Algorithm for Adjacent Vertex Reducible Edge Labeling of Some Special Graphs And Their Associated Graphs

Jingwen Li, Linyu Lan, Shucheng Zhang

***Abstract*—$G(V,E)$ represents the basic chart without circle, if existing a one-to-one mapping $f: E(G) \to \{1,2,...,|E|\}$, for any two vertices in the diagram, in the event that $d(u)=d(v)$, $S(u)=S(v)$,where $S(u)=\sum_{uw\in E(G)} f(uw)$, $d(u)$ represents the degree of the vertex u, then call the mapping f : Adjacent Vertex Reducible Edge Labeling (alluded as AVREL). In graph theory, graph coloring and graph labeling are two research directions of graph theory, and there is little correlation between the two in previous research results. In the process of researching the concept of Adjacent Reducible Edge Coloring proposed by Professor Zhang Zhongfu, we found that there are several graph classes whose coloring number reaches the sum of the number of vertices and edges, so we propose a new concept of Adjacent Reducible Edge Labeling. In the transportation network, the edge weight represents the transportation capacity, and the node transportation capacity is represented by the sum of its associated edges. Two nodes with the same degree of adjacency require the transportation capacity to be as equal as possible, which can be described by the Adjacent Vertex Reducible Edge Coloring model. when the road diversity reaches the extreme value, it can be described by the Adjacent Vertex Reducible Edge Labeling model. In this paper, designing and using Adjacent Vertex Reducible Edge Labeling algorithm (abbreviation: AVREL algorithm). The algorithm recursively looks through the arrangement space of the Adjacent Reducible Edge Label through the underlying label of the edge, lastly sifts through the graph book fulfilling the edge label and results as a label matrix. In the wake of examining the algorithm results, some special graphs such as Petersen-pyramid graphs, Möbius ladder graphs, bicyclic graphs, and some joint graphs in various situations are summed up, the proofs and conjectures are given.**

***Index Terms*—special graph; joint graph; Adjacent Vertex Reducible Edge Labeling; labeling algorithm**

I. Introduction

As a branch of graph theory, graph labeling is of great theoretical and practical importance. Many real-life problems can be transformed into icon-number problems to be solved, such as frequency allocation of communication segments, graph cryptography, resource allocation, traffic scheduling, etc. The concept of icon symbol was first proposed in 1967 by Rosa and others in the beautiful conjecture, that is, "every tree is beautiful". On the basis of the beautiful guess, some scholars put forward the graceful label and the later vertex magic label. In 2009, Professor Zhang Zhongfu proposed the notion of reducible coloring series of graphs.

The new concept of Adjacent Vertex Reducible Edge Labeling proposed in this paper is generated on the magic vertex full labeling and adjacent vertex reducible edge coloring of the graphs. By using the design ideas of random search algorithms such as bee colony algorithm and genetic algorithm, a new algorithm can be designed to solve the adjacent point reducible edge labeling of special graphs and their associated graphs. Then, summarize and prove several labeling theorems for the class of graphs by analyzing the set of labeling results of the algorithm.

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II. Basic Knowledge

This paper mainly discusses special graphs such as road, star, fan, wheel, tree, etc. and the adjacent reducible edge labeling of their associated graphs.

Definition 1: Let $G(V,E)$ be a simple graph, if there is a one-to-one mapping $f: E(G) \to \{1,2,...,|E|\}$,such that for any two points $uv \in E(G)$, if $d(u)=d(v)$, there is $S(u)=S(v)$, where $S(u)=\sum_{uw\in E(G)} f(uw)$, $d(u)$ represents the degree of the vertex u, then f is called Adjacent Vertex Reducible Edge Labeling of G (Adjacent Vertex Reducible Edge Labeling referred to as AVREL) .

Definition 2[4]: Graph kite (($n.t.r$)-K) is a graph consisting of a cycle graph C_n with n vertices and r path graphs P_t with length t.

Definition 3: Petersen-pyramid graph is called graph $Pp_{(n,2)}$, assuming vertex w (not part of the Petersen graph, $P_{(n,2)}$) is connected to all vertices on the outermost circle of the Petersen graph ($P_{(n,2)}$), and vertex h (not part of the Petersen graph, $P_{(n,2)}$) is connected to all vertices on the innermost circle of Petersen graph ($P_{(n,2)}$). Petersen-pyramid graph $Pp_{(n,2)}$ is formed by the set of vertices $V=\{w\}\cup\{h\}\cup\{u_1,u_2,...,u_n\}\cup\{v_1,v_2,...,v_n\}$, and set of edges $E=\{wv_i, i=1,2,...,n\}\cup\{hu_i, i=1,2,...,n\}\cup\{v_iu_i, i=1,2,...,n\}\cup\{u_iu_{i+2}, i=1,2,...,n-2\}\cup\{u_{n-1}u_1\}\cup\{u_nu_2\}\cup\{v_i v_{i+1}, i=1,2,...,n-1\}\cup\{v_nv_1\}$. As shown in figure 1.

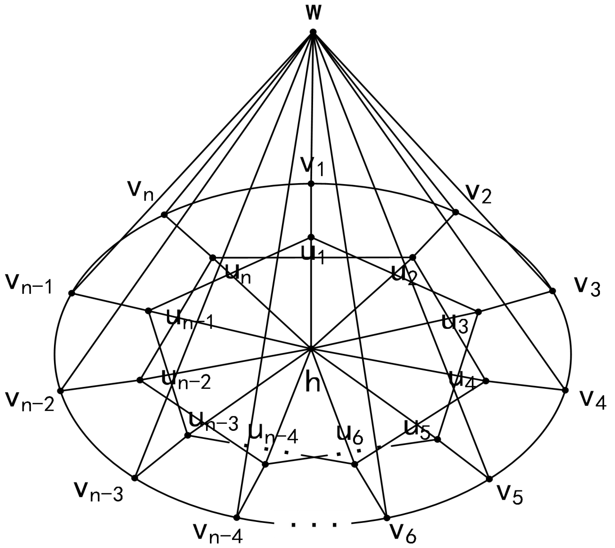


Fig. 1. Example of $Pp_{(n,2)}$ graph

Definition 4: Joint graph $S_m^{mF_n}(m \geq 2, n \geq 4)$, the vertices set of S_m is $V = \{w_0, w_1, ..., w_m\}$, the edges set of S_m is $E = \{w_0 w_i, i = 1,2, ..., m\}$, the vertices set of mF_n is: $V = V(1F_n) \cup V(2F_n) \cup \ldots \cup V(mF_n) = \{w_0\} \cup \{hu_i, h = 1, 2, ..., m, i = 1,2, ..., n\}$, the edges set of $m\ F_n$ is $E = \{w_0 hu_i, h = 1,2, ..., m, i = 1,2, ..., n\} \cup \{hu_i hu_{i+1}, h = 1, 2, ..., m, i = 1,2, ..., n-1\}$. As shown in figure 2.

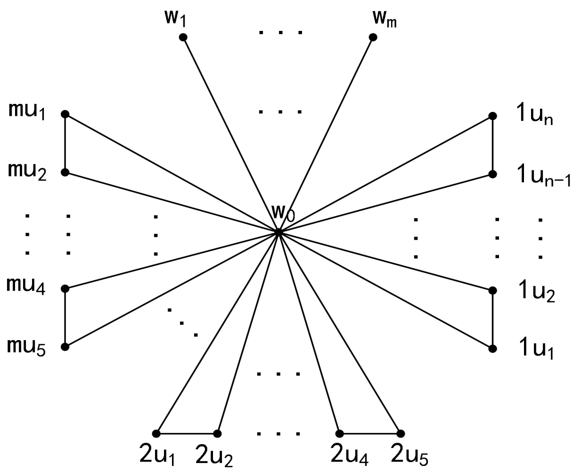


Fig. 2. Examples of $S_m^{mF_n}$ graph

Definition 5[10]: The Möbius ladder graph $ML_n(n \geq 2)$ is obtained by adding edges $u_n v_1$ and $v_n u_1$ to the ladder graph L_n, the vertices set of ML_n is $V = \{u_i, i = 1,2, ..., n\} \cup \{v_i, i = 1,2, ..., n\}$. As shown in figure 3.

