Algorithm for Adjacent Vertex Reducible Edge Labeling of Some Special Graphs And Their Associated Graphs

Jingwen Li, Linyu Lan, Shucheng Zhang

Abstract—$G(V, E)$ represents the basic chart without circle, if existing a one-to-one mapping $f: E(G) \rightarrow \{1, 2, \ldots, |E|\}$, for any two vertices in the diagram, in the event that $d(u) = d(v)$, $S(u) = S(v)$, where $S(u) = \sum_{uv \in E(G)} f(uw)$, $d(u)$ represents the degree of the vertex $u$, then call the mapping $f$ : Adjacent Vertex Reducible Edge Labeling (alluded as AVREL). In graph theory, graph coloring and graph labeling are two research directions of graph theory, and there is little correlation between the two in previous research results. In the process of researching the concept of Adjacent Reducible Edge Coloring proposed by Professor Zhang Zhongfu, we found that there are several graph classes whose coloring number reaches the sum of the number of vertices and edges, so we propose a new concept of Adjacent Reducible Edge Labeling. In the transportation network, the edge weight represents the transportation capacity, and the node transportation capacity is represented by the sum of its associated edges. Two nodes with the same degree of adjacency require the transportation capacity to be as equal as possible, which can be described by the Adjacent Vertex Reducible Edge Coloring model, when the road diversity reaches the extreme value, it can be described by the Adjacent Vertex Reducible Edge Labeling model. In this paper, designing and using Adjacent Vertex Reducible Edge Labeling algorithm (abbreviation: AVREL algorithm). The algorithm recursively looks through the arrangement space of the Adjacent Reducible Edge Label through the underlying label of the edge, lastly sits through the graph book fulfilling the edge label and results as a label matrix. In the wake of examining the algorithm results, some special graphs such as Petersen-pyramid graphs, Möbius ladder graphs, bicyclic graphs, and some joint graphs in various situations are summed up, the proofs and conjectures are given.

Index Terms—special graph; joint graph; Adjacent Vertex Reducible Edge Labeling; labeling algorithm

I. INTRODUCTION

A branch of graph theory, graph labeling is of great theoretical and practical importance. Many real-life problems can be transformed into icon-number problems to be solved, such as frequency allocation of communication segments, graph cryptography, resource allocation, traffic scheduling, etc. The concept of icon symbol was first proposed in 1967 by Rosa and others in the beautiful conjecture, that is, "every tree is beautiful". On the basis of the beautiful guess, some scholars put forward the graceful color conjecture, that is, "every tree is beautiful". On the basis of the beautiful guess, some scholars put forward the graceful label and the later vertex magic label. In 2009, Professor Zhang Zhongfu proposed the notion of reducible coloring series of graphs.

The new concept of Adjacent Vertex Reducible Edge Labeling proposed in this paper is generated on the magic vertex full labeling and adjacent vertex reducible edge coloring of the graphs. By using the design ideas of random search algorithms such as bee colony algorithm and genetic algorithm, a new algorithm can be designed to solve the adjacent point reducible edge labeling of special graphs and their associated graphs. Then summarize and prove several labeling theorems for the class of graphs by analyzing the set of labeling results of the algorithm.

II. BASIC KNOWLEDGE

This paper mainly discusses special graphs such as road, star, fan, wheel, tree, etc. and the adjacent reducible edge labeling of their associated graphs.

Definition 1: Let $G(V, E)$ be a simple graph, if there is a one-to-one mapping $f : E(G) \rightarrow \{1, 2, \ldots, |E|\}$ such that for any two points $uv \in E(G)$, if $d(u) = d(v)$, then $S(u) = S(v)$, where $S(u) = \sum_{uv \in E(G)} f(uw)$, $d(u)$ represents the degree of the vertex $u$, then $f$ is called Adjacent Vertex Reducible Edge Labeling of $G$ (Adjacent Vertex Reducible Edge Labeling referred to as AVREL).

Definition 2[14]: Graph kite ($nt.r.K$) is a graph consisting of a cycle graph $C_n$ with $n$ vertices and $r$ path graphs $P_i$ with length $t$.

Definition 3: Petersen-pyramid graph is called graph $PP(n_2)$, assuming vertex $w$ (not part of the Petersen graph, $P(n_2)$) is connected to all vertices on the outermost circle of the Petersen graph ($P(n_2)$), and vertex $h$ (not part of the Petersen graph, $P(n_2)$) is connected to all vertices on the innermost circle of Petersen graph ($P(n_2)$). Petersen-pyramid graph $PP(n_2)$ is formed by the set of vertices $V = \{w\} \cup \{h\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_r\}$, and set of edges $E = \{wu_i, i = 1, 2, \ldots, n\} \cup \{hu_i, i = 1, 2, \ldots, n\} \cup \{v_iu_i, i = 1, 2, \ldots, n\} \cup \{v_iu_{i+2}, i = 1, 2, \ldots, n - 2\} \cup \{u_{n-1}u_1\} \cup \{u_{n+1}u_2\} \cup \{v_{i+1}, i = 1, 2, \ldots, n - 1\} \cup \{v_nv_1\}$. As shown in figure 1.

