Comparison of Methods to Derive Relative Weights

Scott Shu-Cheng Lin

Abstract—**Identifying critical factors, objectives, criteria, or alternatives in operating research methods is a vital decision-making problem in academic society. Besides, fuzzy theorems, inventory models, and optimization approaches, the analytic hierarchy process is an important tool to decide relative weights for several factors, objectives, criteria, or alternatives. This paper will examine the relationship among several methods to derive the relative weight. Saaty and Vargas provided an example to prove that the relative weight is the same by the Eigenvalue Method (EM) and by the Logarithmic Least Square Method (LLSM). Saaty and Vargas did not know the relative weight by the Least Square Method (LSM). It is an open question proposed by Saaty and Vargas. Saaty is the founding father of the Analytic Hierarchy Process (AHP) and Vargas is one of the main contributors to AHP, respectively. Hence, in this paper, we present a detailed examination of the open question. We obtain that there are two (non-consistent) comparison matrices under the Saaty-Vargas construction which satisfied the property of the relative weights derived by three methods: EM, LLSM, and LSM are the same. We not only solve the open question proposed by Saaty and Vargas but also extend it to a more general setting. We consider all possible conditions for $a \in \{2,3,...,9\}$ such that for $a = 2$, we provided an analytical proof; for $a = 3$, we show a hybrid method such that a discrete sensitivity analysis for the open question proposed by Saaty and Vargas is completely solved by us; for $a \in \{4,5,...,9\}$, we present numerical approaches. Our findings help researchers to decide which evaluation method will be adopted to derive the relative weights for their future research. Our results will help practitioners to decide the relative weights for factors, objectives, criteria, or alternatives in research methods.**

Index Terms—**Eigenvalue method, Logarithmic least square method, Least square method, Analytic hierarchy process**

I. Introduction

To select proper research methods to deal with various encountered and proposed problems is an important task in academic society. To select proper research methods to deal with various problems in modern society. During the planning stage, many factors, objectives, criteria, or alternatives needed to consider, and then to synthesize those mentioned items the most common approach is to apply the weighted arithmetic mean such that the relative weights for those items become a vital research topic. Hence, how to derive relative weights from a comparison matrix is an important issue for practitioners applying the Analytic Hierarchy Process (AHP) concerning their research problems. The most three common approaches to obtain relative weights are (a) the Eigenvalue Method (EM), (b) the Logarithmic Least Square Method (LLSM), and (c) the Least Square Method (LSM).

For a given comparison matrix, denoted as $\lfloor a_{ij} \rfloor_{n\times n}$, the relative weight derived by EM, denoted as w, is the normalized eigenvector corresponding to the largest eigenvalue, denoted as $\lambda_{\max}$, which is the most popular approach in the academic society when applying AHP in their study such that

$$Aw = \lambda_{\max} w, \tag{1.1}$$

that is,

$$\sum_{k=1}^{n} a_{ik} w_k = \lambda_{\max} w_i, \tag{1.2}$$

for $i = 1,2,...,n$, and $\sum_{k=1}^{n} w_k = 1$.

If $u = (u_1, u_2, ..., u_n)$ is the normalized solution for LLSM, and then u is derived by

$$Min \sum_{i,j=1}^{n} \left[\left((\ln a_{ij}) - \ln(u_i/u_j) \right)^2 \right], \tag{1.3}$$

with $\sum_{k=1}^{n} u_k = 1$. Crawford and Williams [1] claimed that the normalized geometric mean of elements in each row, $u_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \Big/ \sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj}}$, for $i = 1,...,n$, is the solution for LLSM.

If $v = (v_1, v_2, ..., v_n)$ is the normalized solution for LSM, and then v is obtained by

$$Min \sum_{i,j=1}^{n} \left[\left(a_{ij} - (v_i/v_j) \right)^2 \right], \tag{1.4}$$

with $\sum_{k=1}^{n} v_k = 1$. Jensen [2] claimed that LSM is difficult to solve because the objective function is non-linear, non-convex and solutions are not unique.

In general, three different approaches: EM, LLSM, and LSM will imply different findings. When a matrix is nearly consistent, then the logarithmic least square method and the eigenvalue method will obtain almost the same solutions, for example, Golden and Wang [3], Saaty [4], and Tone [5]. On

Manuscript received August 26, 2022; revised March 14, 2023.

Scott Shu-Cheng Lin is an associate professor in the Department of Hotel Management, Lee-Ming Institute of Technology, No. 22, Sec. 3, Tai-Lin Rd., Taishan Dist., New Taipei City 243, Taiwan (e-mail: sclin200@mail.lit.edu.tw)

the other hand, Saaty and Vargas [6] developed an inconsistent three-by-three comparison matrix, under the 1-9 scale proposed by Saaty [7], denoted as A, such that

$$A=\left[a_{ij}\right]=\begin{bmatrix}1 & 2 & 0.5\\ 0.5 & 1 & 2\\ 2 & 0.5 & 1\end{bmatrix}, \quad (1.5)$$

such that the second row and the third row are permutations of the first row.

Researchers tried to answer the question: Whether or not W, u and V are the same?

Saaty and Vargas [6] already derived that

$$w=u=\left(1/3,1/3,1/3\right)^T. \quad (1.6)$$

Therefore, there is an open question proposed by Saaty and Vargas [6]:
By LSM, derive the relative weight for the comparison matrix of Equation (1.5).

How to find the exact solution of v by LSM becomes a challenging research topic.

Saaty and Vargas [6] had been cited 196 times to indicate that it is an important paper. Except for Bozóki [8], and Lin [9], the rest papers only mentioned Saaty and Vargas [6] in their introduction such that they did not provide further discussions for the unsolved problem.
To study Bozóki [8], In the following, we have to provide a brief review of four related papers: Bozóki [10], Farkas and Rózsa [11], Bozóki and Lewis [12], and Bozóki [8].