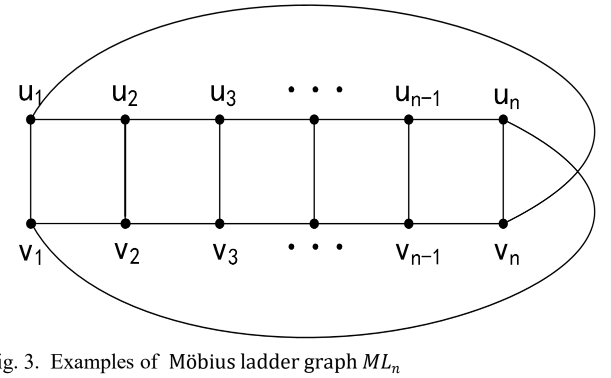


Fig. 3. Examples of Möbius ladder graph ML_n

Definition 6: The associative graph consisting of a 1-order null graph O_1 and a n-order Cycle C_n, is called a wheel graph, denoted W_n, $W_n = O_1 \cup C_n$, where $V(O_1) = \{u_0\}$ and $V(C_n) = \{u_i, i = 1,2, ..., n\}$. The Circle C_n of the wheel graph W_n, with a point cut out on each edge, the joint graph is called a Multi-circle gear graph $MG_n(n \geq 3)$, where the interior point of the path of the join point v_i and v_{i+1} of length 2 is denoted by u_i, and here the subscript takes a module n. The $V(MG_n) = \{u_0\} \cup \{u_i, i = 1,2, ..., n\} \cup \{v_i, i = 1,2, ..., n\}$, $E(MG_n) = \{u_0 u_i, i = 1,2, ..., n\} \cup \{u_i u_{i+1}, i = 1,2, ..., n-1\} \cup \{u_n u_1\} \cup \{u_i v_i, i = 1,2, ..., n\} \cup \{u_i v_{i-1}, i = 2,3, ..., n\} \cup \{u_i v_n\}$. As shown in figure 4.

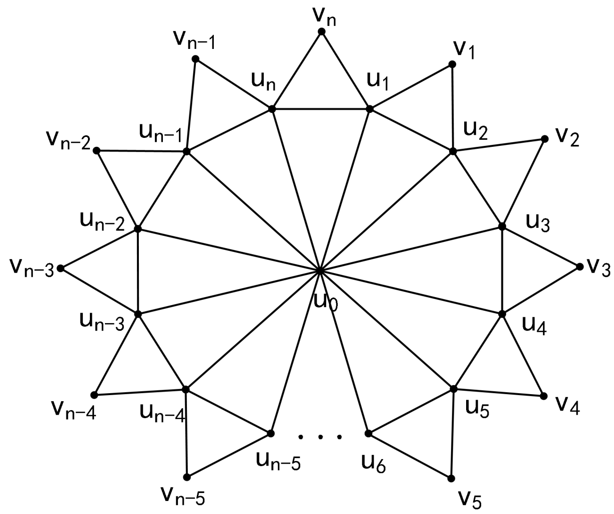


Fig. 4. Examples of Multi-circle gear graph MG_n

III. Algorithm

A. Preparation

As indicated by the meaning of Adjacent Vertex Reducible Edge Labeling, all graphs are separated into two classes, one for graphs with adjacent degree vertices and the other for graphs without adjacent degree vertices. In this paper, we build a graph classification function Classify, which performs a search in the solution space during the process of determining which graph satisfies the labeling condition. The last label success condition is the statement that the labels of adjacent degree vertices are equivalent and that the number of labels is continuous.

B. Principles for the AVREL Algorithm

According to definition 1, $uv \in E(G)$, if $d(u) = d(v)$, then $S(u) = S(v)$, summation formula (1),

$$S(u) = \sum_{uw \in E(G)} f(uw) = S(v) = \sum_{vw \in E(G)} f(vw) \quad (1)$$

The idea of the AVREL algorithm is to transform the adjacency matrix of a graph into an initial eigenmatrix to satisfy the AVREL requirement, the solution space of adjacent reducible edges labeling is recursively searched, and the equilibrium operator is used to judge whether the labeling matrix is in equilibrium, finally, the graph set fulfilling the Adjacent Reducible Edge Labeling is chosen, then, yield as a labeled matrix.

C. AVREL pseudo code

The idea of AVREL algorithm is to transform the adjacency AVREL algorithm description:

AVREL algorithm	
Input:	adjacency matrix of graph $G(p,q)$
Output:	AVREL label matrix or non-AVREL matrix
(1)	read the adjacency matrix (AdjustMatrix) of graph G, initializing the label matrix (LabelAdjust)
(2)	get p, q, degree, isBalance, $\varphi(p,q)$ Classify
(3)	while ($\varphi(p,q)$!= null)
(4)	search $\varphi(p,q)$
(5)	If G.isBalance ← true
(6)	LabelAdjust ← AdjustMatrix
(7)	break
(8)	endif
(9)	endwhile
(10)	if G.isBalance ← false
(11)	Output this graph is a non-AVREL graph
(12)	endif
(13)	else
(14)	Output LabelAdjust
(15)	endelse
(16)	end

IV. CONCLUSION AND PROOF

Theorem 1: For Petersen-pyramid graph $Pp_{(n,2)}$, if $n \geq 5$, all are AVREL graphs.

Proof:

By definition 3, assuming the vertices set of $Pp_{(n,2)}$ is $V = \{w\} \cup \{h\} \cup \{u_i, i = 1,2,\ldots,n\} \cup \{v_i, i = 1,2,\ldots,n\}$, where $V_{adeg=4} = \{v_i, u_i | i = 1,2,\ldots,n | n \geq 5\}$, the edges set of $Pp_{(n,2)}$ is $E = \{wv_i, i = 1,2,\ldots,n\} \cup \{hu_i, i = 1,2,\ldots,n\} \cup \{v_iu_i, i = 1,2,\ldots,n\} \cup \{u_iu_{i+2}, i = 1,2,\ldots,n-2\} \cup \{u_1u_{n-1}\} \cup \{u_nu_2\} \cup \{v_iv_{i+1}, i = 1,2,\ldots,n-1\} \cup \{v_nv_1\}$. Examples of $Pp_{(n,2)}$ are shown in figure 5 (1), (2).

Discuss the following two situations:

Case1: When $n \equiv 1(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(wv_i) = 3n + 1 - i, 1 \leq i \leq n$$

$$f(hu_i) = \begin{cases} \begin{cases} (n+i)/2 + 1, i \equiv 1(mod2) \\ i/2 + 1, i \equiv 0(mod2) \end{cases}, 1 \leq i \leq n-1 \\ 1, i = n \end{cases}$$

$$f(v_iv_{i+1}) = 3n + i + 1, 1 \leq i \leq n-1$$

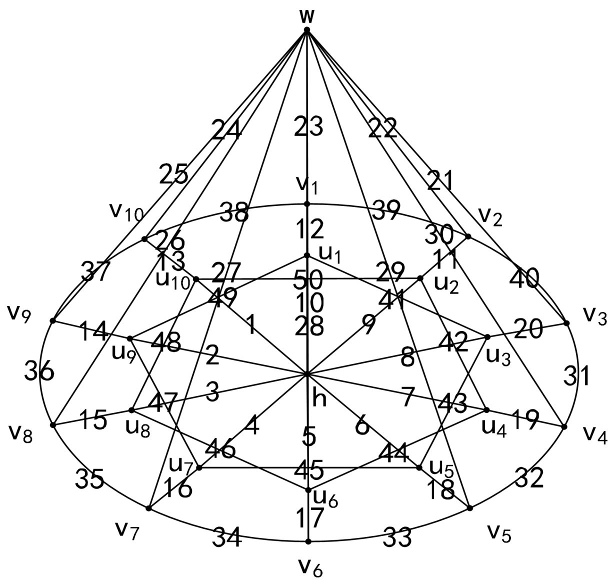
$$f(v_nv_1) = 3n + 1$$

$$f(v_iu_i) = \begin{cases} 2n - i, 1 \leq i \leq n-1 \\ 2n, i = n \end{cases}$$

$$f(u_iu_{i-n+2}) = \begin{cases} \begin{cases} (18n+2)/4, n \equiv 1(mod4) \\ (18n-2)/4, n \equiv 3(mod4) \end{cases}, i = n-1 \\ \begin{cases} (17n+3)/4, n \equiv 1(mod4) \\ (19n+3)/4, n \equiv 3(mod4) \end{cases}, i = n \end{cases}$$