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Definition 4: Joint graph $S^m_F_n(m \geq 2, n \geq 4)$, the vertices set of $S^m_F_n$ is $V = \{w_0, w_1, \ldots, w_m\}$, the edges set of $S^m_F_n$ is $E = \{w_iw_j, i = 1, 2, \ldots, m\}$, the vertices set of $mF_n$ is $V = V(1F_n) \cup V(2F_n) \cup \ldots \cup V(mF_n) = \{w_0\} \cup \{hu_i, h = 1, 2, \ldots, m, i = 1, 2, \ldots, n\}$, the edges set of $mF_n$ is $E = \{w_0hu_i, h = 1, 2, \ldots, m, i = 1, 2, \ldots, n\} \cup \{hu_i, hu_{i+1}, h = 1, 2, \ldots, m, i = 1, 2, \ldots, n - 1\} \cup \{hu_i\}$, As shown in figure 2.

Definition 5: [10] The Moebius ladder graph $M\nu_{\nu}-2$ consists of $\nu$ is obtained by adding edges $u_0u_1$ and $v_0v_1$ to the ladder graph $L_\nu$, the vertices set of $M\nu_{\nu}$ is $V = \{u_i, v_i, i = 1, 2, \ldots, n\}$, As shown in figure 3.

Definition 6: The associative graph consisting of a 1-order null graph $O_{\nu}$ and a $n$-order Cycle $C_n$, is called a wheel graph, denoted $W_n$, $W_n = O_{\nu} \cup C_n$, where $V(O_{\nu}) = \{u_0\}$ and $V(C_n) = \{u_i, i = 1, 2, \ldots, n\}$. The Circle $C_n$ of the wheel graph $W_n$, with a point cut out on each edge, the joint graph is called a Multi-circle gear graph $MG_n(\nu \geq 3)$, where the interior point of the path of the join point $u_i$ and $v_{i+1}$ of length 2 is denoted by $u_i$, and the subscript takes a module $n$. The $V(MG_n) = \{u_0\} \cup \{u_i, i = 1, 2, \ldots, n\} \cup \{v_i, i = 1, 2, \ldots, n\}$, $E(MG_n) = \{u_0u_i, i = 1, 2, \ldots, n\} \cup \{u_iu_{i+1}, i = 1, 2, \ldots, n - 1\} \cup \{u_iu_{i+1}\} \cup \{u_iv_i, i = 1, 2, \ldots, n\}$, As shown in figure 4.

III. ALGORITHM

A. Preparation

As indicated by the meaning of Adjacent Vertex Reducible Edge Labeling, all graphs are separated into two classes, one for graphs with adjacent degree vertices and the other for graphs without adjacent degree vertices. In this paper, we build a graph classification function Classify, which performs a search in the solution space during the process of determining which graph satisfies the labeling condition. The last label success condition is the statement that the labels of adjacent degree vertices are equivalent and that the number of labels is continuous.

B. Principles for the AVREL Algorithm

According to definition 1, $uv \in E(G)$, if $d(u) = d(v)$, then $S(u) = S(v)$, summation formula (1),

$$S(u) = \sum_{uv \in E(G)} f(uv) = S(v) = \sum_{uv \in E(G)} f(uv)$$

The idea of the AVREL algorithm is to transform the adjacency matrix of a graph into an initial eigenmatrix to satisfy the AVREL requirement, the solution space of adjacent reducible edges labeling is recursively searched, and the equilibrium operator is used to judge whether the labeling matrix is in equilibrium, finally, the graph set fulfilling the Adjacent Reducible Edge Labeling is chosen, then, yield as a labeled matrix.

C. AVREL pseudo code

The idea of AVREL algorithm is to transform the adjacency AVREL algorithm description:
AVREL algorithm

Input: adjacency matrix of graph $G(p, q)$
Output: AVREL label matrix or non-AVREL matrix
(1) read the adjacency matrix (AdjustMatrix) of graph $G$, initializing the label matrix (LabelAdjust)
(2) get $p$, $q$, degree, isBalance, $\varphi(p, q)$ Classify
(3) while ($\varphi(p, q) \neq $ null)
(4) search $\varphi(p, q)$
(5) if $G$.isBalance ← true
(6) LabelAdjust ← AdjustMatrix
(7) break
(8) endif
(9) endwhile
(10) if $G$.isBalance ← false
(11) Output this graph is a non-AVREL graph
(12) endif
(13) else
(14) Output LabelAdjust
(15) endelse
(16) end

IV. CONCLUSION AND PROOF

Theorem 1: For Petersen-pyramid graph $P_{(n,2)}$, if $n \geq 5$, all are AVREL graphs.