Bozóki [10] followed Kurosh [13] to solve systems of nonlinear equations to construct a polynomial of degree 24 for a 3 by 3 comparison matrix, and then Bozóki [10] applied a polynomial-solver algorithm to locate all positive real roots. Farkas and Rózsa [11] applied Newton's method of successive approximation to locate a solution for the least square method. However, His method depends on a good starting point to find the solution. Bozóki and Lewis [12] applied the resultant method mentioned by Kurosh [13] to find common roots for a system of multivariate polynomials for 3 by 3 and 4 by 4 comparison matrices. Bozóki [8] used a homotopy method to find common positive zeros of a system of polynomials and then solved by the symbolic computations of Maple with the code of Li [14] and Gao et al. [15]. We can claim that Bozóki [10], Farkas and Rózsa [11], Bozóki and Lewis [12], and Bozóki [8] had provided tremendous improvements to present general solution approaches for the least square method. However, they did not apply their methods to solve the original problem of Equation (1.4) with matrix A of Equation (1.5), proposed by Saaty and Vargas [6]. Moreover, their general solution approach obtained approximated solutions. Lin [9] studied the comparison matrix of Equation (1.5) and then claimed the priority vector derived by LSM is the same as LLSM and EM.
In the next section, we will try to answer the following more generalized problem: Under the construction of Saaty and Vargas [6], how to derive the relative weights by LSM?
To apply the analytic hierarchy process in solving multi-criteria and multi objectives problems, deciding the relative weights for several important factors is the crucial issue that will influence the analysis of reasons and outcomes concerning research topics. Kurttila et al. [16] applied the analytic hierarchy process to build a hybrid process to study forest certifications. Erensal et al. [17] applied the fuzzy analytic hierarchy process to consider technology management with key capabilities in Turkey. Chang et al. [18] built an expert decision-making process with a fuzzy hierarchy of multiple criteria and alternatives. Larimian et al. [19] developed a model with a fuzzy analytic hierarchy process to estimate environmental sustainability. We will follow this trend to study the relationship among factors, objectives, criteria, or alternatives for applying the analytic hierarchy process in research methods.

II. Our Analytical Extension

We abstract handle this problem to change from 2 to a, with $a\in\{2,3,...9\}$ to satisfy the 1-9 scale proposed by Saaty [7], because when $a=1$, the comparison matrix is consistent. Consequently, 0.5 is varied to $1/a$. Hence, we will examine the following comparison matrix,

$$A=\left[a_{ij}\right]=\begin{bmatrix}1 & a & 1/a\\ 1/a & 1 & a\\ a & 1/a & 1\end{bmatrix}, \quad (2.1)$$

such that the second row and the third row still satisfy the rule that are permutations of the first row. We try to find that $v=\left(v_1,v_2,v_3\right)^T$ with v_1, v_2 and v_3 are obtained by solving Equation (1.4) with the comparison matrix A of Equation (2.1), under the constraints of $v_i>0$, for $i=1,2,3$, and $v_1+v_2+v_3=1$. In this section, we will consider the existing problem of the minimum solution.

We are motivated by Lin [9], to convert a three-variables problem with one constraint into a two-variables problem without any constraint. We substitute Equation (2.1)into Equation (1.4), and to simplify the expressions, we assume $x=v_1/v_2$ and $y=v_1/v_3$, to derive that $x/y=v_3/v_2$, then we will handle the following minimum problem, denoted as $F(x,y)$, for $x>0$ and $y>0$,

$$F(x,y)=(a-x)^2+\left(\frac{1}{a}-y\right)^2+\left(\frac{1}{a}-\frac{1}{x}\right)^2 +\left(a-\frac{y}{x}\right)^2+\left(a-\frac{1}{y}\right)^2+\left(\frac{1}{a}-\frac{x}{y}\right)^2, \quad (2.2)$$

where a is an integer with $a\in\{1,2,...,9\}$.

The existing problem of the minimum solution for Equation (2.2), when $a=2$ had been discussed in Lin [9]. We follow his approach to extend his proof to a more general setting of $a\in\{1,2,...,9\}$.

We selectively pick a point, for example (a,a), then

$$F(a,a)=(a-1)^2\left(3+\frac{4}{a}+\frac{3}{a^2}\right), \quad (2.3)$$

which is a finite number.

For later discussion, we assume that

$$A=\frac{1}{(1/a)+\sqrt{F(a,a)}}, \quad (2.4)$$

and

$$B=a+\sqrt{F(a,a)}. \quad (2.5)$$

If $0<x<A$, then

$$\frac{1}{x}>\frac{1}{A}=\frac{1}{a}+\sqrt{F(a,a)}, \quad (2.6)$$

to imply that

$$F(x,y)>\left(\frac{1}{a}-\frac{1}{x}\right)^2>F(a,a), \quad (2.7)$$

for any $y>0$.

If $x>B$, then

$$x>a+\sqrt{F(a,a)}, \quad (2.8)$$

to yield that

$$F(x,y)>(a-x)^2>F(a,a), \quad (2.9)$$

for any $y>0$.

Based on our above discussions, the minimum will not occur if $0<x<A$ or $x>B$ such that the minimum problem will be shrunk under the condition of

$$A\le x\le B. \quad (2.10)$$

By the same argument, for the variable y, we can assume that

$$C=\frac{1}{a+\sqrt{F(a,a)}}, \quad (2.11)$$

and

$$D=\frac{1}{a}+\sqrt{F(a,a)}. \quad (2.12)$$

If $0<y<C$, then

$$\frac{1}{y}>\frac{1}{C}=a+\sqrt{F(a,a)}, \quad (2.13)$$

to imply that

$$F(x,y)>\left(a-\frac{1}{y}\right)^2>F(a,a), \quad (2.14)$$

for any $x>0$.

If $y>D$, then

$$y>\frac{1}{a}+\sqrt{F(a,a)}, \quad (2.15)$$

to yield that

$$F(x,y)>\left(\frac{1}{a}-y\right)^2>F(a,a), \quad (2.16)$$

for any $x>0$.

Based on our above discussions, the minimum will not occur if $0<y<C$ or $y>D$ such that the minimum problem will be shrunk under the condition of

$$C\le y\le D. \quad (2.17)$$

We combine our findings of Equations (2.10) and (2.17) to imply that our minimum problem can be shrunk from an unbounded domain, $\{(x,y):0<x,0<y\}$, to a compact domain, $\{(x,y):A\le x\le B, C\le y\le D\}$.

