Adjacent Vertex Reducible Edge chromatic number.

Definition 2: Suppose that the vertex set of the star graph S_m is $\{w_0,w_1,...,w_m\}$, and the vertex set of m C_n is $\{u_{11},u_{12},...,u_{1n};u_{21},u_{22},...,u_{2n};u_{m1},u_{m2},...,u_{mn}\}$, $S_m^{C_n}$ expresses that m nodes in the star graph S_m except the center node are connected to a cycle graph. The sample diagram is shown in Fig. 1.

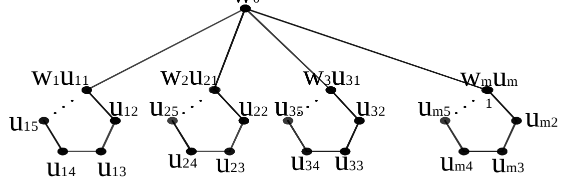


Fig. 1 $S_m^{C_n}$

Definition 3: $S_m^{C_n}\uparrow C_t$ expresses that connect the star-centered node of the joint graph $S_m^{C_n}$ to any vertex of the circle graph C_t. The sample diagram is shown in Fig. 2.

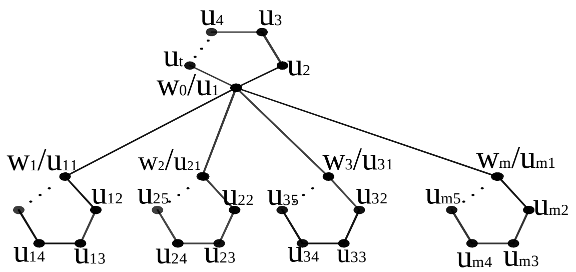
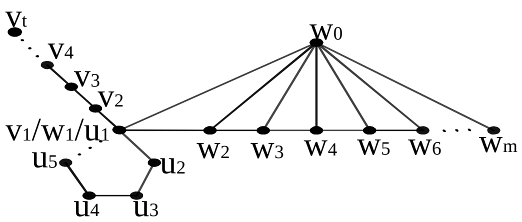
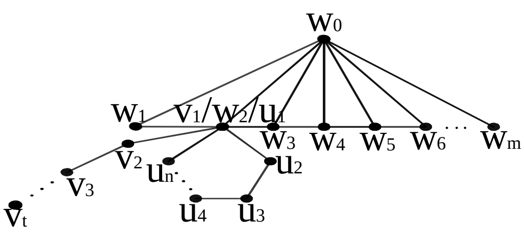


Fig. 2 $S_m^{C_n} \uparrow C_t$

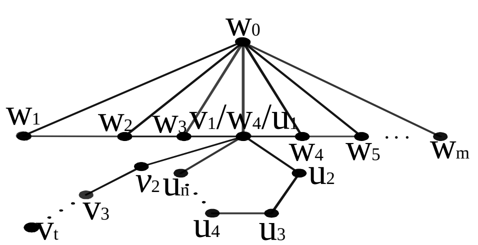
Definition 4: Connect the endpoints of the road graph to any vertex of the circle graph, and set the connection vertex as u_1/v_1, then reconnect a fan graph (non-central node) at the connection vertex, this type of graph is called $(n,t)-k \uparrow F_m(a)$, $(n,t)-k \uparrow F_m(b)$, $(n,t)-k \uparrow F_m(c)$. Some samples are shown in Fig. 3.



(a) The maximum-degree vertex of $(n,t)-k$ connects to the minimum-degree vertex of F_m, which is expressed as $(n,t)-k \uparrow F_m(a)$.



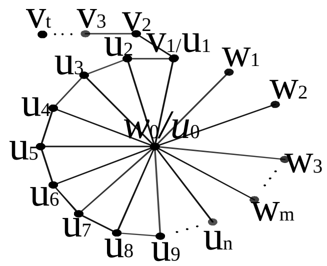
(b) The maximum-degree vertex of $(n,t)-k$ connects to the 3-degree vertex adjacent to the minimum-degree vertex of F_m, which is expressed as $(n,t)-k \uparrow F_m(b)$.



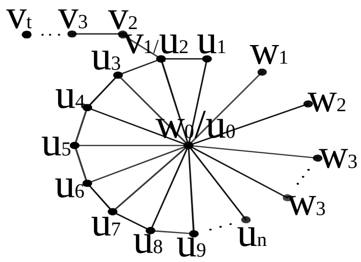
(c) The maximum-degree vertex of $(n,t)-k$ connects to the 3-degree vertex whose distance from the minimum-degree vertex is greater than or equal to 2 of F_m, which is expressed as $(n,t)-k \uparrow F_m(c)$

Fig. 3 $(n,t)-k \uparrow F_m(c)$

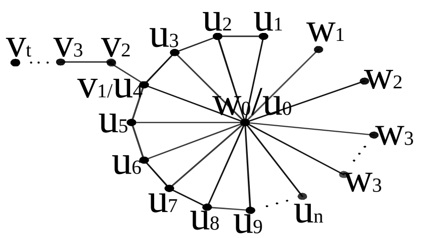
Definition 5: Connect the central node of the star graph S_m to the central node of the fan graph F_n, which is denoted as w_0/u_0. Then connect one end of P_t to any vertex of F_n (except the central node), thus this type of the joint graph is denoted as $S_m \uparrow F_n \downarrow P_t(a)$, $S_m \uparrow F_n \downarrow P_t(b)$, $S_m \uparrow F_n \downarrow P_t(c)$. Some samples are shown in Fig. 4.



(a) The minimum-degree vertex of $S_m \uparrow F_n$ is connected to the 1-degree vertex of P_t, which is expressed as $S_m \uparrow F_n \downarrow P_t(a)$



(b) The 3- degree vertex adjacent to minimum-degree vertex of $S_m \uparrow F_n$ connects the 1-degree vertex of P_t, which is expressed as $S_m \uparrow F_n \downarrow P_t(b)$.



(c) The 3-degree vertex with a distance greater than or equal to 2 from the minimum-degree vertex of the $S_m \uparrow F_n$ connects the 1-degree vertex of P_t, which is expressed as $S_m \uparrow F_n \downarrow P_t(c)$.

Fig. 4 $S_m \uparrow F_n \downarrow P_t(c)$

III. AVREC ALGORITHM

A. Basic principle

The AVREC algorithm is based on the definition of Vertex Reducible Edge Coloring. It uses the adjustment function to break the balance in turn, and then establishes iterations through the balance function, gradually tending to the optimal solution. These graphs can be colored this manner. The main procedure is as follows:

(1) Preprocessing function: input the adjacency matrix of the graph $G(p,q)$ for preprocessing. Calculate the number of edges, the degree sequence, the maximum degree in the graph $G(p,q)$, and divide the classification set by adjacent vertices.