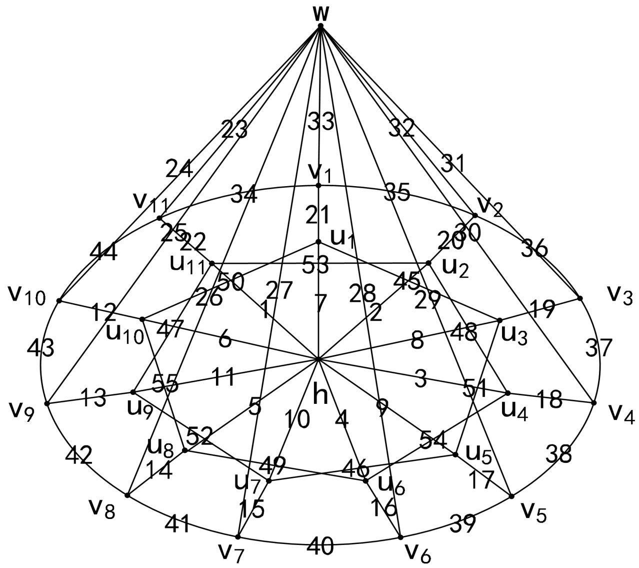
$$f(u_iu_{i+2}) = \begin{cases} \begin{cases} (17n+i+3)/4, i \equiv 0(mod4) \\ (16n+i+3)/4, i \equiv 1(mod4) \\ (19n+i+3)/4, i \equiv 2(mod4) \\ (18n+i+3)/4, i \equiv 3(mod4) \end{cases}, \\ 1 \leq i \leq n-2, n \equiv 1(mod4) \\ \begin{cases} (19n+i+3)/4, i \equiv 0(mod4) \\ (16n+i+3)/4, i \equiv 1(mod4) \\ (17n+i+3)/4, i \equiv 2(mod4) \\ (18n+i+3)/4, i \equiv 3(mod4) \end{cases}, \\ 1 \leq i \leq n-2, n \equiv 3(mod4) \end{cases}$$

From this, the edges labeling set of the Petersen-pyramid graph:



(1) $Pp_{(10,2)}$

$|E(G)| = n + n + n + n + n = 5n$



(2) $Pp_{(11,2)}$

$|E(G)| = n + n + n + n + n = 5n$

Fig.5 $Pp_{(n,2)}$ ($n \geq 5$)

$f(E) = \{3n, 3n-1, \ldots, 2n+1\} \cup \{(n+3)/2, (n+5)/2, \ldots, n, 2, 3, \ldots, (n+1)/2, 1\} \cup \{3n+2, 3n+3, \ldots, 4n\} \cup \{3n+1 \cup \{2n-1, 2n-2, \ldots, n+1, 2n\} \cup \{4n+1, 4n+2, 4n+3, 4n+4, \ldots, 5n-1, 5n\}$.

It can be seen from the exhaustive method that $f(E) \rightarrow [1, 5n]$.

The label sum of $\{V_{adeg=4}\}$ S_4:

$$S_4 = |\{\sum_{vu \in NE(v_i)} f(vu) + f(wv_i)| |\sum_{vu \in NE(u_i)} f(vu) + f(hu_i)| 1 \leq i \leq n\}|$$

$$= |\{\sum_{vu \in NE(v_1)} f(vu) + f(wv_1)\}| |\ldots| |\{\sum_{vu \in NE(v_n)} f(vu) + f(wv_n)\}| |\{\sum_{vu \in NE(u_1)} f(vu) + f(hu_1)\}| | \ldots | |\{\sum_{vu \in NE(u_n)} f(vu) + f(hu_n)\}|$$

$= |\{3n + 2n - 1 + 3n + 1 + 3n + 2\}||...||\{2n + 1 + 3n + 1 + 4n + 2n\}||\{(n + 1)/2 + (16n + 4)/4 + (18n + 2)/4 + 2n - 1 + 1\}||...||\{2n + (17n + 3)/4 + (18n + n - 2 + 3)/4 + 1\}|$
$= |\{11n + 2\}|| ... ||\{11n + 2\}||\{11n + 2\}|| ... ||\{11n + 2\}|$
$= |\{11n + 2\}|$
$= 11n + 2.$

So S_4 is a constant while $n \equiv 1(mod\ 2)$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(wv_i) = \begin{cases} 2n + 3, i = 1 \\ 2n + 2, i = 2 \\ 2n + 1, i = 3 \\ 3n - i + 4, 4 \leq i \leq n \end{cases}$$
$$f(hu_i) = n - i + 1, 1 \leq i \leq n$$
$$f(v_iv_{i+1}) = \begin{cases} 4n - 1, i = 1 \\ 4n, i = 2 \\ 3n + i - 2, 3 \leq i \leq n - 1 \end{cases}$$
$$f(v_nv_1) = 4n - 2$$
$$f(v_iu_i) = \begin{cases} n + 2, i = 1 \\ n + 1, i = 2 \\ 2n - i + 3, 3 \leq i \leq n \end{cases}$$
$$f(u_iu_{i+2}) = 4n + i, 1 \leq i \leq n - 2$$
$$f(u_iu_{i-n+2}) = \begin{cases} 5n - 1, i = n - 1 \\ 5n, i = n \end{cases}$$

From this, the edges labeling set of the Petersen-pyramid graph:

$f(E) = \{2n + 3, 2n + 2, 2n + 1, 3n, 3n - 1, ..., 2n + 4\} \cup \{n, n - 1, ..., 1\} \cup \{4n - 1, 4n, 3n + 1, 3n + 2, ..., 4n - 3 \cup \{4n - 2\} \cup \{n + 2, n + 1, 2n, 2n - 1, ..., n + 3\} \cup \{4n + 1, 4n + 2, ..., 5n - 2\} \cup \{5n - 1, 5n\}$.

It can be seen from the exhaustive method that $f(E) \to [1, 5n]$.

The label sum of $\{V_{adeg=4}\}$ S_4:

$S_4 = |\{ f(wv_i) + \sum_{vu\in NE(v_i)} f(vu) || f(hu_i) + \sum_{vu\in NE(u_i)} f(vu) |1 \leq i \leq n \}|$
$= |\{\sum_{vu\in NE(v_1)} f(vu) + f(wv_1)\}||... ||\{f(wv_n) + \sum_{vu\in NE(v_n)} f(vu) \}||\{\sum_{vu\in NE(u_1)} f(vu) + f(hu_1) \}||\cdots ||\{\sum_{vu\in NE(u_n)} f(vu) + f(hu_n)\}|$
$= |\{2n + 3 + 4n - 1 + 4n - 2 + n + 2\}||...||\{3n - n + 4 + 3n + n - 1 - 2 + 4n - 2 + 2n - n + 3\}||\{n - 1 + 1 + n + 2 + 4n + 1 + 5n - 1\}||\{n - n + 1 + 2n - n + 3 + 4n + n - 2 + 5n\}|$
$= |\{11n + 2\}||...||\{11n + 2\}||\{11n + 2\}||...||\{11n + 2\}|$
$= |\{11n + 2\}|$
$= 11n + 2.$

So S_4 is a constant while $n \equiv 0(mod\ 2)$. From the above, the AVREL scheme is established.

As evidenced by the above, all Petersen-pyramid graphs $Pp_{(n,2)}$ are AVREL graphs.

Theorem 2: For joint graphs $F_n \upharpoonleft S_m$, if $n \geq 3, m \geq 1$, all are AVREL graphs.

Proof:

Assuming the vertices set of $F_n \upharpoonleft S_m$ is $V = \{u_i, i = 0,1, ..., n\} \cup \{v_h, h = 0,1, ..., m\}$, where $V_{adeg=3} = \{u_i, i = 1,2, ..., n - 1\}$ $(n \geq 3, m \geq 1)$, the edges set of $F_n \upharpoonleft S_m$ is $E = \{u_0u_i, i = 1,2, ..., n\} \cup \{v_0v_i, i = 1,2, ..., m\} \cup \{u_iu_{i+1}, i = 1,2, ..., n - 1\}$. An example of $F_n \upharpoonleft S_m$ is shown in figure 6.

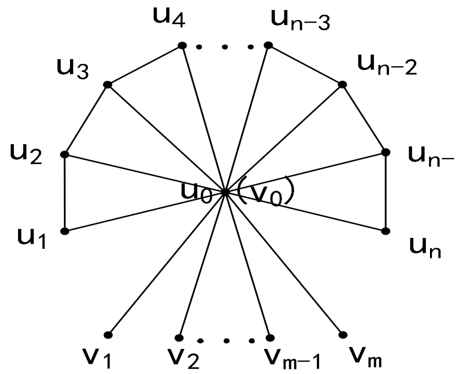


Fig.6 $F_n \upharpoonleft S_m$ $(n \geq 3, m \geq 1)$

Discuss the following two situations:

Case1: When $n \equiv 1(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_0u_i) = 2i - 1, i = 1,2, ..., n$$
$$f(u_iu_{i+1}) = \begin{cases} 2n - 1 - i, i \equiv 1(mod\ 2); \\ n + 1 - i, i \equiv 0(mod\ 2). \end{cases} i = 1, 2, ..., n - 1$$
$$f(v_0v_i) = 2n + h - 1, h = 1,2, ..., m$$

From this, the edges labeling set of the $F_n \upharpoonleft S_m$ graph:

$f(E) = \{1, 3, ..., 2n - 1\} \cup \{2n - 2, 2n - 4, ..., n + 2, n + 1\} \cup \{n - 1, n - 3, ..., 4, 2\} \cup \{2n, 2n + 1, ..., 2n + m - 1\}$.