The continuous function $F(x,y)$ will attain its minimum on a compact set such that we verify that the minimum problem of Equation (2.2) has solutions. We summarize our findings in the next theorem.

Theorem 1. The minimum problem of $F(x,y)$ with Equation (2.2), for $x>0$ and $y>0$, have solutions.

III. The Solution of the First Partial Derivatives

We solve the first partial derivatives concerning *x* and *y* of Equation (2.2) to imply

$$\frac{\partial}{\partial x}F(x,y)=\frac{2}{ax^3y^2}\big(ax^4+ax^4y^2-a^2x^3y^2 -x^3y+a^2xy^3+xy^2-ay^2-ay^4\big), \quad (3.1)$$

and

$$\frac{\partial}{\partial y}F(x,y)=\frac{2}{ax^2y^3}\big(-ax^4+x^3y+a^2x^2y +ax^2y^4-x^2y^3-ax^2-a^2xy^3+ay^4\big). \quad (3.2)$$

Next, we consider the solution for the first partial derivatives system, $\partial F/\partial x=0$ and $\partial F/\partial y=0$, and find

$$ax^4+ax^4y^2+a^2xy^3+xy^2 =a^2x^3y^2+x^3y+ay^2+ay^4, \quad (3.3)$$

and

$$ax^4+x^2y^3+ax^2+a^2xy^3 =x^3y+a^2x^2y+ax^2y^4+ay^4. \quad (3.4)$$

We take the difference between Equations (3.3) and (3.4) to derive that

$$ax^4y^2+xy^2-x^2y^3-ax^2 =a^2x^3y^2+ay^2-a^2x^2y-ax^2y^4. \quad (3.5)$$

We rewrite Equation (3.5) as follows

$$ax^4y^2+xy^2-a^2x^3y^2-ay^2= x^2y^3+ax^2-a^2x^2y-ax^2y^4. \quad (3.6)$$

For the left hand of Equation (3.6), we derive that

$$\begin{aligned}&ax^4y^2+xy^2-a^2x^3y^2-ay^2\\&=y^2\left(ax^4+x-a^2x^3-a\right)\\&=y^2\left(ax^3(x-a)+x-a\right)\end{aligned}$$

$$= y^2\left(a x^3 + 1\right)\left(x - a\right). \quad (3.7)$$

We plug our result of Equation (3.7) into Equation (3.6) and then separate variables *x* and *y* to imply that

$$\frac{\left(a x^3 + 1\right)\left(x - a\right)}{x^2} = \frac{1}{y^2}\left(y^3 + a - a^2 y - a y^4\right). \quad (3.8)$$

Motivated by the left-hand side of Equation (3.8), we can convert the right-hand side of Equation (3.8) as

$$\frac{1}{y^2}\left(y^3 + a - a^2 y - a y^4\right)$$

$$= \frac{y^4}{y^2}\left(\frac{1}{y} - a + \frac{a}{y^3}\left(\frac{1}{y} - a\right)\right)$$

$$= \frac{1}{\left(1/y\right)^2}\left(\left(\frac{a}{y^3} + 1\right)\left(\frac{1}{y} - a\right)\right). \quad (3.9)$$

We plug our findings of Equation (3.9) into Equation (3.8) to obtain that

$$\frac{\left(a x^3 + 1\right)\left(x - a\right)}{x^2} =$$

$$\frac{1}{\left(1/y\right)^2}\left(\left(\frac{a}{y^3} + 1\right)\left(\frac{1}{y} - a\right)\right). \quad (3.10)$$

Based on our derivation of Equation (3.10), we will examine the following auxiliary function, denoted as $f(t)$, where

$$f(t) = \frac{\left(a t^3 + 1\right)\left(t - a\right)}{t^2}, \quad (3.11)$$

for $t > 0$.

Remark. When $a = 2$, a similar result of Equation (3.11) is already derived by Lin [9].

If a point, say (x_0, y_0), satisfies the first partial derivative system of $\frac{\partial F}{\partial x}\Big|_{(x_0, y_0)} = 0$ and $\frac{\partial F}{\partial y}\Big|_{(x_0, y_0)} = 0$, and then by Equations (3.10) and (3.11), we obtain a necessary condition

$$f(x_0) = f\left(\frac{1}{y_0}\right). \quad (3.12)$$

We summarize our results in the following theorem.

Theorem 2. If (x_0, y_0) satisfies the first partial derivation system of $F(x, y)$, then $f(x_0) = f(1/y_0)$ must be held.

IV. THE PROPERTY FOR AUXILIARY FUNCTIONS

The discussion in this section already appeared in Lin [9], when $a = 2$. However, when $a = 3$, we still need the property of the auxiliary function, $f(x)$ under different conditions. Hence, we still provide a further examination of the property for auxiliary functions, with $a \in \{1, 2, ..., 9\}$. Hence, for our latter discussion, we begin to study the monotonic property of $f(t)$.

We derive that

$$\frac{df(t)}{dt} = \frac{1}{t^3}\left(2a t^4 - a^2 t^3 - t + 2a\right). \quad (4.1)$$

Our goal is to find the conditions for *a* to guarantee that $f(t)$ is a strictly increasing function which is to show that $df(t)/dt > 0$, for $t > 0$. Therefore, we assume the numerator of the right-hand side of Equation (4.1) as a new auxiliary function, denoted as $N(t)$, to imply that

$$N(t) = 2a t^4 - a^2 t^3 - t + 2a, \quad (4.2)$$

for $t > 0$. Our goal is to find conditions of *a* to guarantee that

$$N(t) > 0, \quad (4.3)$$

for $t > 0$. We find that

$$\frac{dN(t)}{dt} = 8a t^3 - 3a^2 t^2 - 1, \quad (4.4)$$

and

$$\frac{d^2 N(t)}{dt^2} = 24a t^2 - 6a^2 t, \quad (4.5)$$

for $t > 0$. From Equation (4.5), we solve $d^2 N(t)/dt^2 = 0$ to find two inflection points: $t = 0$, and $t = a/4$. Under the domain of $t > 0$, we derive that there is only an inflection point, that occurred at $t = a/4$ such that $N(t)$ is concave down for $0 < t < a/4$ and $N(t)$ is concave up for $t > a/4$.