(2) Prepare three preparatory functions before coloring:

balance function, adjustment function, continuous function, and set an intermediate matrix: Adjust.

(3) Start coloring from the first non-zero edge chromatic number of the upper triangle of the matrix. When the coloring meets the adjustment function or does not meet the continuous function, the previous chromatic number needs to be restored to the latest state, and coloring must begin from the next edge. Use the intermediate matrix Adjust to record this matrix when the adjacent vertices of the same degree meet the balance function.

(4) Output the middle balance matrix Adjust. When all the edges in the matrix cannot increase the chromatic number. And this is what we need.

B. Pseudocode

Input: The adjacency matrix of the graph $G(p,q)$

output: The adjacency matrix that satisfies the adjacent vertex reducible edge coloring

begin

solve the distance matrix M1 of the graph G according to M, and initialize a RecordMatrix

while (the edges in the graph G are not all colored)

for i $\leftarrow$ 0 to n

ei++

if (ei++ does not satisfy the condition of AVREC)

ei--

end if

if (M1) satisfies the balance function isBalance)

RecordMatrix $\leftarrow$ M1

end if

if (M1 satisfies the balance function isBalance and meets the maximum chromatic number of coloring)

output balance matrix RecordMatrix

end for

end while

output the adjacency matrix RecordMatrix that finally satisfies the adjacent vertex reducible edge coloring

end

C. The results of the Adjacent Vertex Reducible Edge Coloring algorithm

We analyzes the algorithm results from the chromatic number, edge density, and average degree[21]. Fig. 5 shows the curve about the maximum chromatic number and total number of graphs. As can be seen from Fig. 5, due to the influence of edges, increase first and then decrease phenomenon occurs. Fig. 6 shows the edge density with the number of edges from 7 vertices to 10 vertices and the maximum chromatic number. Fig. 7 shows the average degree with the number of edges from 7 vertices to 10 vertices and the proportion of AVREC graphs that can be colored to the maximum chromatic number to the total number of graphs. It can be seen from Fig. 7, the proportion of AVREC graphs always fixes in a range.