Proof:

By definition 3, assuming the vertices set of $P_{(n,2)}$ is $V = \{w\} \cup \{t\} \cup \{u_i, i = 1, 2, \ldots, n\} \cup \{v_i, i = 1, 2, \ldots, n\}$, where $V_{\text{address}} = \{v_i, u_i | i = 1, 2, \ldots, n| n \geq 5\}$, the edges set of $P_{(n,2)}$ is $E = \{(wv_i, i = 1, 2, \ldots, n) \cup \{(hu_i, i = 1, 2, \ldots, n) \cup \{v_iu_i, i = 1, 2, \ldots, n \cup \{u_iu_{i+2}, i = 1, 2, \ldots, n-2 \cup \{u_1u_n\} \cup \{v_iu_{i+1}, i = 1, 2, \ldots, n-1 \cup \{v_nv_1\} \}$. Examples of $P_{(n,2)}$ are shown in figure 5 (1), (2).

Discuss the following two situations:
Case1: When $n \equiv 1 \pmod{2}$:
Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(wv_i) = 3n + i - 1, 1 \leq i \leq n$$
$$f(hu_i) = \begin{cases} 
(i + 1)/2 + 1, i \equiv 1 \pmod{2} \\
1, i = n 
\end{cases}, 1 \leq i \leq n - 1$$
$$f(v_{i+1}) = 3n + i + 1, 1 \leq i \leq n - 1$$
$$f(v_{i+1}) = 3n + i - 1, 1 \leq i \leq n - 1$$
$$f(u_{i+2}) = \begin{cases} 
(i + 3)/2, i \equiv 1 \pmod{2} \\
2n, i = n 
\end{cases}, 1 \leq i \leq n - 1$$
$$f(u_{i+2}) = \begin{cases} 
(i + 3)/4, i \equiv 1 \pmod{4} \\
16n + i + 3)/4, i \equiv 1 \pmod{4} \\
19n + i + 3)/4, i \equiv 1 \pmod{4} \\
18n + i + 3)/4, i \equiv 1 \pmod{4} \\
1 \leq i \leq n - 2, n \equiv 1 \pmod{4} \\
19n + i + 3)/4, i \equiv 1 \pmod{4} \\
16n + i + 3)/4, i \equiv 1 \pmod{4} \\
18n + i + 3)/4, i \equiv 1 \pmod{4} \\
1 \leq i \leq n - 2, n \equiv 3 \pmod{4} 
\end{cases}, 1 \leq i \leq n - 2, n \equiv 3 \pmod{4}$$

From this, the edges labeling set of the Petersen-pyramid graph:

$$|E(G)| = n + n + n + n + n = 5n$$

Fig.5 $P_{(n,2)}$ ($n \geq 5$)

$$f(E) = \{3n, 3n-1, \ldots, 2n+1\} \cup \{(n+3)/2, (n+5)/2, \ldots, n, 2, n+2, 3n+3, \ldots, 4n \} \cup \{3n+1 \cup \{2n-1, 2n-2, n+1, 2n\} \cup \{4n+1, 4n+2, 4n+3, 4n+4, 4n, \ldots, 5n-1, 5n\}.
$$

It can be seen from the exhaustive method that $f(E) \rightarrow [1, 5n]$.

The label sum of $V_{\text{address}}$ $S_4$:

$$S_4 = \| \sum_{v \in V_{\text{address}}} f(vu) + f(wv_i)\| \sum_{u_i \in V_{\text{address}}} f(u_i) + f(hu_i)\|$$

$$= \| \sum_{v \in V_{\text{address}}} f(vu) + f(wv_i)\| \sum_{u_i \in V_{\text{address}}} f(u_i) + f(hu_i)\|$$

$$\sum_{v \in V_{\text{address}}} f(vu) + f(hu_i)\|$$

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As evidenced by the above, all Petersen-pyramid graphs $P_{(n,2)}$ are AVREL graphs.

**Theorem 2:** For joint graphs $F_n \uparrow S_m$, if $n \geq 3, m \geq 1$, all are AVREL graphs.

**Proof:**
Assuming the vertices set of $F_n \uparrow S_m$ is $V = \{ u_i, i = 0,1, ..., n \} \cup \{ v_{hi}, h = 0,1, ..., m \}$, where $V_{adeg=3} = \{ u_i, i = 1,2, ..., n-1 \}$ ($n \geq 3, m \geq 1$), the edges set of $F_n \uparrow S_m$ is $E = \{ u_iu_i, i = 1,2, ..., n \} \cup \{ v_{hi}v_{h+1}, h = 1,2, ..., m \} \cup \{ u_iu_{i+1}, i = 1,2, ..., n-1 \}$. An example of $F_n \uparrow S_m$ is shown in Figure 6.