It can be seen from the exhaustive method that $f(E) \to [1, 2n + m - 1]$.

The label sum of $\{V_{adeg=3}\}$ S_3:

$S_3 = |\{\sum_{vu\in NE(u_i)} f(vu) |1 \leq i \leq n - 1\}|$
$= |\{f(u_0u_i) + f(u_{i-1}u_i) + f(u_iu_{i+1})|1 \leq i \leq n - 1\}|$
$= |\{2i - 1 + 2n - 1 - i + 1 + n + 1 - i\}|$
$= |\{3n\}|$
$= 3n.$

So S_3 is a constant while $n \equiv 1(mod\ 2)$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_0u_i) = 2i, i = 1,2, ..., n - 1$$
$$f(u_0u_n) = 2n - 1$$
$$f(u_iu_{i+1}) = \begin{cases} n - i, i \equiv 1(mod\ 2); \\ 2n - 1 - i, i \equiv 0(mod\ 2); \end{cases} i = 1,2, ..., n - 1$$
$$f(v_0v_i) = 2n + h - 1, h = 1,2, ..., m$$

From this, the edges labeling set of the $F_n \upharpoonleft S_m$ graph:

$f(E) = \{2, 4, ..., 2n - 2\} \cup \{2n - 1\} \cup \{n - 1, n - 3, ..., 3,1\} \cup \{2n - 3, 2n - 5, ..., n + 3, n + 1\} \cup \{2n, 2n +$

$1, ..., 2n+m-1\}$.

It can be seen from the exhaustive method that $f(E) \rightarrow [1,2n+m-1]$.

The label sum of $\{V_{adeg=3}\}$ S_3:

$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu)\,|1 \leq i \leq n-1\}|$
$= |\{f(u_0u_i) + f(u_{i-1}u_i) + f(u_iu_{i+1})|1 \leq i \leq n-1\}|$
$= |\{2i + n - i + 1 + 2n - 1 - i\}|$
$= |\{3n\}|$
$= 3n.$

So S_3 is a constant while $n \equiv 0(mod\ 2)$. From the above, the AVREL scheme is established.

As evidenced by the above, all joint graphs $F_n \uparrow S_m$ are AVREL graphs.

Theorem 3: For joint graphs $S_m^{F_n}$, if $m \geq 2, n \geq 3, m \neq n+1$, all are AVREL graphs.

Proof:

By definition 4, assuming the vertices set of $S_m^{F_n}$ $(m \geq 1, n \geq 3)$ is $V = \{u_i, i = 0, 1, ..., n\} \cup \{w_h, h = 0, 1, ..., m\}$, where $V_{adeg=3} = \{hu_i | n-1 \geq i \geq 2, m \geq h \geq 1\}$ $(m \geq 1, n \geq 3)$, the edges set of $S_m^{F_n}$ is $E = \{u_0hu_i, i = 1,2, ..., n, m \geq h \geq 1\} \cup \{hu_ihu_{i+1}, i = 1,2, ..., n-1, m \geq h \geq 1\} \cup \{w_0w_h, h = 1, ..., m\}$. Examples of $S_m^{F_n}$ are shown in figure 7 (1), (2).

Discuss the following two situations:

Case1: When $n \equiv 1(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(w_0w_i) = 2hn, h = i = 1, 2, \dots, m$$

$$f(hu_0hu_i) = 2i - 1 + 2(h-1)n, h = 1, 2, 3, ..., m, i = 1, 2, \dots, n$$

$$f(hu_ihu_{i+1}) = \begin{cases} 2hn - 1 - i, h = 1, 2, 3, ..., m, \\ \quad i \equiv 1(mod\ 2) \\ (2h-1)n + 1 - i, h = 1, 2, 3, ..., m, \\ \quad i \equiv 0(mod\ 2) \end{cases}, i = 1, 2, \dots, n-1$$

From this, the edges labeling set of the $S_m^{mF_n}$ graph:
$f(E) = \{2n, 4n, ..., 2mn\} \cup \{1, 1+2n, ..., 2mn - 1\} \cup \{2n-2, 4n-4, ..., 2mn-n+1\} \cup \{n-1, 3n-3, ..., 2mn-2n+2\}$.

It can be seen from the exhaustive method that f(E) → [1, $2mn$].

The label sum of $\{V_{adeg=3}\}$ S_3:

$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu)\,|\, 2 \leq i \leq n-1, 1 \leq h \leq m\}|$
$= |\{f(hu_{i-1}hu_i) + f(hu_ihu_{i+1}) + f(hu_0hu_i)|2 \leq i \leq n-1, 1 \leq h \leq m\}|$
$= |\{2hn - 1 - i + 1 + 2hn - n + 1 - i + 2i - 1 + 2hn - 2n\}|$
$= |\{(6h-3)n\}|$
$= (6h-3)n.$

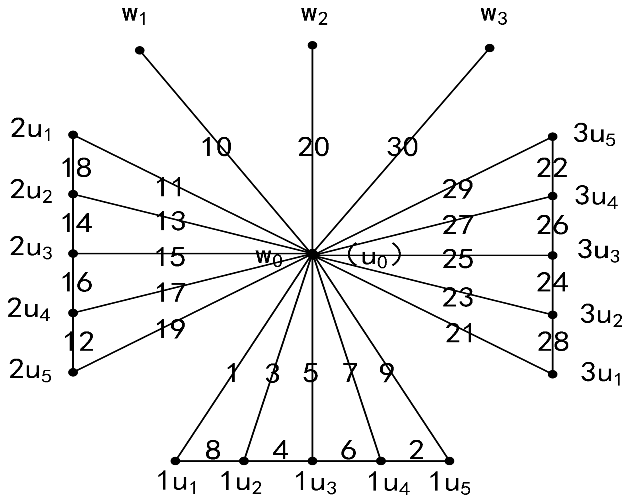
So S_3 is a constant while $n \equiv 1(mod\ 2)$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

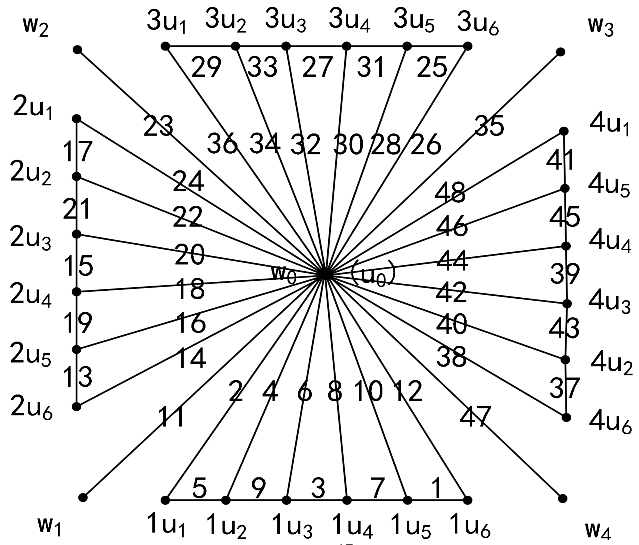
$$f(w_0w_i) = 2hn - 1, h = i = 1, 2, ..., m$$

$$f(hu_0hu_i) = 2i + 2(h-1)n, h = 1, 2, 3, ..., m, i = 1,2, ..., n$$



(1) $S_3^{3F_5}$

$|E(G)| = m + m(n + (n-1)) = 2mn$



(2) $S_4^{4F_6}$

$|E(G)| = m + m(n + (n-1)) = 2mn$

Fig.7 $S_m^{mF_n}$ $(m \geq 2, n \geq 3)$

$$f(hu_ihu_{i+1}) = \begin{cases} (2h-1)n - i,\ h = 1, 2, 3, ..., m, \\ \quad i \equiv 1(mod\ 2) \\ 2hn - 1 - i, h = 1, 2, 3, ..., m, \\ \quad i \equiv 0(mod\ 2) \end{cases}, i = 1,2, .. n-1$$

From this, the edges labeling set of the $S_m^{F_n}$ graph:
$f(E) = \{2n-1, 4n-1, ..., 2mn-1\} \cup \{2, 4+2n, ..., 2mn\} \cup \{n-1, 3n-3, ..., 2mn-2n+1\} \cup \{2n-3, 2n-5, ..., 2mn-n+1\}$.

It can be seen from the exhaustive method that $f(E) \rightarrow [1, 2mn]$.