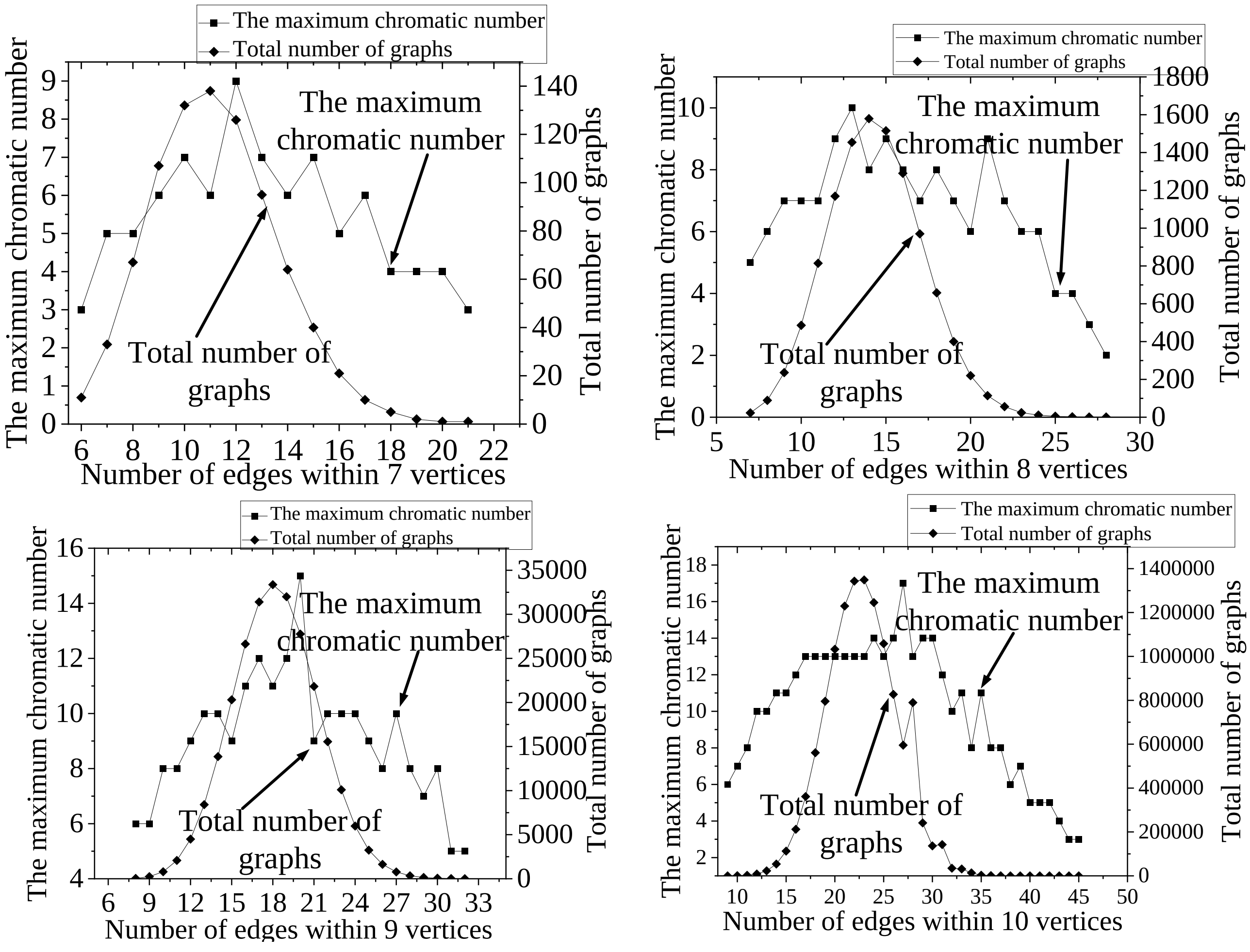


Fig. 5 Line chart of chromatic number

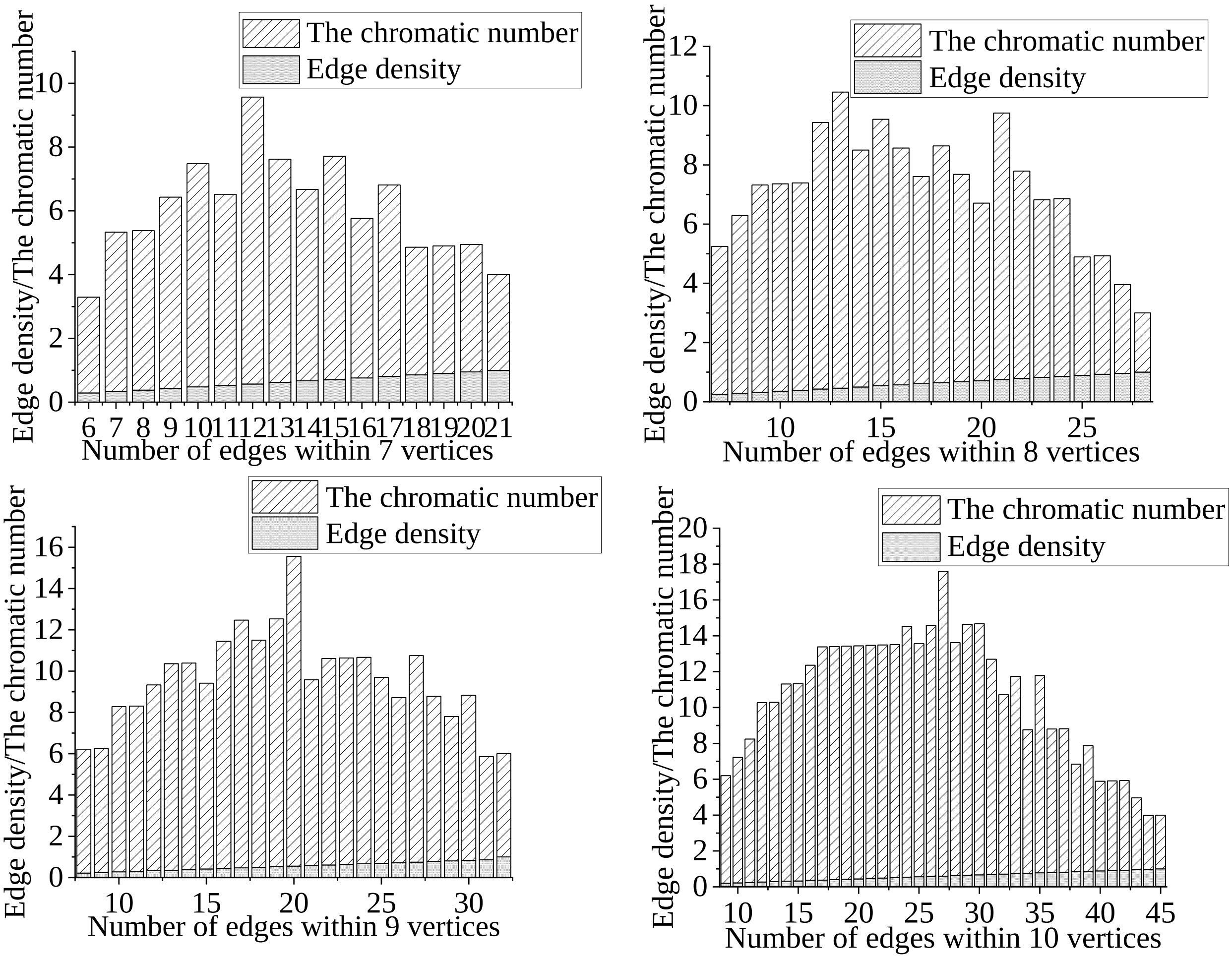


Fig. 6 Bar chart of edge density

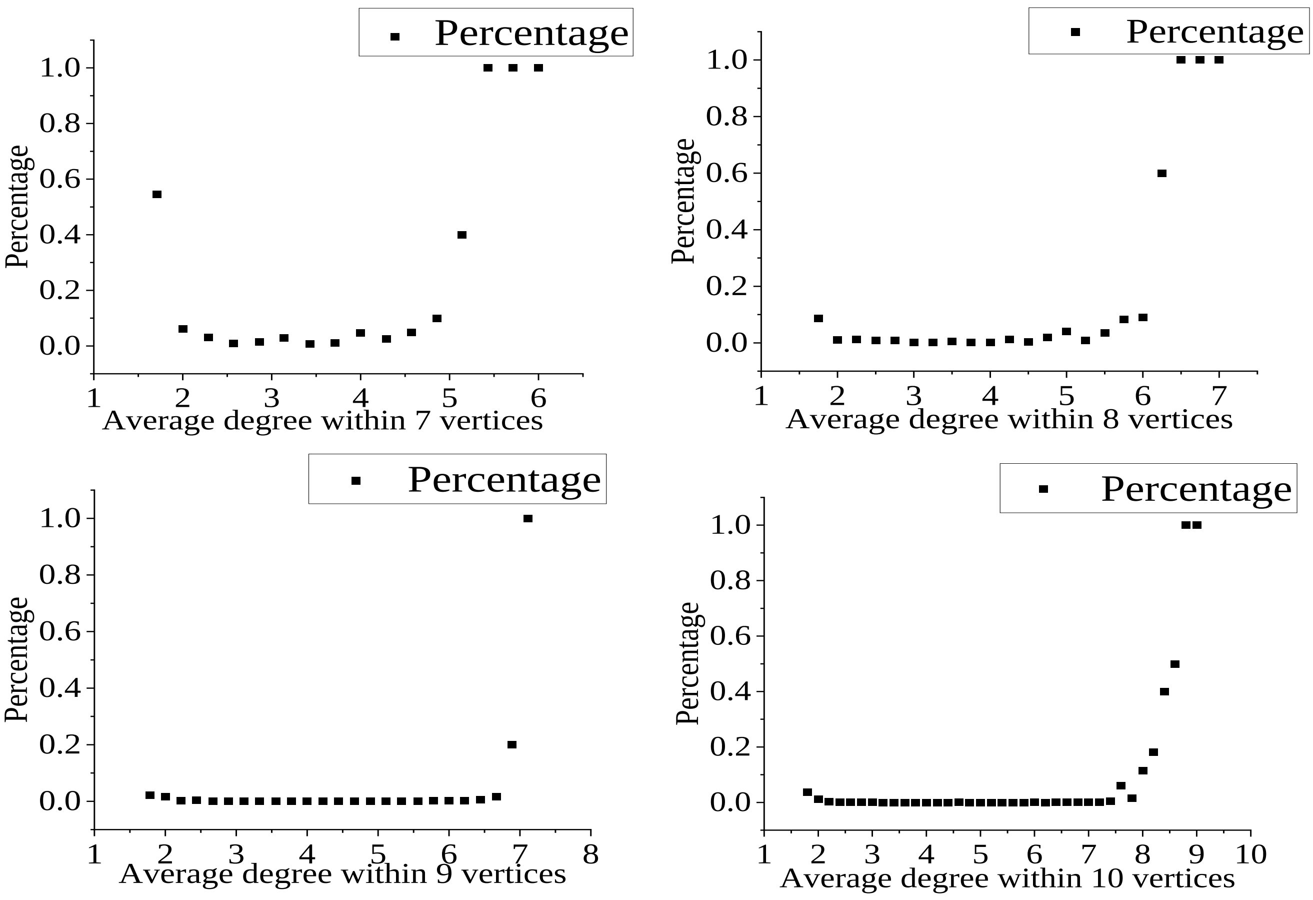


Fig. 7 Dot chart of percentage of graphs

IV. EXPERIMENTAL RESULTS

A. *Data statistics*

The experimental results are listed in Table I, Table II, Table III

TABLE I. COLORING SITUATION OF GRAPHS FROM 3 VERTICES TO 6 VERTICES

(p, q)	Total	Δ	k	(p, q)	Total	Δ	k
(3,2)	1	2	2	(5,10)	1	4	2
(3,3)	1	2	1	(6,5)	6	5	3
(4,3)	2	3	1	(6,6)	13	5	5
(4,4)	2	3	3	(6,7)	19	5	5
(4,5)	1	3	3	(6,8)	22	5	5
(4,6)	1	3	3	(6,9)	20	5	6
(5,4)	3	4	3	(6,10)	14	5	5
(5,5)	5	4	3	(6,11)	9	5	5
(5,6)	5	4	3	(6,12)	5	5	6
(5,7)	4	4	4	(6,13)	2	5	4
(5,8)	2	4	4	(6,14)	1	5	3
(5,9)	1	4	3	(6,15)	1	5	3

(Δ=Maximum degree)

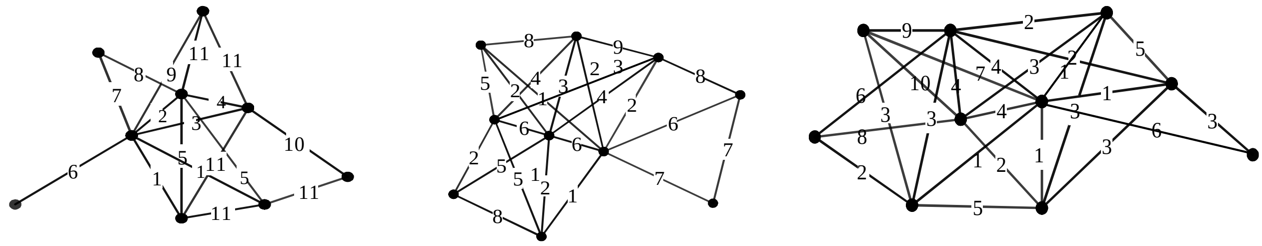
TABLE II. COLORING SITUATION OF DOUBLE CIRCLE GRAPH WITHIN 10 VERTICES

Double circle graph	Total	Coloring situation of double circle graph/k							
		k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
(4,5)	1	——	——	1					
(5,6)	5	——	3	1	1				
(6,7)	19	——	7	8	3	1			
(7,8)	67	2	24	25	14	2			
(8,9)	236	3	88	79	52	10	1	3	
(9,10)	797	11	312	278	122	56	11	5	2
(10,11)	2687	52	974	961	423	177	65	18	8

TABLE III. COLORING SITUATION OF SINGLE CIRCLE GRAPH WITHIN 10 VERTICES

Single Circle graph	Total	Coloring situation of single circle graph/k									
		k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
(3,3)	1	1									
(4,4)	2	——	1	1							
(5,5)	5	1	1	3							
(6,6)	13	——	5	7	——	1					
(7,7)	33	1	16	10	4	2					
(8,8)	89	1	35	30	18	4	1				
(9,9)	240	3	105	67	44	17	4				
(10,10)	657	14	257	207	109	41	21	8			

B. *Give example of some graphs*



G (9,17)　　G (10,24)　　G (10,25)

Fig. 8 The examples of graphs

C. Comparison

1. Comparison of two AVREC algorithms

We analyze and compare 12205208 result sets within 10 vertices using two AVREC algorithms. The AVREC algorithm has two edge constraints based on the objective functions: the edge constraint function and the color set constraint function. As a result, the edge constraint function Ye and the color set constraint function Yc jointly determine the final objective function Y, where Y=Ye+Yc. However, as the number of edges and vertices increases, the calculation of the objective function becomes increasingly difficult. Thus, we have improved the algorithm and designed a step-by-step optimization algorithm, as well as a balance operator and an adjustment operator, which ensured that each step was optimal. By doing so, we can greatly increase our efficiency.