![Figure 6](image)

**Fig.6** $F_n \uparrow S_m$ ($n \geq 3, m \geq 1$)

Discuss the following two situations:

**Case1:** When $n \equiv 0 \pmod{2}$:
Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_iu_i) = 2i - 1, i = 1,2, ..., n$$
$$f(u_iu_{i+1}) = 2n + 1 - i, i \equiv 1 \pmod{2}$$
$$f(v_{hi}v_{h+1}) = 2n + h - 1, h = 1,2, ..., m$$

From this, the edges labeling set of the $F_n \uparrow S_m$ graph:

$$f(\{u_iu_i\}) = \{2i, 4, ..., 2n-2, n\} \cup \{2n+1, 2n+3, ..., 2n+2m\}$$

It can be seen from the exhaustive method that $f(E) \rightarrow [1,2n+m-1]$.

The label sum of $\{V_{adeg=3}\} S_3$:

$$S_3 = |\{ f(v_{hi}) [1 \leq i \leq n-1] \} + \sum_{v_{h} \in NE(v_{h})} f(v_{h}) + \sum_{v_{h} \in NE(u_{h})} f(u_{h}) \} |$$

It is shown in figure 6.

**Case2:** When $n \equiv 1 \pmod{2}$:
Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_iu_i) = 2i - 1, i = 1,2, ..., n$$
$$f(u_iu_{i+1}) = 2n + 1 - i, i \equiv 0 \pmod{2}$$
$$f(v_{hi}v_{h+1}) = 2n + h - 1, h = 1,2, ..., m$$

From this, the edges labeling set of the $F_n \uparrow S_m$ graph:

$$f(\{u_iu_i\}) = \{2i, 4, ..., 2n-2, n\} \cup \{2n+1, 2n+3, ..., 2n+2m\}$$

It can be seen from the exhaustive method that $f(E) \rightarrow [1,2n+m-1]$.

The label sum of $\{V_{adeg=3}\} S_3$:

$$S_3 = |\{ f(v_{hi}) [1 \leq i \leq n-1] \} + \sum_{v_{h} \in NE(v_{h})} f(v_{h}) + \sum_{v_{h} \in NE(u_{h})} f(u_{h}) \} |$$

It is shown in figure 6.
1, ..., 2n + m − 1).

It can be seen from the exhaustive method that \( f(E) \rightarrow [1, 2n + m − 1] \).

The label sum of \( \{V_{\text{adeq}=3}\} \) \( S_3 \):

\[
S_3 = \left\{ \sum_{(v_i, w_j) \in \text{NE}(u)} f(v_i) | 2 \leq i \leq n, 1 \leq h \leq m \right\}
\]

\[
= \left| \{ f(u_i, w_{i+1}) + f(h_{i-1}w_i) + f(h_{i+1}w_i) | 2 \leq i \leq n - 1, 1 \leq h \leq m \} \right|
\]

\[
= \left| \{ 2hn - 1 - i + 1 + 2hn - n - 1 + i + 2i - 1 + 2hn - 2n \} \right|
\]

\[
= \left| \{ (6h - 3)n \} \right|
\]

\[
= (6h - 3)n.
\]

So \( S_3 \) is a constant while \( n \equiv 1 (\text{mod} 2) \). From the above, the AVREL scheme is established.

Case 2: When \( n \equiv 0 (\text{mod} 2) \):

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

\[
f(w_{0i}) = 2hn - 1, h = 1, 2, ..., m
\]

\[
f(h_{i-1}w_i) = 2i - 1 + 2(h - 1)n, h = 1, 2, 3, ..., m, i = 1, 2, ..., n
\]

\[
f(h_iw_{i+1}) = \begin{cases} 
2hn - 1 - i, h = 1, 2, 3, ..., m, \\
2hn - 1 - i, h = 1, 2, 3, ..., m, i = 1, 2, ..., n - 1 
\end{cases}
\]

From this, the edges labeling set of the \( S_{m}^{F_{n}} \) graph:

\[
f(E) = \{ 2hn - 1, 4n - 4, ..., 2mn - 2n - 2 \} \cup \{ (6h - 3)n \} \cup \{ 2n - 2, 4n - 4, ..., 2mn - n + 1 \} \cup \{ n - 1, 3n - 3, ..., 2mn - n + 1 \} \cup \{ 2n - 2, 4n - 4, ..., 2mn - n + 1 \}.
\]

It can be seen from the exhaustive method that \( f(E) \rightarrow [1, 2mn] \).