The label sum of $\{V_{adeg=3}\}$ S_3:

$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu)\,|\, 2 \leq i \leq n-1, 1 \leq h \leq m\}|$
$= |\{f(hu_{i-1}hu_i) + f(hu_ihu_{i+1}) + f(hu_0hu_i)|2 \leq i \leq n-1, 1 \leq h \leq m\}|$
$= 2hn - n - i + 1 + 2hn - 1 - i + 2i + 2hn - 2n$
$= |\{(6h-3)n\}|$
$= (6h-3)n.$

So S_3 is a constant while $n \equiv 0(mod\ 2)$. From the above, the AVREL scheme is established.

As evidenced by the above, all joint graphs $F_n \uparrow S_m$ are AVREL graphs.

Theorem 4: For joint graphs $F_n \uparrow (3, 2, 2) - K$, if $n \geq 3$, all are AVREL graphs.

Proof:

Assuming the vertices set of $F_n \uparrow (3, 2, 2) - K$ $(n \geq 3)$ is $V = \{w_0(u_1), w_1, ..., w_n\} \cup \{u_2(v_1), u_3(v_1')\} \cup \{v_2, v_2'\}$, where $V_{adeg=3} = \{w_i | n-1 \geq i \geq 2\}$ $(n \geq 3)$, $V'_{adeg=3} = \{u_2(v_1), u_3(v_1')\}$, the edges set of $F_n \uparrow (3, 2, 2) - K$ is $E = \{w_0 w_i, i = 1,2, ..., n\} \cup \{w_i w_{i+1}, i = 1,2, ..., n-1\} \cup \{u_1 u_2, u_1 u_3, u_2 u_3\} \cup \{v_1 v_2, v_1' v_2'\}$. Examples of $F_n \uparrow (3, 2, 2) - K$ are shown in figure 8(1), (2).

Discuss the following two situations:

Case1: When $n \equiv 1(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(w_0 w_i) = 2i - 1, i = 1,2, ..., n$$

$$f(w_i w_{i+1}) = \begin{cases} 2n - i - 1, i \equiv 1(mod\ 2) \\ n - i + 1, i \equiv 0(mod\ 2) \end{cases}, i = 1,2, ..., n-1$$

$$f(u_i u_{i+1}) = \begin{cases} 2n, i = 1 \\ 2n + 2, i = 2 \end{cases}$$

$$f(u_1 u_3) = 2n + 1$$

$$f(v_1 v_2) = 2n + 4$$

$$f(v_1' v_2') = 2n + 3$$

From this, the edges labeling set of the $F_n \uparrow (3, 2, 2) - K$ graph:

$f(E) = \{1, 3, ..., 2n - 1\} \cup \{2n - 2, 2n - 4, ..., n + 1\} \cup \{n - 1, n - 3, ..., 2\} \cup \{2n, 2n + 2\} \cup \{2n + 1\} \cup \{2n + 4\} \cup \{2n + 3\}$.

It can be seen from the exhaustive method that $f(E) \rightarrow [1, 2n + 4]$.

The label sum of $\{V_{adeg=3}\}$ S_3, $\{V'_{adeg=3}\}$ S'_3:

$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu) \mid 2 \leq i \leq n - 1, 1 \leq h \leq m\}|$
$= |\{f(w_{i-1} w_i) + f(w_i w_{i+1}) + f(w_0 w_i) | 2 \leq i \leq n - 1, 1 \leq h \leq m\}|$
$= |\{2n - i + 1 - 1 + n - i + 1 + 2i - 1\}|$
$= |\{3n\}|$
$= 3n.$

$S'_3 = |\{\sum_{vu \in NE(u_2(v_1))} f(vu) \mid\mid \sum_{vu \in NE(u_3(v'_1))} f(vu)\}|$
$= |\{f(u_1 u_2) + f(u_2 u_3) + f(v_1 v_2)\}||\{f(u_1 u_3) + f(u_2 u_3) + f(v_1' v_2')\}|$
$= |\{2n + 2n + 2 + 2n + 4\}||\{2n + 1 + 2n + 2 + 2n + 3\}|$
$= |\{6n + 6\}||\{6n + 6\}|$
$= |\{6n + 6\}|$
$= 6n + 6.$

So S_3 and S'_3 are constant while $n \equiv 1(mod\ 2)$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

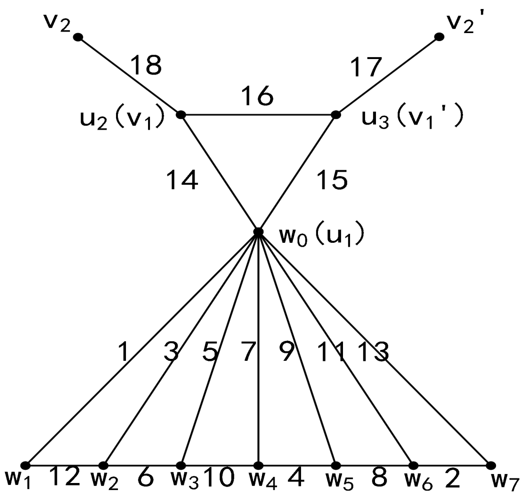
$$f(w_0 w_i) = 2i, i = 1,2, .., n - 1$$

$$f(w_0 w_n) = 2n - 1$$

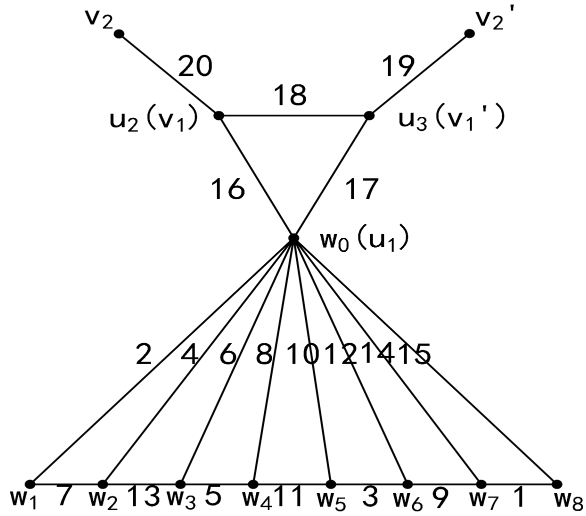
$$f(w_i w_{i+1}) = \begin{cases} n - i \quad i \equiv 1(mod\ 2) \\ 2n - i - 1 \quad i \equiv 0(mod\ 2) \end{cases}, i = 1,2, ..., n - 1$$

$$f(u_i u_{i+1}) = \begin{cases} 2n, i = 1 \\ 2n + 2, i = 2 \end{cases}$$

$$f(u_1 u_3) = 2n + 1$$



(1) $F_7 \uparrow (3, 2, 2) - K$
$|E(G)| = n + (n - 1) + 3 + 2 = 2n + 4$



(2) $F_8 \uparrow (3, 2, 2) - K$
$|E(G)| = n + (n - 1) + 3 + 2 = 2n + 4$

Fig.8 $F_n \uparrow (3, 2, 2) - K$ $(n \geq 3)$

$$f(v_1 v_2) = 2n + 4$$

$$f(v_1' v_2') = 2n + 3$$

From this, the edges labeling set of the $F_n \uparrow (3, 2, 2) - K$ graph:

$f(E) = \{2, 4, ..., 2n - 2\} \cup \{2n - 1\} \cup \{n - 1, n - 3, ..., 1\} \cup \{2n - 3, 2n - 5, ..., n + 1\} \cup \{2n, 2n + 1\} \cup \{2n + 2\} \cup \{2n + 3\} \cup \{2n + 4\}$.

It can be seen from the exhaustive method that $f(E) \rightarrow [1, 2n + 4]$.

The label sum of $\{V_{adeg=3}\}$ S_3, $\{V'_{adeg=3}\}$ S'_3:

$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu) \mid 2 \leq i \leq n - 1, 1 \leq h \leq m\}|$
$= |\{f(w_{i-1} w_i) + f(w_i w_{i+1}) + f(w_0 w_i) | 2 \leq i \leq n - 1, 1 \leq h \leq m\}|$
$= |\{2n - i + 1 - 1 + n - i + 1 + 2i - 1\}|$
$= |\{3n\}|$
$= 3n.$

$S'_3 = |\{\sum_{vu \in NE(u_2(v_1))} f(vu) \mid\mid \sum_{vu \in NE(u_3(v'_1))} f(vu)\}|$
$= |\{f(u_1 u_2) + f(u_2 u_3) + f(v_1 v_2)\}||\{f(u_1 u_3) + f(u_2 u_3) + f(v_1' v_2')\}|$
$= |\{2n + 2n + 2 + 2n + 4\}||\{2n + 1 + 2n + 2 + 2n + 3\}|$
$= |\{6n + 6\}||\{6n + 6\}|$
$= |\{6n + 6\}|$

$= 6n + 6.$

So S_3 and S'_3 are constant while $n \equiv 0(mod\ 2)$. From the above, the AVREL scheme is established.