Owing to $\frac{d^2 N(t)}{dt^2} = \frac{d}{dt}N'(t) < 0$ for $0 < t < a/4$ and $\frac{d^2 N(t)}{dt^2} = \frac{d}{dt}N'(t) > 0$ for $t > a/4$, we imply that $N'(t)$ is decreasing for $0 < t < a/4$ and $N'(t)$ is increasing for $t > a/4$. $N'(t)$ decreases from $\lim_{t \to 0} N'(t) = -1$ to $N'(a/4) = -1 - \left(a^4/16\right)$, and then $N'(t)$ increases from $N'(a/4) = -1 - \left(a^4/16\right)$ to $\lim_{t \to \infty} N'(t) = \infty$. Hence, there is a unique point, say t^Δ, that satisfies

$$N'\left(t^\Delta\right) = 0. \quad (4.6)$$

We find that $N'(t) < 0$ for $0 < t < t^\Delta$, and $N'(t) > 0$, for $t > t^\Delta$. We imply that $N(t)$ decreases for $0 < t < t^\Delta$, and $N(t)$ increases for $t > t^\Delta$. Moreover, t^Δ is the global minimum point for $N(t)$, under the domain, $t > 0$, that is, $N\left(t^\Delta\right)$ is the global minimum value for $N(t)$.

Consequently, our goal of Equation (4.3) to show the positivity of $N(t)$ is converted to verify that

$$N\left(t^{\Delta}\right) > 0. \tag{4.7}$$

We summarize our results in the next theorem.

Theorem 3. To prove $N(t) > 0$, for $t > 0$ is equivalent to show that $N\left(t^{\Delta}\right) > 0$, where t^{Δ} is the unique solution for $N'(t) = 0$.

We recall Equation (4.4) for $N'(t)$ and Equation (4.6) with $N'\left(t^{\Delta}\right) = 0$, and then we know that

$$8a\left(t^{\Delta}\right)^3 = 3a^2\left(t^{\Delta}\right)^2 + 1. \tag{4.8}$$

Now, we compute $N\left(t^{\Delta}\right)$, and then

$$N\left(t^{\Delta}\right) = 2a\left(t^{\Delta}\right)^4 - a^2\left(t^{\Delta}\right)^3 - t^{\Delta} + 2a. \tag{4.9}$$

Based on Equation (4.8), we obtain that

$$2a\left(t^{\Delta}\right)^4 = \frac{3a^2\left(t^{\Delta}\right)^3 + t^{\Delta}}{4}. \tag{4.10}$$

We plug Equation (4.10) into Equation (4.9) to derive that

$$N\left(t^{\Delta}\right) = -\frac{a^2}{4}\left(t^{\Delta}\right)^3 - \frac{3}{4}t^{\Delta} + 2a. \tag{4.11}$$

We apply Equation (4.8) again to simplify $N\left(t^{\Delta}\right)$ as

$$N\left(t^{\Delta}\right) = \frac{3}{32}\left(21a - 8t^{\Delta} - a^3\left(t^{\Delta}\right)^2\right). \tag{4.12}$$

From Equation (4.8), we obtain that

$$t^{\Delta} = \frac{3}{8}a + \frac{1}{8a\left(t^{\Delta}\right)^2}. \tag{4.13}$$

Motivated by Equation. (4.13), we try to find an upper bound of $1/8a\left(t^{\Delta}\right)^2$.

For our later discussion, we need the following assertion,

$$t^{\Delta} > \frac{1}{a}. \tag{4.14}$$

We recall that $N'(t) < 0$ for $0 < t < t^{\Delta}$, and $N'(t) > 0$ for $t > t^{\Delta}$ such that to verify our assertion of Equation (4.14) is equivalent to proving that

$$N'\left(\frac{1}{a}\right) < 0. \tag{4.15}$$

Hence, using Equation (4.4), we compute $N'\left(\frac{1}{a}\right)$, then

$$N'\left(\frac{1}{a}\right) = \frac{8}{a^2} - 4. \tag{4.16}$$

We recall that the domain of a is $\{2,3,...,9\}$ such that $a^2 \geq 4$ to imply Equation (4.15) is valid. Consequently, our assertion of Equation (4.14) also is verified.

Based on our result of Equation (4.14), we obtain that

$$\frac{a}{8} > \frac{1}{8a\left(t^{\Delta}\right)^2}. \tag{4.17}$$

We plug our finding of Equation (4.17) into Equation (4.13) to find an upper bound of t^{Δ} as

$$t^{\Delta} < \frac{a}{2}. \tag{4.18}$$

We summarize our results in the following theorem.

Theorem 4. We find a lower bound and an upper bound to estimate t^{Δ}, and then $\frac{3}{8}a < t^{\Delta} < \frac{a}{2}$.

V. Result for a=2

We must confess that the result of $a = 2$ already mentioned in Lin [9]. However, we still provide a simplified version for $a = 2$ to help readers to realize the solution procedure among (i) $a = 2$, (ii) $a = 3$, and (iii) $a \in \{4,5,...,9\}$ are different.

Using our result of an upper bound in Theorem 4, we estimate $N\left(t^{\Delta}\right)$ of Equation (4.12), and then

$$N\left(t^{\Delta}\right) > \frac{3a}{128}\left(68 - a^4\right). \tag{5.1}$$

Based on our findings of Equation (5.1), our problem is separated into two cases: $a \in \{2\}$ and $a \in \{3,4,..,9\}$ to discuss.

If $a \in \{2\}$, from Equation (5.1), we show that $N\left(t^{\Delta}\right) > 0$. According to our Theorem 3, $N(t) > 0$, for $t > 0$ then we recall Equations (4.1) and (4.2) to derive that $f'(t) > 0$ and $f(t)$ is a strictly increasing function.