The label sum of \( \{V_{\text{adeq}=3}\} \) \( S_3 \):

\[
S_3 = \left\{ \sum_{(v_i, w_j) \in \text{NE}(u)} f(v_i) | 2 \leq i \leq n, 1 \leq h \leq m \right\}
\]

\[
= \left| \{ f(u_i, w_{i+1}) + f(h_{i-1}w_i) + f(h_{i+1}w_i) | 2 \leq i \leq n - 1, 1 \leq h \leq m \} \right|
\]

\[
= \left| \{ (6h - 3)n \} \right|
\]

\[
= (6h - 3)n.
\]

So \( S_3 \) is a constant while \( n \equiv 0 (\text{mod} 2) \). From the above, the AVREL scheme is established.
As evidenced by the above, all joint graphs \( F_n \uparrow S_m \) are AVREL graphs.

**Theorem 4:** For joint graphs \( F_n \uparrow (3, 2, 2) - K \), if \( n \geq 3 \), all are AVREL graphs.

**Proof:**

Assuming the vertices set of \( F_n \uparrow (3, 2, 2) - K \) \( (n \geq 3) \) is 
\[ V = \{ w_0(u_1),w_1,...,w_n \} \cup \{ u_2(v_1),u_3(v_1') \} \cup \{ v_2,v_2' \} \],
where \( V_{\text{adeq}=3} = \{ w_i | n-1 \geq i \geq 2 \} (n \geq 3) \), \( V'_{\text{adeq}=3} = \{ u_2(v_1),u_3(v_1') \} \), the edges set of \( F_n \uparrow (3, 2, 2) - K \) is 
\[ E = \{ w_0w_i, i = 1,2,...,n \} \cup \{ w_iw_{i+1}, i = 1,2,...,n-1 \} \cup \{ u_1u_2, u_3u_3' \} \cup \{ v_1v_2, v_1'v_2' \} \]. Examples of \( F_n \uparrow (3, 2, 2) - K \) are shown in figure 8(1), (2).

Discuss the following two situations:

**Case 1:** When \( n \equiv 1 \mod(2) \):

Through formula (1) and the algorithm results, consider the following edge labeling scheme:
\[
\begin{align*}
f(w_0w_i) & = 2i - 1, \quad i = 1,2,...,n \\
f(w_iw_{i+1}) & = \begin{cases} 
2n - i - 1, i \equiv 1 \mod(2) \\
2n - i + 1, i \equiv 0 \mod(2) 
\end{cases}, \quad i = 1,2,...,n-1 \\
f(u_1u_2) & = 2n + 1 \\
f(v_1v_2) & = 2n + 4 \\
f(v_1'v_2') & = 2n + 3
\end{align*}
\]

From this, the edges labeling set of the \( F_n \uparrow (3, 2, 2) - K \) graph:
\[
f(E) = \{1,3,...,2n-1\} \cup \{2n-2,2n-4,...,n+1\} \cup \{n-1,n-3,...,2\} \cup \{2n,2n+2\} \cup \{2n+1\} \cup \{2n+4\} \cup \{2n+3\}.
\]

It can be seen from the exhaustive method that \( f(E) \rightarrow [1,2n+4] \).

The label sum of \( \{V_{\text{adeq}=3}\} \) \( S_3 \), \( \{V'_{\text{adeq}=3}\} \) \( S'_3 \):
\[
\begin{align*}
S_3 & = |\sum_{v\in V} f(vu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{w_0w_i} f(wu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{w_iw_{i+1}} f(wu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{u_1u_2} f(uu)| \\
S'_3 & = |\sum_{v\in V'} f(vu)| \sum_{v\in V'} f(vu)| \\
& = |\sum_{v_1v_2} f(vv)| (2) \\
& = |\sum_{v_1'v_2'} f(vv)| (2)
\end{align*}
\]

From this, the edges labeling set of the \( F_n \uparrow (3, 2, 2) - K \) graph:
\[
f(E) = \{2,4,...,2n-2\} \cup \{2n-1\} \cup \{n-1,n-3,...,1\} \cup \{2n-3,2n-5,...,n+1\} \cup \{2n,2n+1\} \cup \{2n+2\} \cup \{2n+3\} \cup \{2n+4\}.
\]

It can be seen from the exhaustive method that \( f(E) \rightarrow [1,2n+4] \).

The label sum of \( \{V_{\text{adeq}=3}\} \) \( S_3 \), \( \{V'_{\text{adeq}=3}\} \) \( S'_3 \):
\[
\begin{align*}
S_3 & = |\sum_{v\in V} f(vu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{w_0w_i} f(wu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{w_iw_{i+1}} f(wu)| 2 \leq i \leq n-1, 1 \leq h \leq m| \\
& = |\sum_{u_1u_2} f(uu)| \\
S'_3 & = |\sum_{v\in V'} f(vu)| \sum_{v\in V'} f(vu)| \\
& = |\sum_{v_1v_2} f(vv)| (2) \\
& = |\sum_{v_1'v_2'} f(vv)| (2)
\end{align*}
\]
\[ = 6n + 6. \]

So \( S_3 \) and \( S'_3 \) are constant while \( n \equiv 0 \pmod{2} \). From the above, the AVREL scheme is established.