As evidenced by the above, all joint graphs $F_n \uparrow (3,2,2) - K$ are AVREL graphs.

Theorem 5: For bicyclic graphs, if $4 \le p \le 15$, are AVREL graphs.

Proof:

The results from the AVREL algorithm of bicyclic graphs are shown in Table 1.

Table 1 AVREL for bicyclic graphs when $4 \le n \le 19$

$(p, p+1)$	Graph number	Graph AVREL	Graph Non-AVREL
$(4,5)$	1	1	0
$(5,6)$	5	3	2
$(6,7)$	19	7	12
$(7,8)$	67	23	44
$(8,9)$	236	73	163
$(9,10)$	797	219	578
$(10,11)$	2678	694	1984
$(11,12)$	8833	2046	6787
$(12,13)$	28908	6106	22802
$(13,14)$	93569	17988	75581
$(14,15)$	300748	52372	248376
$(15,16)$	959374	150546	808828

From the data in Table 1, the proportion of the bicyclic graphs that satisfying the AVREL is shown in figure 9.

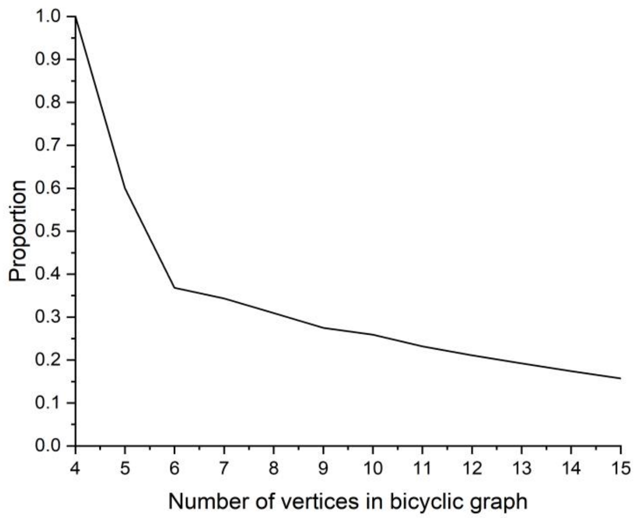


Fig.9 Proportion of bicyclic graphs that meet the AVREL

Bicyclic graphs labeling results are shown in figure 10.

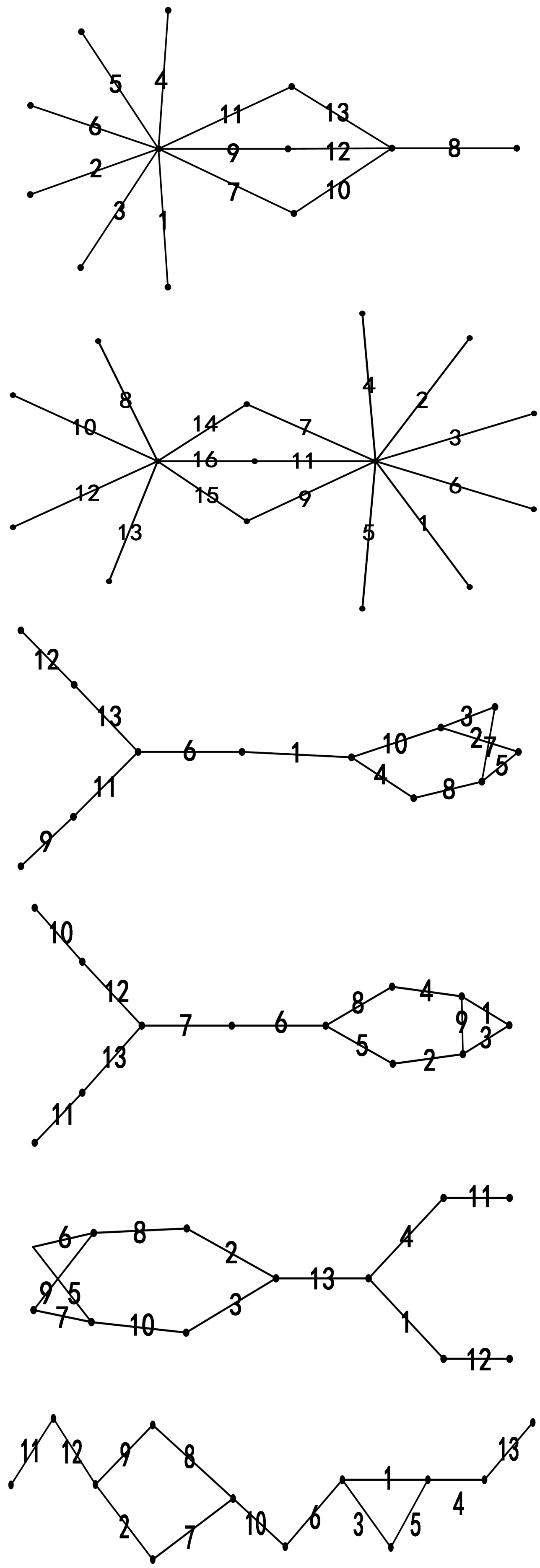
As per the outcomes acquired by the algorithm, it can be inferred that all graphs within 15 vertices fulfill this AVREL condition. However, because of the constraints of the PC and the efficiency of the algorithm, the larger vertices of bicyclic graphs are not tested.

Conjecture 1: For bicyclic graphs, if $n \ge 16$, are AVREL graphs.

Theorem 6: For Möbius ladder graphs ML_n, if $n \equiv 1(mod\ 2)$, all are AVREL graphs, if $2 \le n \le 19$ and $n \equiv 0(mod\ 2)$, all are not AVREL graphs.

Proof:

Assuming the vertices set of Möbius ladder graph $ML_n\ (n \equiv 1(mod\ 2))$ is $V = \{u_i, v_i | n \ge i \ge 1\}(n \equiv 1(mod$



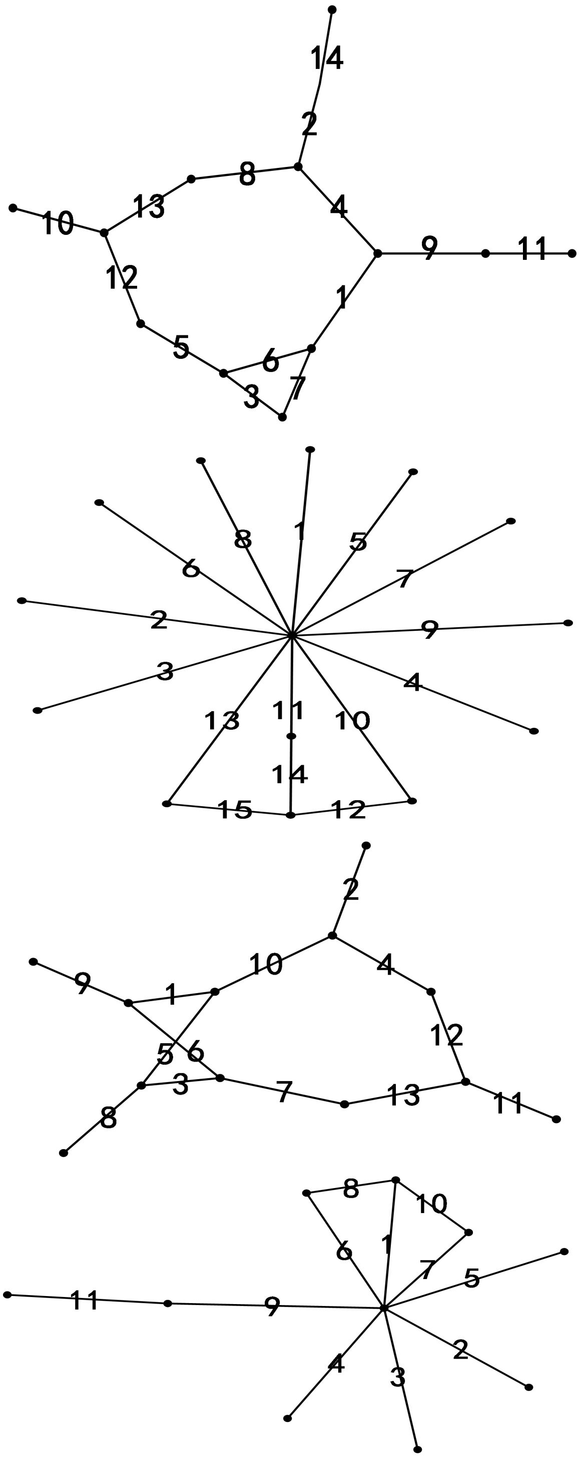


Fig.10 Partial Bicyclic graphs labeling results

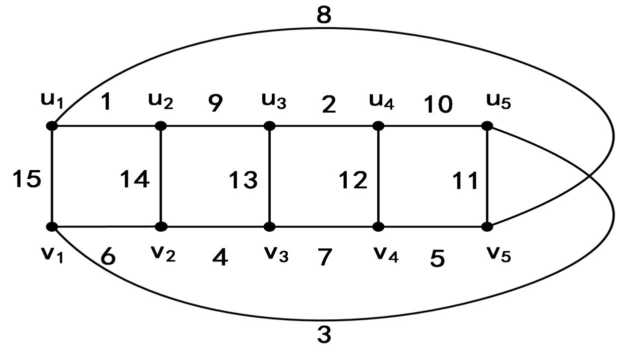
2)), where $V_{adeg=3} = \{u_i, v_i | n \geq i \geq 1\}$ $(n \geq 3)$, the edges set of ML_n is $E = \{u_iv_i, i = 1,2,...,n\} \cup \{u_iu_{i+1}, i = 1,2,...,n-1\} \cup \{v_iv_{i+1}, i = 1,2,...,n-1\} \cup \{u_1v_n, v_1u_n\}$.Exampl es of ML_n are shown in figure 11 (1), (2).