Recall our Theorem 2, if (x_0, y_0) satisfies the first partial derivation system of $F(x, y)$, and $f(t)$ is a strictly increasing function for $a = 2$ then we obtain the necessary condition

$$y_0 = \frac{1}{x_0}. \tag{5.2}$$

Hence, we plug our results of Equation (5.2) into $F(x, y)$ to convert the expression from $F(x, y)$ into $F(x, y = 1/x)$, then

$$F\left(x, y = \frac{1}{x}\right) = 2(a - x)^2 + 2\left(\frac{1}{a} - \frac{1}{x}\right)^2 + \left(a - \frac{1}{x^2}\right)^2 + \left(\frac{1}{a} - x^2\right)^2. \tag{5.3}$$

We summarize our results in the following theorem.

Theorem 5. When $a \in \{2\}$, the original two-variable minimum problem, $F(x,y)$, of Equation (2.2) is converted to a one-variable minimum problem, $F(x,y=1/x)$ of Equation (5.3).

We begin to find the minimum of $F(x,y=1/x)$, for $x>0$. we derive that

$$\frac{d}{dx}F\left(x,y=\frac{1}{x}\right)=\frac{4}{a\,x^5}g(x), \qquad (5.4)$$

where $g(x)$ is a new auxiliary function, with

$$g(x)=a\,x^8+(a-1)x^6 - a^2x^5+x^3+(a^2-a)x^2-a. \qquad (5.5)$$

We rewrite Equation (5.5) as

$$g(x)=(x-1)h(x), \qquad (5.6)$$

with the third auxiliary function $h(x)$ that satisfies

$$h(x)=a\left(\frac{x^8-1}{x-1}\right)-x^3\left(\frac{x^3-1}{x-1}\right) - a^2x^2\left(\frac{x^3-1}{x-1}\right)+a\,x^2\left(\frac{x^4-1}{x-1}\right). \qquad (5.7)$$

Our goal is to verify that $x=1$ is a unique positive solution for $g(x)=0$. Hence, we will begin to show that

$$h(x)>0, \qquad (5.8)$$

for $x>0$.

When $a=2$, we find that

$$h(x)=2x^7+2x^6+3x^5-x^4-x^3+2x+2. \qquad (5.9)$$

When $x\ge 1$, we obtain that

$$\begin{aligned} h(x) &> 2x^6+3x^5-x^4-x^3 \\ &> x^6+x^5-x^4-x^3 \\ &= x^4(x^2-1)+x^3(x^2-1)>0. \end{aligned} \qquad (5.10)$$

When $0<x<1$, we know that

$$h(x)>2-x^4-x^3>0. \qquad (5.11)$$

Based on Equations (5.10) and (5.11), we derive that the assertion of Equation (5.8) is valid when $a=2$.

When $a=2$, we show that the minimum problem of $F(x,y=1/x)$ occurs at $x^*=1$ such that $y^*=1/x^*=1$. Hence, we obtain

$$v_1^*=v_2^*=v_3^*. \qquad (5.12)$$

By the normalization $v_1^*+v_2^*+v_3^*=1$, we derive that

$$v_1^*=v_2^*=v_3^*=\frac{1}{3}, \qquad (5.13)$$

for LSM, with $a=2$.

We recall the results of Equation (1.6), by Saaty and Vargas [6] for EM and LLSM and then summarize our findings in the next theorem.

Theorem 6. We provide a positive answer for the open question proposed by Saaty and Vargas [6] such that when $a=2$, the relative weights of Equation (1.5) are derived by EM, LLSM, and LSM are the same.

Remark. We admit that Theorem 6 already appeared in Lin [9].

VI. Result for a=3

When $a=3$, Equation (5.1) cannot help to show that $N(t^{\Delta})>0$. By the numerical method, by Equation (4.8), we obtain that

$$t^{\Delta}=1.156171, \qquad (6.1)$$

and then we derive that

$$N(t^{\Delta})=1.655>0. \qquad (6.2)$$

Hence, similar to the previous case of $a=2$, when $a=3$, $f(t)$ also is a strictly increasing function. If (x,y) satisfies the first partial derivative system of $\frac{\partial F}{\partial x}\Big|_{(x,y)}=0$ and $\frac{\partial F}{\partial y}\Big|_{(x,y)}=0$, then we derive that $x=1/y$. Hence, we can convert $F(x,y)$ to $F(x,y=1/x)$. We combine our results in the next theorem.

Theorem 7. When $a=3$, the original two-variable minimum problem, $F(x,y)$, of Equation (2.2) is converted to a one-variable minimum problem, $F(x,y=1/x)$ of Equation (5.3).

Consequently, we consider Equation (5.7) with $a=3$, to find that

$$h(x)=3x^7+3x^6+5x^5 -4x^4-4x^3-3x^2+3x+3. \qquad (6.3)$$

When $x\ge 1$, we obtain that

$$\begin{aligned} h(x) &> 3x^7+3x^6+5x^5-4x^4-4x^3-3x^2 \\ &= 3x^4(x^3-1)+3x^3(x^3-1) \\ &\quad +3x^2(x^3-1)+x^4(x-1)+x^3(x^2-1)\ge 0. \end{aligned} \qquad (6.4)$$

When $0<x\le 0.6$, we derive that

$$\begin{aligned} h(x) &> -4(0.6)^4-4(0.6)^3-3(0.6)^2+3 \\ &= 0.538>0. \end{aligned} \qquad (6.5)$$

For $0.6<x<1$, it is difficult to show that $h(x)>0$ by an analytic method. Hence, we will apply a numerical approach. We list our results for $h'(x)$ in Table 1, and for $h(x)$ in Table 2.

Based on Table 1, we know that an estimated solution

TABLE I
Numerical results for $h'(x)$, with $a=3$.

x	0.7	0.72	0.729	0.7297	0.7298	0.730	0.8
$h'(x)$	-1.07	-0.386	-0.031	-2.39×10^{-3}	1.75×10^{-3}	0.01	3.971

TABLE II
Numerical results for $h(x)$, with $a=3$.

x	0.7	0.72	0.729	0.7297	0.7298	0.730	0.8
$h(x)$	2.737	2.723	2.721273	2.721262	2.721262	2.721263	2.847578

of $h'(x)=0$ can be treated as 0.7298. From $h'(x)<0$, with $x<0.7298$ and $h'(x)>0$, with $x>0.7298$, we can take 0.7298 as an approximated minimum solution. Our assertion is supported by the numerical result of Table 2 to compare values for $h(x)$.