As evidenced by the above, all joint graphs \( F_n \uparrow (3, 2, 2) - K \) are AVREL graphs.

**Theorem 5:** For bicyclic graphs, if \( 4 \leq p \leq 15 \), are AVREL graphs.

**Proof:**

The results from the AVREL algorithm of bicyclic graphs are shown in Table 1.

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<th>((p, p + 1))</th>
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<th>Graph ( \text{Non-AVREL} )</th>
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<tr>
<td>((15, 16))</td>
<td>959374</td>
<td>150546</td>
<td>808828</td>
</tr>
</tbody>
</table>

From the data in Table 1, the proportion of the bicyclic graphs that satisfying the AVREL is shown in figure 9.

Bicyclic graphs labeling results are shown in figure 10.

As per the outcomes acquired by the algorithm, it can be inferred that all graphs within 15 vertices fulfill this AVREL condition. However, because of the constraints of the PC and the efficiency of the algorithm, the larger vertices of bicyclic graphs are not tested.

**Conjecture 1:** For bicyclic graphs, if \( n \geq 16 \), are AVREL graphs.

**Theorem 6:** For Möbius ladder graphs \( ML_n \), if \( n \equiv 1 \pmod{2} \), all are AVREL graphs, if \( 2 \leq n \leq 19 \) and \( n \equiv 0 \pmod{2} \), all are not AVREL graphs.

**Proof:**

Assuming the vertices set of Möbius ladder graph \( ML_n \) \((n \equiv 1 \pmod{2})\) is \( V = \{u_i, v_i|n \geq i \geq 1\}(n \equiv 1\pmod{2}) \).
where \( \mathcal{C}_d = 3 \) is \( \{ \mathcal{C}_i \mathcal{C}_j \mid \mathcal{C}_i \geq \mathcal{C}_j \geq 1 \} \) (\( \mathcal{C}_j \geq 3 \)), the edges set of \( M_{\mathcal{C}} \) is \( \mathcal{E} = \{ \mathcal{C}_i \mathcal{C}_{i+1}, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\mathcal{C}_j - 1} \} \cup \{ \mathcal{C}_1 \mathcal{C}_2, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\mathcal{C}_j - 1} \} \cup \{ \mathcal{C}_1 \mathcal{C}_2, \mathcal{C}_1 \} \). Examples of \( M_{\mathcal{C}} \) are shown in figure 11 (1), (2).

Discuss the following two situations:

**Case 1:** When \( \mathcal{C} \equiv 1 \pmod{2} 

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

\[
\begin{align*}
    f(\mathcal{C}_i \mathcal{C}_{i+1}) &= \begin{cases} 
    (2n + i + 1)/2, & i \equiv 1 \pmod{2}, \\
    (n + i + 1)/2, & i \equiv 0 \pmod{2}, \\
    \end{cases} \\
    f(\mathcal{C}_i \mathcal{C}_{i+1}) &= \begin{cases} 
    (2n + i + 1)/2, & i \equiv 1 \pmod{2}, \\
    (n + i + 1)/2, & i \equiv 0 \pmod{2}, \\
    \end{cases} \\
    f(\mathcal{C}_i \mathcal{C}_{i+1}) &= \begin{cases} 
    (2n + i + 1)/2, & i \equiv 1 \pmod{2}, \\
    (n + i + 1)/2, & i \equiv 0 \pmod{2}, \\
    \end{cases} \\
    f(\mathcal{C}_i \mathcal{C}_{i+1}) &= \begin{cases} 
    (2n + i + 1)/2, & i \equiv 1 \pmod{2}, \\
    (n + i + 1)/2, & i \equiv 0 \pmod{2}, \\
    \end{cases}
\end{align*}
\]

From this, the edges labeling set of the \( M_{\mathcal{C}} \) (\( n \equiv 1 \pmod{2} \)) graph:

\[
\begin{align*}
    f(E) &= \{3n, 3n - 1, \ldots, 2n + 1\} \cup \{(n + 1)/2\} \cup \{(3n + 1)/2\} \cup \{(1, \ldots, (n - 1)/2, (3n + 5)/2, \ldots, 2n) \cup \{(n + 1, n + 2, \ldots, (3n - 1)/2, (n + 3)/2, \ldots, n\}.
\end{align*}
\]

It can be seen from the exhaustive method that \( f(E) \rightarrow [1, 3n] \).