In this study, we compare and analyze the coloring results from the performance and efficiency of the algorithms.

(1) From the performance of the algorithms: we mainly analyze the accuracy of the algorithm (referring to the proportion of the result set that is closest to the theoretical chromatic value in the atlas). It can be seen from Fig. 9, the accuracy of AVREC based on stepwise optimization is higher.The accuracy of two AVREC algorithms as can be seen in Fig. 9.

(2) From the efficiency of the algorithm: we mainly analyze the time complexity

a. AVREC algorithm based on objective function: For a complete graph with n vertices, the number of exchanges is $n\times(n-1)\times(n-2)$, and the worst case in the mutation process is to mutate n times, so the total number of mutations is $n\times n$ at most. The time complexity is $T=[n\times(n-1)\times(n-2)+n\times n]=O(n^3)$.

b. AVREC algorithms based on stepwise optimization: Since the maximum number of mutations of an element is n, and the maximum number of mutations is $\max\times n$ times. The time complexity is $T=[n\times(\max\times n)]=O(n^2)$.

2. The comparison of different algorithms for the same joint graph

The same joint graph is colored using different algorithms, and the results are also different. Nevertheless, from a practical point of view, we must also consider the carrying capacity of the current situation, and the k which is colored should not be excessive.The experimental data information is listed in Table IV.

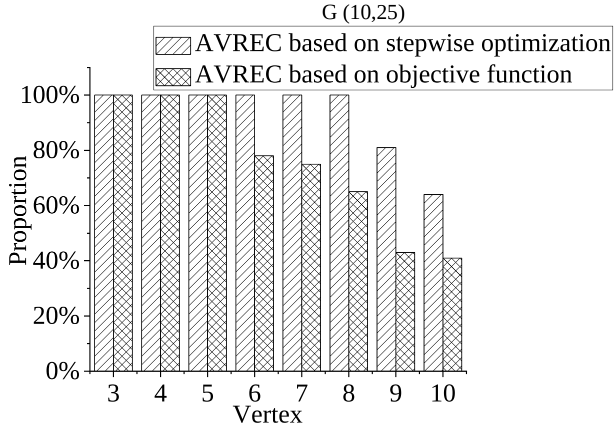


Fig. 9 The accuracy of algorithm

TABLE IV. A COMPARISON OF THE MAXIMUM CHROMATIC NUMBER OF DIFFERENT ALGORITHMS IN SEVERAL TYPES OF JOINT GRAPHS

The type of graph	Maximum chromatic number of different algorithms	
	AVREC	AVSR
$S_m^{C_n}$	3m	3m
$S_m^{C_n}\uparrow C_t$	3m+2	3m+2
$(n,t)-k\uparrow F_m$	13	$2m+3$
$S_m\uparrow F_n\downarrow P_t$	$11+m$	$2n+m$

D. Conclusions

Theorem 1: For joint graphs $S_m^{C_n}(m\geq 4,n\geq 3)$, there is

$$\chi'_{avrec}\left(S_m^{C_n}\right)=3m, m\geq 4, n\geq 3$$

Proof: Suppose the vertex set of S_m is $\{w_0,w_1,...,w_m\}$, and the vertex set of m C_n is $\{u_{11},u_{12},...,u_{1n};u_{21},u_{22},....,u_{2n};\ u_{m1},u_{m2},....,u_{mn}\}$, where $w_m=u_{h1}, i=h=1,2,....,m$.