Owing to the minimum value,

$$h(0.7298)=2.721262>0, \tag{6.6}$$

we obtain that

$$h(x)>0, \tag{6.7}$$

for $0.6<x<1$.

We combine our findings of Equation (6.4) for $x\geq1$, Equation (6.5) for $0<x\leq0.6$, and Equation (6.7) for $0.6<x<1$, to conclude that

$$h(x)>0, \tag{6.8}$$

for $0<x$.

Hence, our assertion of Equation (5.8) is valid for $a=3$ such that $g(x)=0$ of Equation (5.5) has a unique solution at $x=1$, when $a=3$.

Similar to our discussion for Equation (5.12), we obtain our findings for $a=3$ in the next theorem.

Theorem 8. When $a=3$, the relative weights of Equation (2.1) derived by EM, LLSM, and LSM are the same.

VII. Our Numerical Results

When $a=4$, based on Equation (4.8), we numerically find that

$$t^{\Delta}=1.513639, \tag{7.1}$$

and then by Equation (4.9), we obtain

$$N(t^{\Delta})=-7.0069<0. \tag{7.2}$$

Hence, when $a=4$, we will show that $f(t)$ is not a strictly increasing function in the following figure 1.

Based on the above Figure 1, we numerically know that $f(t)$ increases for $0<t<0.93$, and decreases for

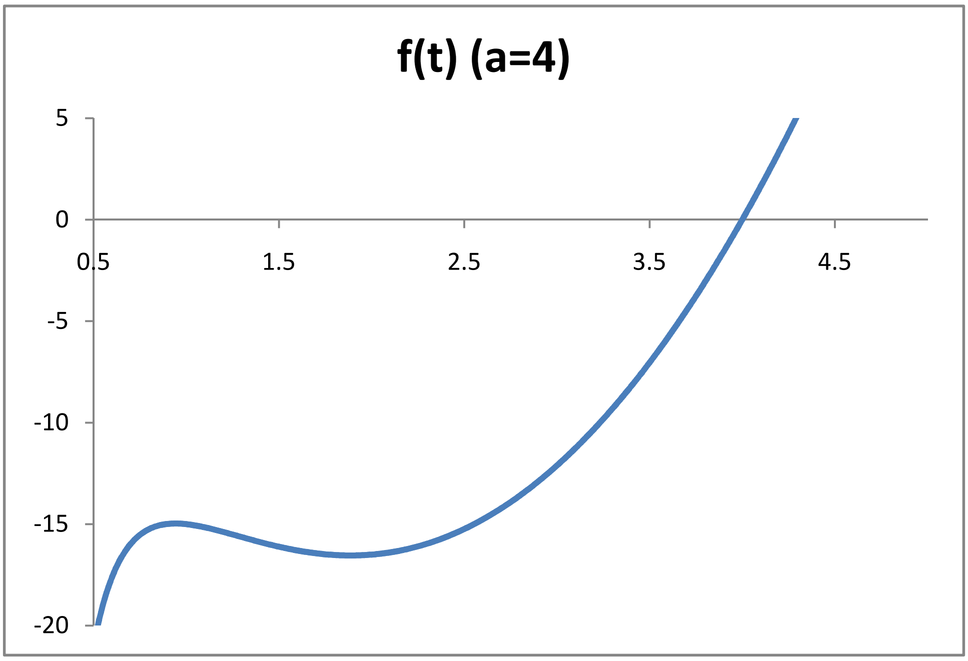

Fig. 1 Graph of f(t), with a = 4.

$0.93 < t < 1.88$, and then increases for $t > 1.88$. Based on Figure 1, we obtain that $f(t)$ is not a one-to-one function. Hence, our previous treatment to convert the objective function from $F(x, y)$ to $F(x, y = 1/x)$ is no longer valid.

We numerically find the minimum value for $F(x, y)$ at 28.445342, that occurs at three solutions:

$$(x_1, y_1) = (0.676972, 0.458291), \quad (7.3)$$

$$(x_2, y_2) = (2.182012, 1.477161), \quad (7.4)$$

and

$$(x_3, y_3) = (0.676972, 1.477161). \quad (7.5)$$

Based on Equations (7.3-7.5), we find that the uniqueness of the minimum point for the non-consistent comparison matrix proposed by Saaty and Vargas [6] with $a = 4$, by LSM, is not valid.

Similarly, for $a \in \{4,5,\ldots,9\}$, we find t^{Δ} by Equation (4.8), and then we compute $N(t^{\Delta})$ by Equation (4.9). Our findings are listed in the following Table 3.

From Table 3, when $a \in \{4,5,\ldots,9\}$, we always obtain that $N(t^{\Delta}) < 0$. Therefore, we cannot mimic our approach for $a \in \{2,3\}$ to change the expression of our minimum problem from $F(x, y)$ to $F(x, y = 1/x)$. On the other hand, we follow our method for $a = 4$, adopting numerical methods to find minimum values and minimum points. We list our findings in the following Table 4.

From Table 4, we know that the minimum values of F(x, y) attained at three different points that are not x = 1 and y = 1, when $a \in \{4,5,\ldots,9\}$. On the other hand, by EM, we know the maximum eigenvalue is $1 + a + (1/a)$ and the corresponding normalized relative weight is $(1/3, 1/3, 1/3)$ for EM. Consequently, the respected solution for Equation (2.2), is derived as x = 1 and y = 1.

Moreover, Saaty and Vargas [6] mentioned that the relative weight derived by LLSM $(u_1, u_2, \ldots, u_n)$ satisfies that

$$u_i = \sqrt[n]{\left(\prod_{j=1}^{n} a_{ij}\right)} \Bigg/ \sum_{k=1}^{n} \sqrt[n]{\left(\prod_{j=1}^{n} a_{kj}\right)}. \quad (7.6)$$

Based on Equation (7.6), we find that the corresponding normalized relative weight is $(1/3, 1/3, 1/3)$ for LLSM. Consequently, the respected solution for Equation (2.2), is derived as $x = 1$ and $y = 1$, for $a \in \{4,5,\ldots,9\}$.