The label sum of \( \{\mathcal{C}_{\mathcal{C}_1}\} \) is:

\[
\begin{align*}
    s_3 &= |\sum_{\mathcal{C}_1 \in E} f(\mathcal{C}_1) - \sum_{\mathcal{C}_1 \in E} f(\mathcal{C}_1)| \\
    &= |(3n - i + 1 + (i + 1)/2 + (n + i - 1 + 1)/2)| \\
    &= |(3n + 3)/2| \\
    &= (3n + 3)/2.
\end{align*}
\]

The calculated set of edge labels is \( f(E) \):

It is a constant while \( n \equiv 1 \pmod{2} \). From the above, the AVREL scheme is established.

**Case 2:** When \( n \equiv 0 \pmod{2} \):

The results from the AVREL algorithm of the Möbius ladder graphs \( M_{\mathcal{C}} \) (\( 2 \leq \mathcal{C} \leq 19 \)) are shown in Table 2:

### Table 2: AVREL for \( M_{\mathcal{C}} \) while \( 2 \leq \mathcal{C} \leq 19 \)

<table>
<thead>
<tr>
<th>( M_{\mathcal{C}} )</th>
<th>AVREL(( \mathcal{C} ))</th>
<th>Non-AVREL(( \mathcal{C} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 \leq \mathcal{C} \leq 19</td>
<td>3, 5, 7, 9, 11, 13, 15, 17, 19</td>
<td>2, 4, 6, 8, 10, 12, 14, 16, 18</td>
</tr>
</tbody>
</table>
From Table 2, the proportion of the Möbius ladder graphs that satisfy the AVREL is shown in figure 12.

![Image](image_url)

**Fig.12** Proportion of Möbius ladder graphs that meet the labeling

From figure 12, it can be clearly seen that the results for ladder graphs up to 19 vertices show a distinct odd-even distribution. The results of the odd-numbered vertex graphs all meet the condition AVREL, and the even-numbered vertex graphs are not AVREL graphs.

As evidenced by the above, all Möbius ladder graphs $ML_n (n \equiv 1 (mod 2))$ are AVREL; Möbius ladder graphs $ML_n (n \equiv 0 (mod 2), 2 \leq n \leq 19)$ are not AVREL graphs.

**Conjecture 2:** For Möbius ladder graphs $ML_n$, if $n \equiv 0 (mod 2)$, all are not AVREL graphs.

**Theorem 7:** For Multi-circle gear graph $MG_n$, if $n \geq 3$, $n \neq 5$, all are AVREL graphs, when $n = 5$, the Multi-circle gear graph $MG_5$ is not AVREL graph.

**Proof:**

Assuming the vertices set of Multi-circle gear graph $MG_n$ $(n \geq 3)$ is $V = \{u_0, v_1, n \geq i \geq 1\} \cup \{u_0\} (n \equiv 1 (mod 2))$, where $V_{avg = 5} = \{u_0 \mid n \geq i \geq 1\} (n \geq 3, n \neq 5)$ or $V_{avg = 5} = \{u_0 \mid n \geq i \geq 1\} \cup \{u_0\} (n = 5)$ the edges set of $MG_n$ is $E = \{u_0u_{i+1} \mid i = 1, 2, \ldots, n\} \cup \{u_{i+1}u_i, i = 1, 2, \ldots, n - 1\} \cup \{u_n, u_1\} \cup \{v_i, i = 1, 2, \ldots, n\} \cup \{v_{i-1}, i = 2, 3, \ldots, n\} \cup \{u_0, v_n\}$. Examples of $MG_n$ are shown in figure 13 (1), (2).

Discuss the following three situations:

**Case 1:** When $n \equiv 1 (mod 2), n \neq 5$.

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

\[
\begin{align*}
&f(u_0u_{i+1}) = 2(i - 1), i = 1, 2, \ldots, n - 1 \\
&f(u_0v_i) = 2n - 1 \\
&f(u_{i+1}v_i) = 2n - 2i + 2, i = 1, 2, \ldots, n \\
&f(u_i,v_i) = \begin{cases} 
3n - i + 2, n \equiv 1 (mod 2) \\
4n - i + 2, n \equiv 0 (mod 2)
\end{cases}, n \geq i \geq 1 \\
&f(u_{i-1},v_i) = \begin{cases} 
4n - i + 2, n \equiv 1 (mod 2) \\
3n - i + 2, n \equiv 0 (mod 2)
\end{cases}, n \geq i \geq 2 \\
&f(u_0v_n) = 2n - 1
\end{align*}
\]