The joint graph $S_m^{C_n}$ satisfies f coloring rule:

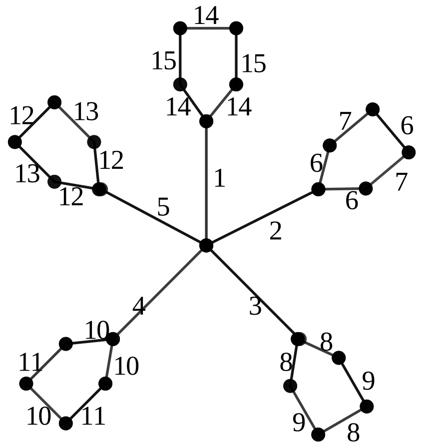
$$f\left(w_0w_i\right)=i, i=1,2,...,m$$

$$f\left(u_{h1}u_{hn}\right)=\begin{cases}m+2h+1, n\equiv 1\left(\bmod 2\right)\\ m+2h+2, n\equiv 0\left(\bmod 2\right)\end{cases} h=1,2,...,m$$

$$f\left(u_{hi}u_{hi+1}\right)=\begin{cases}m+2h-1, i\equiv 1\left(\bmod 2\right)\\ m+2h, i\equiv 0\left(\bmod 2\right)\end{cases} h=1,2,...,m;$$

$$i=1,2,...,n-1$$

In the joint graph $S_m^{C_n}$, the distance between all 3-degree vertices is greater than 1, and the distance between all 2-degree vertices is 1. And because adjacent vertices in the same degree must have the same color sets, where chromatic number is the maximum, the color sets of the 3-degree vertices are different. According to the definition of AVREC, color sets of all 2-degree must be the same, so $\chi'_{avrec}\left(S_m^{C_n}\right)=3m$. Fig. 10 shows the result of $S_m^{C_n}$.



$S_5^{C_5}$

Fig. 10 $S_m^{C_n}$

Theorem 2: For joint graphs $S_m^{C_n}\uparrow C_t\,(n\geq 3,t\geq 3,m\geq 2)$ there is

$$\chi'_{avrec}\left(S_m^{C_n}\uparrow C_t\right)=3m+2,n\geq 3,t\geq 3,m\geq 3$$

Proof: Suppose the vertex set of S_m is $\{w_0,w_1,...,w_m\}$, and the vertex set of C_t is $\{u_0,u_1,\ldots,u_t\}$, and the vertex set of m C_n is $\{u_{11},u_{12},...,u_{1n};u_{21},u_{22},...,u_{2n};u_{m1},u_{m2},...,u_{mn}\}$, where $w_i=u_{h1},w_0=u_1,i=h=1,2,....,m$.

The joint graph $S_m^{C_n}\uparrow C_t$ satisfies f coloring rule:

$$f\left(w_0w_i\right)=i,i=1,2,...,m$$

$$f\left(u_1u_t\right)=\begin{cases}m+1,t\equiv 1\left(\bmod 2\right)\\ m+2,t\equiv 0\left(\bmod 2\right)\end{cases}$$

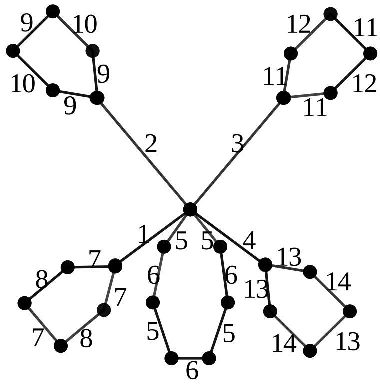
$$f\left(u_iu_{i+1}\right)=\begin{cases}m+1,i\equiv 1\left(\bmod 2\right)\\ m+2,i\equiv 0\left(\bmod 2\right)\end{cases}$$

$$f\left(u_{h1}u_{hn}\right)=\begin{cases}m+2h+1,n\equiv 1\left(\bmod 2\right)\\ m+2h+2,n\equiv 0\left(\bmod 2\right)\end{cases}$$

$$f\left(u_{hi}u_{hi+1}\right)=\begin{cases}m+2h+1,i\equiv 1\left(\bmod 2\right)\\ m+2h+2,i\equiv 0\left(\bmod 2\right)\end{cases}h=1,2,...,m$$

$$i=1,2,....,n-1$$

It is known that the joint graph $S_m^{C_n}\uparrow C_t$ has a maximum-degree vertex, that the distance between *m* 3-degree vertices is 2, and that the distance between all 2-degree vertices is 1 in C_t and C_n. On the one hand, adjacent vertices of the same degree should have the same color sets, On the other hand, the chromatic number to be colored should be maximized, so the color sets of m 3-degree vertices must be different. According to the definition of AVREC, the color sets of all the 2-degree vertices must be same; such as $f\left(u_3u_4\right)=3m+3$, at the moment, at least two 2-degree vertices u_3,u_4 and $t-2$ 2-degree vertices have different color sets, and the chromatic numbers is discontinuous, which contradicts the hypothesis, so $\chi'_{avrec}\left(S_m^{C_n}\uparrow C_t\right)=3m+2$. Fig. 11 shows the result of $S_m^{C_n}\uparrow C_t$.



$S_4^{C_5}\uparrow C_7$

Fig. 11 $S_m^{Cn}\uparrow C_t$

Theorem 3: For joint graphs $\left(n,t\right)-k\uparrow F_m(n\geq 3,t\geq 3,m\geq 6)$, there are

$$\chi'_{avrec}\left(\left(n,t\right)-k\uparrow F_m\left(a\right)\right)=9,n\geq 3,t\geq 3,m\geq 6$$

$$\chi'_{avrec}\left(\left(n,t\right)-k\uparrow F_m\left(b\right)\right)=11,n\geq 3,t\geq 3,m\geq 6$$

$$\chi'_{avrec}\left(\left(n,t\right)-k\uparrow F_m\left(c\right)\right)=13,n\geq 3,t\geq 3,m\geq 6$$

Proof: Suppose the vertex set of $\left(n,t\right)-k$ is $V=V\left(C_n\right)\bigcup V\left(P_t\right)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_t\}$, and the vertex set of F_m is $V=\{w_0,w_1,\ldots,w_m\}$.