We compare our results among EM, LLSM, and LSM for $a \in \{4,5,\ldots,9\}$ to conclude the next theorem.

Theorem 9. When $a \in \{4,5,\ldots,9\}$, the relative weights of Equation (2.1) derived by EM, and LLSM are the same. On the other hand, the relative weights obtained by LSM are different from EM.

We merge our findings from Theorems 6, 8, and 9 to summarize our final theorem.

Theorem 10. The open question proposed by Saaty and Vargas [6] to construct an inconsistent comparison matrix, under the structure of Equation (2.1) only has two positive answers: a = 2, and a = 3.

TABLE III

The results of t^{Δ} and $N(t^{\Delta})$, for $a \in \{5,6,\ldots,9\}$.

a	5	6	7	8	9
t^{Δ}	1.882057877	2.254100269	2.627586413	3.001734106	3.376218446
$N(t^{\Delta})$	-33.077268	-92.767678	-210.20295	-419.00087	-763.85364

TABLE IV

When $a \in \{4,5,\ldots,9\}$, minimum values, and minimum points of $F(x, y)$.

		Min points		
a	Min value	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)
5	47.7154	(3.4077, 1.8460)	(0.5417,0.2935)	(0.5417, 1.8460)
6	71.0252	(4.5026, 2.1219)	(0.4713, 0.2221)	(0.4713, 2.1219)
7	98.3734	(5.5602, 2.3580)	(0.4241, 0.1798)	(0.4241, 2.3580)
8	129.7545	(6.6004, 2.5691)	(0.3892, 0.1515)	(0.3892, 2.5691)
9	165.1636	(7.6308, 2.7624)	(0.3620, 0.1310)	(0.3620, 2.7624)

VIII. Direction for Future Research

We check our numerical results for a ∈ {4,5, … ,9} to find out some interesting results. When $a=4$, after we find the first pair for the minimum point, denoted as (x_1, y_1), the second and the third minimum point satisfy that

$$(x_2, y_2) = (1/y_1, 1/x_1), \quad (8.1)$$

and

$$(x_3, y_3) = (x_1, 1/x_1). \quad (8.2)$$

On the other hand, for a ∈ {4,5, … ,9}, if we assume the first pair for the minimum point, denoted as (x_1, y_1), the second and third minimum points satisfy that

$$(x_2, y_2) = (1/y_1, 1/x_1), \quad (8.3)$$

and

$$(x_3, y_3) = (1/y_1, y_1). \quad (8.4)$$

If $F(x_1, y_1)$ attains the minimum value, then we compute

$$\begin{aligned} F(1/y_1, 1/x_1) &= (a-(1/y_1))^2 + ((1/a)-x_1)^2 \\ &+ ((1/a)-y_1)^2 + (a-(y_1/x_1))^2 + (a-x_1)^2 \\ &+ ((1/a)-(x_1/y_1))^2 = F(x_1, y_1), \end{aligned} \quad (8.5)$$

to derive that $F(1/y_1, 1/x_1)$ also attains the minimum value. Hence, we provide a reasonable explanation of why the second pair satisfy Equation (8.1) for $a=4$, and Equation (8.3) for a ∈ {4,5, … ,9}.

However, up to now, we still cannot provide a logical justification for the diversity of Equations (8.2) and (8.4). When $a=4$, If we directly compare (x_1, y_1) of Equation (7.3) and (x_3, y_3) of Equation (7.5), then we find that $x_1 = x_3$, and $y_3 = 1/x_3$ such that the relation of Equation (5.2), $y = 1/x$ that we used to develop our solution approach for a ∈ {2,3}, seems to appear again.

Similarly, for a ∈ {4,5, … ,9}, we consider the third and the fifth columns to find that $y_1 = y_3$ and $y_3 = 1/x_3$ such that the relation of Equation (5.2), $y = 1/x$ that we constructed to solve the minimum solution for a ∈ {2,3}, surprisingly appears.

We predict that providing a reasonable explanation for one minimum solution still satisfies $y = 1/x$, for a ∈ {4,5, … ,9} will be a good research issue for future researchers.

We recall that Tolga et al. [20] combined the analytic hierarchy process and fuzzy replacement analysis to execute operating system selection. By analytic network approach, Kahraman et al. [21] constructed a fuzzy optimization model for the quality function deployment planning process. Vaidya and Kumar [22] provided a literature review for applications with the analytic hierarchy process. Dagdeviren and Yuksel [23] established a fuzzy analytic hierarchy process system to examine safety management under behavior constraints. We will follow this trend to develop new models to decide the relative weights among factors, objectives, criteria, or alternatives for applying the analytic hierarchy process in research methods.

Several related papers by Okonkwo and Ade-Ibijola [24], Chen et al. [25], Unyapoti and Pochai [26], Ren et al. [27], Kusuma et al. [28], and Geng et al. [29], are worthy to mention to reveal the current research trend.

IX. Restricted Equivalence Functions in Fuzzy Sets and Systems

We apply a similar mathematical method to study the pending questionable findings of Bustince et al. [30].