From this, the edges labeling set of the $MG_n (n \equiv 1 (mod 2), n \neq 5)$ graph:

\[
f(E) = \{1, 3, \ldots, 2n - 3\} \cup \{2n - 1\} \cup \{2n, 2n - 2, \ldots, 2\} \cup \{3n + 1, 3n - 1, \ldots, 2n + 2, 4n, 4n - 2, \ldots, 3n + 3\} \\
\cup \{4n + 1, 4n - 1, \ldots, 3n + 2, 3n, 3n - 2, \ldots, 2n + 3\} \cup \{2n + 1\}.
\]

**Case 2:** When $n \equiv 0 (mod 2)$.

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

\[
\begin{align*}
&f(u_0u_{i+1}) = 2i - 1, i = 1, 2, \ldots, n - 1 \\
&f(u_0v_i) = 2n - 1 \\
&f(u_{i+1}v_i) = 2n - 2i + 2, i = 1, 2, \ldots, n \\
&f(u_i,v_i) = \begin{cases} 
3n - i + 2, n \equiv 1 (mod 2) \\
4n - i + 2, n \equiv 0 (mod 2)
\end{cases}, n \geq i \geq 1 \\
&f(u_{i-1},v_i) = \begin{cases} 
4n - i + 2, n \equiv 1 (mod 2) \\
3n - i + 2, n \equiv 0 (mod 2)
\end{cases}, n \geq i \geq 2 \\
&f(u_0v_n) = 2n - 1
\end{align*}
\]

From this, the edges labeling set of the $MG_n (n \equiv 1 (mod 2), n \neq 5)$ graph:

\[
f(E) = \{1, 3, \ldots, 2n - 3\} \cup \{2n - 1\} \cup \{2n, 2n - 2, \ldots, 2\} \cup \{3n + 1, 3n - 1, \ldots, 2n + 2, 4n, 4n - 2, \ldots, 3n + 3\} \\
\cup \{4n + 1, 4n - 1, \ldots, 3n + 2, 3n, 3n - 2, \ldots, 2n + 3\} \cup \{2n + 1\}.
\]
According to the formula (1), the following conditions must be satisfied: \( S(\mu_1) = S(\mu_2) = \ldots = S(\mu_n) = S(\upsilon_0) \), and the edge label sum of \( MG_n \) is \( \text{Sum} = (20 \times 21)/2 = 210 \).

We assume that \( A = f(\upsilon_0\upsilon_1) + f(\upsilon_1\upsilon_2) + f(\upsilon_2\upsilon_3) + f(\upsilon_3\upsilon_4) + f(\upsilon_4\upsilon_5) + f(\upsilon_5\upsilon_0) \), the sum label of the remaining edges is \( B, \) \( \text{Sum} = A + B \).

Based on the geometric properties of the graph \( MG_n \), \( S(\mu_1) + S(\mu_2) + S(\mu_3) + S(\mu_4) + S(\upsilon_0) = S_1 = 2B + A, \) and we get the following formula (3):

\[
    (\text{Sum} \times 2 - A) \text{mod} 6 = 0
\]

That is: \( (0 - \text{Amod6}) \text{mod} 6 = 0 \) (4)

According to formula (3), the following is obtained: \( \text{Amod6} = 0 \) (5)

Assuming that \( S_1 \) is maximized, then \( A \) is minimized, the label set of the set of edges covered by \( A \) is \( \{1, 2, \ldots, 9, 10\} \), the label sum is \( A = 55 \), however, according to formula (5), we could get the \( A_{\min} = 60 \), then the \( S_{1\max} = 360 \), while \( B_{\max} = S_1/2 = 180 \).

According to formula (2), the \( B_{\max} = 210 - A = 150 \). For \( B_{\max} \neq B_{\max} \) and \( B_{\max} = B_{\max} \) the result contradicts the premise of the hypothesis, the hypothesis does not hold. So when \( n = 5 \), the Multi-circle gear graph \( MG_n \) is not AVREL graph.

As evidenced by the above, all Multi-circle gear graphs \( MG_n (n \geq 3, n \neq 5) \) are AVREL graphs; when \( n = 5 \), the Multi-circle gear graph \( MG_5 \) is not AVREL graph.

V. CONCLUSION

The findings introduced in this article rich our understanding of reducible series, a new concept of Adjacent Vertex Reducible Edge Labeling was proposed based on the current concept of reducible labeling, and a new heuristic algorithm was proposed which depended on the intelligent algorithm, and the Adjacent Vertex Reducible Edge Labeling algorithm has been utilized to calculate all non-isomorphic graphs within finite vertices, some theorems and guesses were given.

REFERENCES

SHUCHENG ZHANG was born in Binzhou, Shandong Province, China in 1995. He received his BA in software engineering from the University of Ludong in 2018. Now he has received his M. Phil degree in software engineering from Lanzhou Jiaotong University. His major field of study is graph theory algorithm with applications. Mr. Zhang became a member (M) of CCF in 2021.