(1) Proof of class (a) graph

The joint graph $\left(n,t\right)-k\uparrow F_m\left(a\right)$ satisfies f coloring rule:

$$f\left(w_0w_1\right)=5,\ f\left(w_0w_m\right)=9$$

$$f\left(w_0w_i\right)=\begin{cases}6,i\equiv 0\left(\bmod 3\right)\\ 8,i\equiv 1\left(\bmod 3\right)\\ 7,i\equiv 2\left(\bmod 3\right)\end{cases}i=2,3,...,m-1$$

$$f\left(w_iw_{i+1}\right)=\begin{cases}6,i\equiv 1\left(\bmod 3\right)\\ 8,i\equiv 2\left(\bmod 3\right)\\ 7,i\equiv 0\left(\bmod 3\right)\end{cases}i=2,3,...,m-1$$

$$f\left(v_iv_{i+1}\right)=\begin{cases}1,i\equiv 1\left(\bmod 2\right)\\ 3,i\equiv 2\left(\bmod 2\right)\end{cases}i=1,2,...,t-1$$

$$f\left(u_1u_n\right)=\begin{cases}2,n\equiv 0\left(\bmod 2\right)\\ 4,n\equiv 1\left(\bmod 2\right)\end{cases}$$

$$f\left(u_iu_{i+1}\right)=\begin{cases}4,i\equiv 1\left(\bmod 2\right)\\ 2,i\equiv 0\left(\bmod 2\right)\end{cases}i=1,2,...,n-2$$

The joint graph $\left(n,t\right)-k\uparrow F_m(a)$ has one maximum degree vertex, $m-2$ 3-degree vertices, $n+t-2$ 2-degree vertices and two 1-degree vertices. The color set of any adjacent 3-degree vertex in F_m must be the same, and the color set of any adjacent 2-degree vertex in C_n, P_t must be the same according to the definition of AVREC.

This kind of graph can be colored up to 9 different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is 12, there will be different color set of adjacent vertices in the same degree; such as $f(w_0w_2)=10$, there is at least one 3-degree vertex w_2 and $m-3$ 3-degree vertices color sets differ, and does not satisfy the coloring rule of 8, 6, 7, and the chromatic number is discontinuous, which contradicts the assumption, therefore $\chi'_{avrec}((n,t)-k\uparrow F_m(a))=9$. Fig. 12 shows the result of $(n,t)-k\uparrow F_m(a)$.

(2) Proof of class (b) graph

The joint graph $(n,t)-k\uparrow F_m(b)$ satisfies f coloring rule:

$$f(w_0w_1)=5, f(w_0w_2)=7, f(w_1w_2)=6, f(w_0w_m)=11$$

$$f(v_iv_{i+1})=\begin{cases}1, i\equiv 1(\mathrm{mod}\,2)\\ 3, i\equiv 0(\mathrm{mod}\,2)\end{cases} i=1,2,...,t-1$$

$$f(u_1u_n)=\begin{cases}2, n\equiv 0(\mathrm{mod}\,2)\\ 4, n\equiv 1(\mathrm{mod}\,2)\end{cases}$$

$$f(u_iu_{i+1})=\begin{cases}2, i\equiv 1(\mathrm{mod}\,2)\\ 4, i\equiv 0(\mathrm{mod}\,2)\end{cases} i=1,2,...,m-1$$

$$f(w_0w_i)=\begin{cases}9, i\equiv 0(\mathrm{mod}\,3)\\ 8, i\equiv 1(\mathrm{mod}\,3)\\ 10, i\equiv 2(\mathrm{mod}\,3)\end{cases} i=1,2,...,m-1$$

$$f(w_iw_{i+1})=\begin{cases}8, i\equiv 2(\mathrm{mod}\,3)\\ 9, i\equiv 0(\mathrm{mod}\,3)\\ 10, i\equiv 1(\mathrm{mod}\,3)\end{cases} i=2,3,...,m-1$$

The joint graph $(n,t)-k\uparrow F_m(b)$ has one maximum degree vertex, $m-3$ 3-degree vertices, $n+t-1$ 2-degree vertices and two 1-degree vertices. The color set of any adjacent 3-degree vertex in F_m must be the same, and also the color set of any adjacent 2-degree vertex in C_n, P_t must be the same according to the definition of AVREC.

This kind of graph can be colored up to 11 different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is 12, there will be different color set of adjacent vertices in the same degree; for example $f(u_2u_3)=12$, there are at least two 2-degree vertices u_2, u_3 and $n-3$ 2-degree vertices color sets differ, which contradicts the assumption. If $f(w_0w_4)=12$, there is at least one 3-degree vertex w_2 and $m-4$ 3-degree vertices color sets differ, and $f(w_0w_3), f(w_0w_4),..., f(w_0w_{m-t})$ does not satisfy the coloring regulation of 9, 8, 10, and the chromatic number is discontinuous, therefore $\chi'_{avrec}((n,t)-k\uparrow F_m(b))=11$ .Fig. 13 shows the result of $(n,t)-k\uparrow F_m(b)$.

(3) Proof of class (c) graph

The joint graph $(n,t)-k\uparrow F_m(c)$ satisfies f coloring rule:

$$f(w_0w_1)=5, f(w_0w_m)=13$$

$$f(w_0w_i)=9(v_1/w_t, i=t, 2<t<n-1)$$

$$f(u_1u_n)=\begin{cases}2, n\equiv 1(\mathrm{mod}\,2)\\ 4, n\equiv 0(\mathrm{mod}\,2)\end{cases}$$

$$f(u_iu_{i+1})=\begin{cases}2, i\equiv 1(\mathrm{mod}\,2)\\ 4, i\equiv 0(\mathrm{mod}\,2)\end{cases} i=1,2,...,n-1$$

$$f(v_iv_{i+1})=\begin{cases}1, i\equiv 1(\mathrm{mod}\,2)\\ 3, i\equiv 0(\mathrm{mod}\,2)\end{cases} i=1,2,...,t-1$$

$$f(w_0w_i)=\begin{cases}7, i\equiv 2(\mathrm{mod}\,3)\\ 6, i\equiv 0(\mathrm{mod}\,3)\\ 8, i\equiv 1(\mathrm{mod}\,3)\end{cases} 2<i<t$$