Discuss the following two situations:

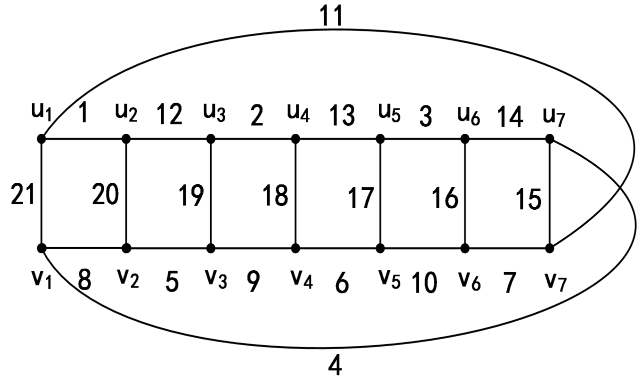
Case1: When $n \equiv 1(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_iv_i) = 3n - i + 1, i = 1,2,...,n$$
$$f(u_nv_1) = (n+1)/2$$
$$f(v_nu_1) = (3n+1)/2$$
$$f(u_iu_{i+1}) = \begin{cases} (i+1)/2, i \equiv 1(mod\ 2); \\ (3n+i+1)/2, i \equiv 0(mod\ 2). \end{cases} i = 1,2,...,n-1$$



(1) ML_5
$|E(G)| = n + n + n = 3n$



(2) ML_7
$|E(G)| = n + n + n = 3n$

Fig.11 ML_n $(n \equiv 1(mod\ 2))$

$$f(v_iv_{i+1}) = \begin{cases} (2n+i+1)/2, i \equiv 1(mod\ 2); \\ (n+i+1)/2, i \equiv 0(mod\ 2). \end{cases} i = 1,2,...,n-1$$

From this, the edges labeling set of the ML_n $(n \equiv 1(mod\ 2))$ graph:

$f(E) = \{3n, 3n-1, ..., 2n+1\} \cup \{(n+1)/2\} \cup \{(3n+1)/2\} \cup \{1,2,...,(n-1)/2, (3n+3)/2, (3n+5)/2, ..., 2n\} \cup \{n+1, n+2, ..., (3n-1)/2, (n+3)/2, (n+5)/2, ..., n\}$.

It can be seen from the exhaustive method that $f(E) \to [1, 3n]$.

The label sum of $\{V_{adeg=3}\}$ S_3:

$$S_3 = |\{\sum_{vu \in NE(u_i)} f(vu) \,||\, \sum_{vu \in NE(u_3(v_i))} f(vu)\}|$$
$$= |\{f(u_iv_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) \,||\, f(u_iv_i) + f(v_{i-1}v_i) + f(v_iv_{i+1})\}|$$
$$= |\{3n - i + 1 + (i+1)/2 + (3n+i-1+1)/2 \,||\, 3n - i + 1 + (2n+i+1)/2 + (n+i-1+1)/2\}|$$
$$= |\{(9n+3)/2 || (9n+3)/2\}|$$
$$= |\{(9n+3)/2\}|$$
$$= (9n+3)/2.$$

The calculated set of edge labels is $f(E)$:

So S_3 is a constant while $n \equiv 1(mod\ 2)$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

The results from the AVREL algorithm of the Möbius ladder graphs ML_n $(2 \leq n \leq 19)$ are shown in Table 2:

Table 2 AVREL for ML_n while $2 \leq n \leq 19$

ML_n	AVREL(n)	Non − AVREL(n)
$2 \leq n \leq 19$	3, 5, 7, 9, 11, 13, 15, 17, 19	2, 4, 6, 8, 10, 12, 14, 16, 18

From Table 2, the proportion of the Möbius ladder graphs that satisfy the AVREL is shown in figure 12.

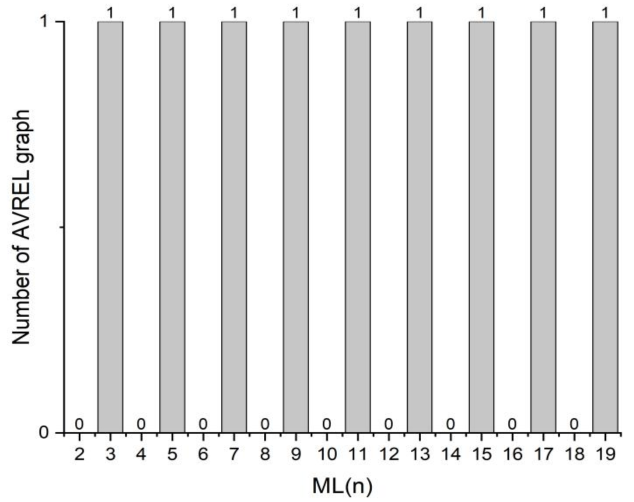


Fig.12 Proportion of Möbius ladder graphs that meet the labeling

From figure 12, it can be clearly seen that the results for ladder graphs up to 19 vertices show a distinct odd-even distribution. The results of the odd-numbered vertex graphs all meet the condition AVREL, and the even-numbered vertex graphs are not AVREL graphs.

As evidenced by the above, all Möbius ladder graphs ML_n $(n \equiv 1(mod\ 2))$ are AVREL graphs; Möbius ladder graphs ML_n $(n \equiv 0(mod\ 2), 2 \le n \le 19)$ are not AVREL graphs.

Conjecture 2: For Möbius ladder graphs ML_n, if $n \equiv 0(mod\ 2)$, all are not AVREL graphs.

Theorem 7: For Multi-circle gear graph MG_n, if $n \ge 3, n \ne 5$, all are AVREL graphs, when $n = 5$, the Multi-circle gear graph MG_5 is not AVREL graph.

Proof:

Assuming the vertices set of Multi-circle gear graph MG_n $(n \ge 3)$ is $V = \{u_i, v_i | n \ge i \ge 1\} \cup \{u_0\}$ $(n \equiv 1(mod\ 2))$, where $V_{adeg=5} = \{u_i | n \ge i \ge 1\}$ $(n \ge 3, n \ne 5)$ or $V_{adeg=5} = \{u_i | n \ge i \ge 1\} \cup \{u_0\}$ $(n = 5)$ the edges set of MG_n is $E = \{u_0u_i, i = 1,2,...,n\} \cup \{u_iu_{i+1}, i = 1,2,...,n-1\} \cup \{u_n u_1\} \cup \{u_iv_i, i = 1,2,...,n\} \cup \{u_iv_{i-1}, i = 2,3,...,n\} \cup \{u_iv_n\}$. Examples of $\mathrm{MG_n}$ are shown in figure 13 (1), (2).