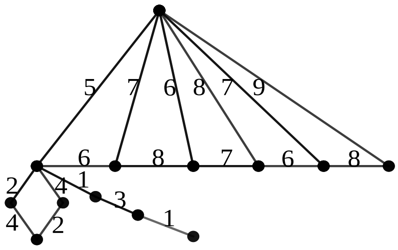
$$f(w_0w_i)=\begin{cases}11, i\equiv 0(\mathrm{mod}\,3)\\ 10, i\equiv 1(\mathrm{mod}\,3)\\ 12, i\equiv 2(\mathrm{mod}\,3)\end{cases} t<i<n-1$$

$$f(w_iw_{i+1})=\begin{cases}6, i\equiv 1(\mathrm{mod}\,3)\\ 8, i\equiv 2(\mathrm{mod}\,3)\\ 7, i\equiv 0(\mathrm{mod}\,3)\end{cases} 1<t<i$$

$$f(w_iw_{i+1})=\begin{cases}10, i\equiv 2(\mathrm{mod}\,3)\\ 12, i\equiv 0(\mathrm{mod}\,3)\\ 11, i\equiv 1(\mathrm{mod}\,3)\end{cases} i<t<n-1$$

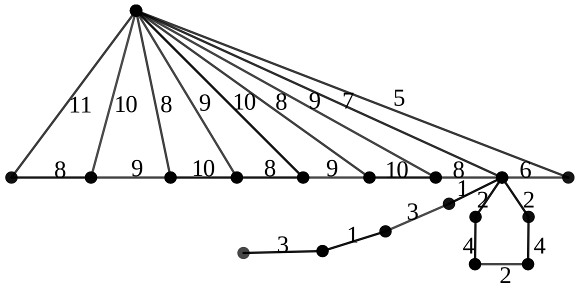
The joint graph $(n,t)-k\uparrow F_m(c)$ has one maximum degree vertex, $m-3$ 3-degree vertices, $n+t-1$ 2-degree vertices and two 1-degree vertices. The color sets of any adjacent 3-degree vertex in F_m must be the same, and the color sets of any adjacent 2-degree vertex in C_n and P_t must be the same according to the definition of AVREC.

This kind of graph can be colored up to 13 different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is 14, there will be different color set of adjacent vertices in the same degree; for example $f(u_3u_4)=14$, there are at least two 2-degree vertices u_3, u_4 and $n-3$ 2-degree vertices color sets differ, which contradicts the assumption. If $f(w_0w_7)=14$, there is at least one 3-degree vertex w_2 and $m-4$ 3-degree vertices color sets differ, and $f(w_0w_4), f(w_0w_5),..., f(w_0w_{m-t})$ does not satisfy the coloring rule of 10, 11, 12, and the chromatic number is discontinuous, therefore $\chi'_{avrec}((n,t)-k\uparrow F_m(c))=13$ .Fig. 14 shows the result of $(n,t)-k\uparrow F_m(c)$.



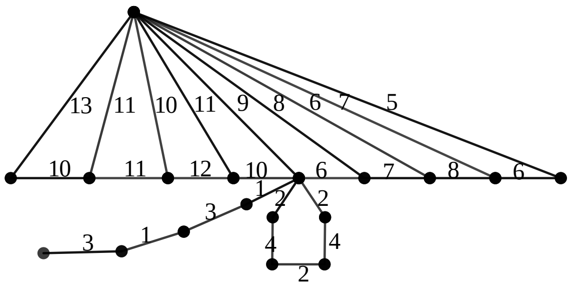
$(4,4)-k\uparrow F_6(a)$

Fig. 12 $(n,t)-k\uparrow F_m(a)$



$(5,5)-k\uparrow F_9(b)$

Fig. 13 $(n,t)-k\uparrow F_m(b)$



$(5,5)-k\uparrow F_9(c)$

Fig. 14 $(n,t)-k\uparrow F_m(c)$

Theorem 4: For joint graph $S_m\uparrow F_n\downarrow P_t(n\ge 3,t\ge 3,m\ge 2)$, there are

$$\chi'_{avrec}\left(S_m\uparrow F_n\downarrow P_t(a)\right)=5,n\ge 3,t\ge 3,m\ge 2$$

$$\chi'_{avrec}\left(S_m\uparrow F_n\downarrow P_t(b)\right)=9+m,n\ge 3,t\ge 3,m\ge 2$$

$$\chi'_{avrec}\left(S_m\uparrow F_n\downarrow P_t(c)\right)=11+m,n\ge 3,t\ge 3,m\ge 2$$

Proof: Suppose the vertex set of S_m is $V=\{w_0,w_1,\ldots,w_m\}$, and the vertex set of F_n is $V=\{u_1,u_2,\ldots,u_n\}$, and the vertex set of P_t is $V=\{v_1,v_2,\ldots,v_t\}$

(1) Proof of class (a) graph

The joint graph $S_m\uparrow F_n\downarrow P_t(a)$ satisfies f coloring rule:

$$f(u_0u_1)=3,\ f(u_0u_n)=5$$

$$f(w_0w_i)=5+i,i=1,2,\ldots\ldots,m$$

$$f(v_iv_{i+1})=\begin{cases}2,i\equiv 0(\bmod 2)\\1,i\equiv 1(\bmod 2)\end{cases} i=1,2,\ldots,t-1$$

$$f(u_0u_i)=\begin{cases}3,i\equiv 1(\bmod 3)\\1,i\equiv 2(\bmod 3)\\4,i\equiv 0(\bmod 3)\end{cases} i=1,2,\ldots,n-1$$

$$f(u_iu_{i+1})=\begin{cases}4,i\equiv 1(\bmod 3)\\3,i\equiv 2(\bmod 3)\\1,i\equiv 0(\bmod 3)\end{cases} i=1,2,\ldots,n-1$$

The joint graph $S_m\uparrow F_n\downarrow P_t(a)$ has one maximum degree vertex, $n-1$ 3-degree vertices, $t-1$ 2-degree vertices and m 1-degree vertices. The color sets of any adjacent 3-degree vertices must be the same, and the color sets of any adjacent 2-degree vertices in P_t must be the same according to definition of AVREC.

This kind of graph can be colored up to $5+m$ different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is $6+m$, there will be different color set of adjacent vertices in the same degree; for example $f(u_1u_2)=6+m$, there are at least two 2-degree vertices u_1, u_2 and $n-3$ 3-degree vertices color sets differ, and does not satisfy the coloring regulation of 1, 3, 4, which contradict the assumption. If $f(v_2v_3)=6+m$, there is at least two 2-degree vertices v_1,v_2 and $t-2$ 2-degree vertices color sets differ and the chromatic number is discontinuous, therefore $\chi'_{avrec}\left(S_m\uparrow F_n\downarrow P_t(a)\right)=5+m$ .Fig. 15 shows the result of $S_m\uparrow F_n\downarrow P_t(a)$.

(2) Proof of class (b) graph

The joint graph $S_m\uparrow F_n\downarrow P_t(b)$ satisfies f coloring rule:

$$f(u_0u_1)=7,\ f(u_0u_2)=8,\ f(u_1u_2)=6$$

$$f(w_0w_i)=9+i,i=1,2,\ldots m$$

$$f(v_iv_{i+1})=\begin{cases}1,i\equiv 1(\bmod 2)\\2,i\equiv 0(\bmod 2)\end{cases} i=1,2,\ldots,t-1$$

$$f(u_0u_i)=\begin{cases}3,i\equiv 0(\bmod 3)\\5,i\equiv 1(\bmod 3)\\4,i\equiv 2(\bmod 3)\end{cases} i=3,4,\ldots,n-1$$

$$f(u_iu_{i+1})=\begin{cases}4,i\equiv 0(\bmod 3)\\3,i\equiv 1(\bmod 3)\\5,i\equiv 2(\bmod 3)\end{cases} i=2,3,\ldots,n-1$$

The joint graph $S_m\uparrow F_n\downarrow P_t(b)$ has one maximum degree vertex, one 4-degree vertices, $n-3$ 3-degree vertices , t 2-degree vertices and $m+1$ 1-degree vertices. The color sets of any adjacent 3- degree vertices in F_n must be the same, and the color sets of 2-degree vertex in P_t must be the same according to definition of AVREC.

This kind of graph can be colored up to $9+m$ different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is $10+m$, there will be different color sets of adjacent vertices in the same degree; for example $f(v_3v_4)=10+m$, there are at least two 2-degree vertices v_3 , v_4 and $t-4$ 2-degree vertices color sets differ, which contradicts the assumption. If $f(u_0u_7)=10+m$, there is at least one 3-degree vertex u_7 and $m-4$ 3-degree vertices color sets differ, and $f(u_0u_3), f(u_0u_4), \ldots, f(u_0u_{m-t})$ does not satisfy the coloring regulation of 3, 5, 4, and the color number is discontinuous, therefore $\chi'_{avrec}(S_m \uparrow F_n \downarrow P_t(b)) = 9+m$.Fig. 16 shows the result of $S_m \uparrow F_n \downarrow P_t(b)$.

(3) Proof of class (c) graph

The joint graph $S_m \uparrow F_n \downarrow P_t(c)$ satisfies f coloring rule:

$$f(u_0u_1)=10$$

$$f(u_0u_n)=11, f(u_0u_i)=6, i=t, 2<t<n-1$$

$$f(w_0w_i)=11+m, i=1,2,\ldots,m$$

$$f(v_iv_{i+1})=\begin{cases}2, i\equiv 0(\bmod 2)\\ 1, i\equiv 1(\bmod 2)\end{cases} i=1,2,\ldots,t-1$$

$$f(u_0u_i)=\begin{cases}5, i\equiv 2(\bmod 3)\\ 4, i\equiv 0(\bmod 3)\\ 3, i\equiv 1(\bmod 3)\end{cases} 2<i<t$$

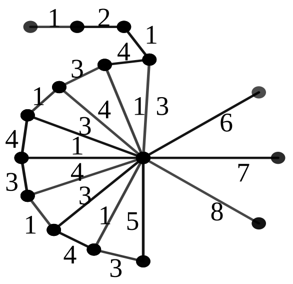
$$f(u_0u_i)=\begin{cases}8, i\equiv 1(\bmod 3)\\ 7, i\equiv 2(\bmod 3)\\ 9, i\equiv 0(\bmod 3)\end{cases} t<i<n-1$$

$$f(u_iu_{i+1})=\begin{cases}3, i\equiv 1(\bmod 3)\\ 5, i\equiv 2(\bmod 3)\\ 4, i\equiv 0(\bmod 3)\end{cases} 1<i<t-1$$

$$f(u_iu_{i+1})=\begin{cases}7, i\equiv 0(\bmod 3)\\ 9, i\equiv 1(\bmod 3)\\ 8, i\equiv 2(\bmod 3)\end{cases} t+1<i<n-1$$

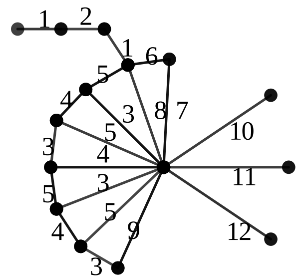
The joint graph $S_m \uparrow F_n \downarrow P_t(c)$ has one maximum degree vertex, one 4-degree vertices, $n-3$ 3-degree vertices, t 2-degree vertices and $m+1$ 1-degree vertices. The color sets of the 3-degree vertices must be the same, and the color sets of the 2-degree vertex in P_t must be the same according to the definition of AVREC.

This kind of graph can be colored up to $11+m$ different colors in accordance with the fundamental coloring rule. Assuming that any edge chromatic number is $12+m$, there will be different color sets of adjacent vertices in the same degree and the chromatic number is discontinuous; for example $f(w_0w_3)=12+m$,there are at least one 1-degree vertex w_3 and $m-1$ 1-degree vertices color sets different and the color number is discontinuous, which contradicts the assumption. If $f(u_3u_4)=12+m$, there is at least two 3-degree vertices u_3, u_4 and $n-5$ 3-degree vertices color sets differ, therefore $\chi'_{avrec}(S_m \uparrow F_n \downarrow P_t(c)) = 11+m$. Fig. 17 shows the result of $S_m \uparrow F_n \downarrow P_t(c)$.



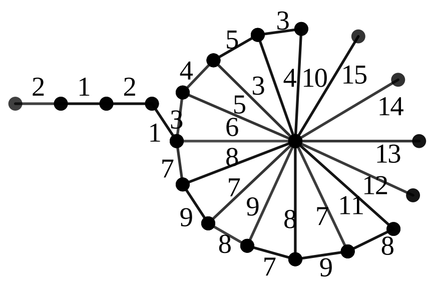
$S_3 \uparrow F_9 \downarrow P_4(a)$

Fig. 15 $S_m \uparrow F_n \downarrow P_t(a)$



$S_3 \uparrow F_8 \downarrow P_4(b)$

Fig. 16 $S_m \uparrow F_n \downarrow P_t(b)$



$S_4 \uparrow F_{11} \downarrow P_5(c)$

Fig. 17 $S_m \uparrow F_n \downarrow P_t(c)$

V. Conclusion

In this paper, we designed a coloring algorithm for adjacent vertex reducible edge coloring for random graphs. And use this algorithm to find the coloring rules on special joint graph within 15 vertices, analyze the results to derive the staining properties of several types of joint graphs and prove them.

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