Discuss the following three situations:

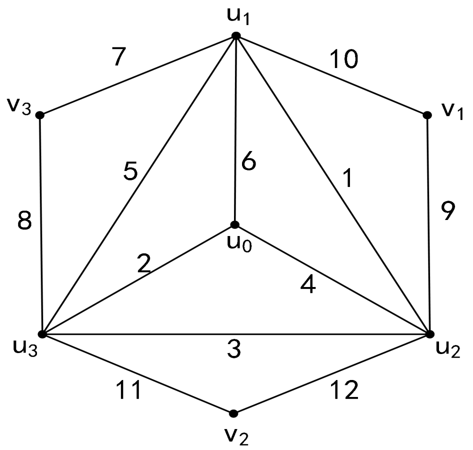
Case1: When $n \equiv 1(mod\ 2), n \ne 5$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

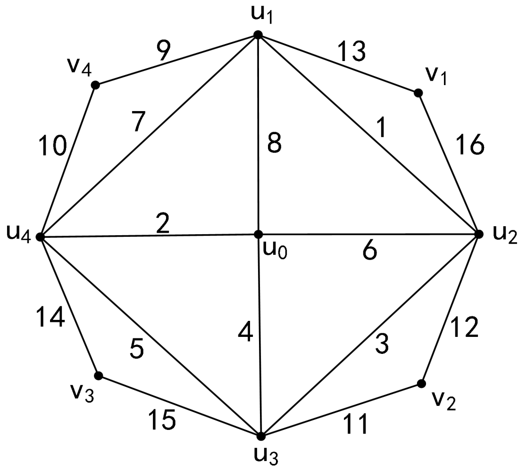
$$f(u_iu_{i+1}) = 2i - 1, i = 1,2,...,n-1$$
$$f(u_nv_1) = 2n - 1$$
$$f(u_0u_i) = 2n - 2i + 2, i = 1,2,...,n$$
$$f(u_iv_i) = \begin{cases} 3n - i + 2, n \equiv 1(mod\ 2) \\ 4n - i + 2, n \equiv 0(mod\ 2) \end{cases}, n \ge i \ge 1$$
$$f(u_iv_{i-1}) = \begin{cases} 4n - i + 2, n \equiv 1(mod\ 2) \\ 3n - i + 2, n \equiv 0(mod\ 2) \end{cases}, n \ge i \ge 2$$
$$f(u_1v_n) = 2n + 1$$

From this, the edges labeling set of the MG_n ($n \equiv 1(mod\ 2), n \ne 5$) graph:

$f(E) = \{1,3,...,2n-3\} \cup \{2n-1\} \cup \{2n, 2n-2,...,2\} \cup \{3n+1, 3n-1,...,2n+2, 4n, 4n-2,...,3n+3\} \cup \{4n+1, 4n-1,...,3n+2, 3n, 3n-2,...,2n+3\} \cup \{2n+1\}$.



(1) MG_3
$|E(G)| = n + n + 2n = 4n$



(2) MG_4
$|E(G)| = n + n + 2n = 4n$

Fig.13 MG_n $(n \ge 3)$

It can be seen from the exhaustive method that $f(E) \to [1, 4n]$.

The label sum of $\{V_{adeg=5}\}$ S_5:

$S_5 = |\{\sum_{vu \in NE(u_i)} f(vu)\ |n \ge i \ge 1, n \ge 3, n \ne 5\}|$
$= |\{f(u_0u_i) + f(u_nu_1) + f(u_iu_{i+1}) + f(u_1v_n) + f(u_iv_i) | i = 1 || f(u_0u_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_{i-1}) + f(u_iv_i)\} | n-1 \ge i \ge 2 || f(u_0u_i) + f(u_{n-1}u_n) + f(u_nu_1) + f(u_iv_{i-1}) + f(u_iv_i) | i = n||$
$= |\{2n + 2n - 1 + 1 + 2n + 1 + 3n + 1 || 2n - 2i + 2 + 2i - 3 + 2i - 1 + 3n - i + 2 + 4n - i + 2 || 2 + 2n - 3 + 2n - 1 + 4n - n + 2 + 3n - n + 2\}|$
$= |\{9n + 2 || 9n + 2 || 9n + 2\}|$
$= |\{9n + 2\}|$
$= 9n + 2.$

The calculated set of edge labels is $f(E)$:

So S_5 is a constant while $n \equiv 1(mod\ 2), n \ne 5$. From the above, the AVREL scheme is established.

Case2: When $n \equiv 0(mod\ 2)$:

Through formula (1) and the algorithm results, consider the following edge labeling scheme:

$$f(u_iu_{i+1}) = 2i - 1, i = 1,2,...,n-1$$
$$f(u_nv_1) = 2n - 1$$
$$f(u_0u_i) = 2n - 2i + 2, i = 1,2,...,n$$
$$f(u_iv_{i-1}) = \begin{cases} 3n - i + 2, n \equiv 1(mod\ 2) \\ 4n - i + 2, n \equiv 0(mod\ 2) \end{cases}, n \ge i \ge 2$$

$$f(u_iv_i) = \begin{cases} 3n+1, i=1 \\ 4n-i+2, n \equiv 1(mod\ 2) \\ 3n-i+2, n \equiv 0(mod\ 2) \end{cases}, n \geq i \geq 2$$
$$f(u_1v_n) = 2n+1$$

From this, the edges labeling set of the MG_n $(n \equiv 0(mod\ 2))$ graph:

$f(E) = \{1,3,...,2n-3\} \cup \{2n-1\} \cup \{2n, 2n-2,...,2\} \cup \{3n+1, 4n-1, 4n-3,...,3n+3, 3n, 3n-2..., 2n+2\} \cup \{3n-1, 3n-3,...,2n+3, 4n, 4n-2,...,3n+2\} \cup \{2n+1\}$.

It can be seen from the exhaustive method that $f(E) \to [1, 4n]$.

The label sum of $\{V_{adeg=5}\}$ S_5:

$S_5 = |\{\sum_{vu\in NE(u_i)} f(vu)\ |n \geq i \geq 1, n \geq 3\}|$
$= |\{f(u_0u_i) + f(u_nu_1) + f(u_iu_{i+1}) + f(u_1v_n) + f(u_iv_i)|i = 1||f(u_0u_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_{i-1}) + f(u_iv_i)\}|n-1 \geq i \geq 2||f(u_0u_i) + f(u_{n-1}u_n) + f(u_nu_1) + f(u_iv_{i-1}) + f(u_iv_i)|i = n||$
$= |\{2n + 2n - 1 + 1 + 2n + 1 + 3n + 1||2n - 2i + 2 + 2i - 3 + 2i - 1 + 4n - i + 2 + 3n - i + 2||2 + 2n - 3 + 2n - 1 + 4n - n + 2 + 3n - n + 2\}|$
$= |\{9n + 2||9n + 2||9n + 2\}|$
$= |\{9n + 2\}|$
$= 9n + 2.$

The calculated set of edge labels is $f(E)$:

So S_5 is a constant while $n \equiv 0(mod\ 2)$. From the above, the AVREL scheme is established.

Case3: When $n = 5$:

Prove by contradiction: when $n = 5$, the Multi-circle gear graph MG_5 is not AVREL graph.

Assume when $n = 5$, the Multi-circle gear graph MG_5 is AVREL graph. Then the $V_{adeg=5} = \{u_i|5 \geq i \geq 1\} \cup \{u_0\}$, according to the formula (1), the following conditions must be satisfied: $S(u_1) = S(u_2) = S(u_3) = S(u_4) = S(u_5) = S(u_0)$, and the edge label sum of MG_5 is $Sum = (20 \times 21)/2 = 210$.

We assume that $A = f(u_1v_1) + f(u_2v_1) + f(u_2v_2) + f(u_3v_2) + f(u_3v_3) + f(u_4v_3) + f(u_4v_4) + f(u_5v_4) + f(u_5v_5) + f(u_1v_5)$, the label sum of the remaining edges is B,

$$Sum = A + B \quad (2)$$

Based on the geometric properties of the graph MG_5, $S(u_1) + S(u_2) + S(u_3) + S(u_4) + S(u_5) + S(u_0) = S_1 = 2B + A = Sum \times 2 - A$, get the following formula (3):

$$(Sum \times 2 - A) mod 6 = 0 \quad (3)$$

That is

$$(0 - A mod 6) mod 6 = 0 \quad (4)$$

According to formula (3), the following is obtained:

$$A mod 6 = 0 \quad (5)$$

Assuming that S_1 is maximized, then A is minimized, the label set of the set of edges covered by A is $\{1, 2, ..., 9, 10\}$, the label sum is $A = 55$, however, according to formula (5), we could get the $A_{min} = 60$, then the $S_{1max} = 360$, while $B_{max} = S_1/2 = 180$.

According to formula (2), the $B'_{max} = 210 - A = 150$. For $B_{max} \neq B'_{max}$ and $B_{max} > B'_{max}$, the result contradicts the premise of the hypothesis, the hypothesis does not hold. So when $n = 5$, the Multi-circle gear graph MG_5 is not AVREL graph.

As evidenced by the above, all Multi-circle gear graph MG_n $(n \geq 3, n \neq 5)$ are AVREL graphs; when $n = 5$, the Multi-circle gear graph MG_5 is not AVREL graph.

V. Conclusion

The findings introduced in this article rich our understanding of reducible series, a new concept of Adjacent Vertex Reducible Edge Labeling was proposed based on the current concept of reducible labeling, and a new heuristic algorithm was proposed which depended on the intelligent algorithm, and the Adjacent Vertex Reducible Edge Labeling algorithm has been utilized to calculate all non-isomorphic graphs within finite vertices, some theorems and guesses were given.

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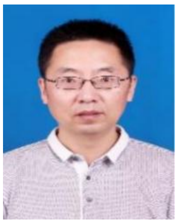

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