We recall the following assertion of Bustince et al. [30], they mentioned that the expected value of $\mu_{Q_t}(q)$ for $q = 0,\ldots,t$ is

$$E(\mu_{Q_t}(q)) = \frac{m_b(t)}{L-1}. \quad (9.1)$$

They assumed that

$$REF(x,y) = 1 - |x-y|, \quad (9.2)$$

and

$$\varphi(x) = x, \quad (9.3)$$

such that we derive that

$$\mu_{Q_t}(q) = 1 - \left|\frac{q - m_b(t)}{L-1}\right|. \quad (9.4)$$

Based on Equations (9.1-9.3), we compute that

$$\begin{aligned} 1 - E(\mu_{Q_t}(q)) &= \frac{\sum_{q=0}^{t} h(q)\left|\frac{q-m_b(t)}{L-1}\right|}{\sum_{q=0}^{t} h(q)} \\ &= \frac{\sum_{q=0}^{t_0} h(q)(q-m_b(t)) + \sum_{q=t_0}^{t} h(q)(m_b(t)-q)}{(L-1)\sum_{q=0}^{t} h(q)} \\ &= \frac{\sum_{q=0}^{t_0} qh(q) + \sum_{q=t_0}^{t} qh(q) - 2\sum_{q=t_0}^{t} qh(q) + m_b(t)\left[\sum_{q=t_0}^{t} h(q) - \sum_{q=0}^{t_0} h(q)\right]}{(L-1)\sum_{q=0}^{t} h(q)} \\ &= \frac{m_b(t)}{L-1} + \alpha, \end{aligned} \quad (9.5)$$

where α is an abbreviation with

$$\alpha = \frac{2\sum_{q=t_0}^{t} qh(q) - m_b(t)\left[\sum_{q=t_0}^{t} h(q) - \sum_{q=0}^{t_0} h(q)\right]}{(L-1)\sum_{q=0}^{t} h(q)}. \quad (9.6)$$

We check our results of Equation (9.5) to reveal the assertion of Bustince et al. [30] mentioned in Equation (9.1) that contained questionable findings.

X. An Independent Set in Pattern Recognition

In this section, we study the pending problem to construct a set of three or four independent vectors which is a preliminary work to develop a solution procedure for pattern recognition problems as proposed by Chu et al. [31] and Yen et al. [32].

First, we recall a theorem by Chu et al. [31] that mentioned that if $x + y + z = 1$, and $w \neq 0$, according to

$$\det\begin{bmatrix} x & y & z \\ x+2w & y+w & z-3w \\ x+w & y+2w & z-3w \end{bmatrix} = 3w^2(x+y+z) = 3w^2 \neq 0, \quad (10.1)$$

then three vectors (x, y, z), $(x+2w,\ y+w,\ z-3w)$, and $(x+w,\ y+2w,\ z-3w)$ are independent of three vectors.

Following Equation (10.1) of Chu et al. [31], we compute

$$\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1+\alpha & w_2-\alpha & w_3 \\ w_1+\alpha & w_2 & w_3-\alpha \end{bmatrix} = \det\begin{bmatrix} w_1 & w_2 & w_3 \\ \alpha & -\alpha & 0 \\ \alpha & 0 & -\alpha \end{bmatrix}$$

$$= w_1\alpha^2 - w_2(-\alpha^2) + w_3\big(0-(-\alpha^2)\big) = (w_1+w_2+w_3)\alpha^2 = \alpha^2, \quad (10.2)$$

under the condition of $w_1 + w_2 + w_3 = 1$.

Based on Equation (10.2), we know that if $w_1 + w_2 + w_3 = 1$ and $\alpha \neq 0$, then three vectors (w_1, w_2, w_3), $(w_1+\alpha, w_2-\alpha, w_3)$, and $(w_1+\alpha, w_2, w_3-\alpha)$ are independent.

The pending problem is to generalize the above results to any finite-dimensional vector space.

We will extend our results from three independent vectors to generalize to four independent vectors.

The determinant of four vectors: (w_1, w_2, w_3, w_4), $(w_1+\alpha, w_2-\alpha, w_3, w_4)$, $(w_1+\alpha, w_2, w_3-\alpha, w_4)$ and $(w_1+\alpha, w_2, w_3, w_4-\alpha)$ is evaluated as follows,

$$\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1+\alpha & w_2-\alpha & w_3 & w_4 \\ w_1+\alpha & w_2 & w_3-\alpha & w_4 \\ w_1+\alpha & w_2 & w_3 & w_4-\alpha \end{bmatrix}. \quad (10.3)$$

We apply the row operation to simplify Equation (10.3) to yield that

$$\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ \alpha & -\alpha & 0 & 0 \\ \alpha & 0 & -\alpha & 0 \\ \alpha & 0 & 0 & -\alpha \end{bmatrix}. \quad (10.4)$$

We apply the first-row expansion of Equation (10.4) to derive that the determinant,

$$w_1(-\alpha^3) - w_2\alpha^3 + w_3(-\alpha^3) - w_4\alpha^3 = (w_1+w_2+w_3+w_4)(-\alpha^3) = -\alpha^3, \quad (10.5)$$

under the condition of $w_1 + w_2 + w_3 + w_4 = 1$.

Based on our above derivations, we will propose the following open question for the future researcher: To show that the determinant n vectors: $(w_1, w_2, \ldots, w_n)$, $(w_1+\alpha, w_2-\alpha, \ldots, w_4)$, $(w_1+\alpha, w_2, w_3-\alpha, \ldots, w_4)$ and $(w_1+\alpha, w_2, w_3, \ldots, w_{n-1}, w_4-\alpha)$ are linearly independent, by the determinant composed of above mentioned n vectors is

$$(-1)^{n-1}\alpha^{n-1}(w_1 + w_2 + \ldots + w_n). \quad (10.6)$$

XI. Conclusion

We examine the open question left by Saaty and Vargas [6]. We extend their open question to a more general setting such that we consider all possible cases under the construction proposed by Saaty and Vargas [6] from a special case $a = 2$ to a more generalized environment as $a \in \{2,3,\ldots,9\}$.

We not only provide a positive answer for their open question when $a = 2$ but also find another inconsistent comparison matrix with $a = 3$ that also has the property: the relative weights derived by EM, LLSM, and LSM are the same.

Moreover, when $a \in \{4,5,\ldots,9\}$, we apply a numerical method to show that the relative weights derived by LSM are not unique such that under the construction of Saaty and Vargas [6], to obtain relative weights being identical for EM, LLSM, and LSM are only valid for $a = 2$, and $a = 3$.

On the other hand, we prove that an assertion mentioned by Bustince et al. [30] contained questionable findings. Moreover, we solve a pending problem to find a family of independent vectors, for three and four-dimensional vector spaces.

Our results will help researchers develop a relationship among factors, objectives, criteria, or alternatives to accommodate the analytic hierarchy process in their future research.

References

[1] G. Crawford, C. Williams, "A note on the analysis of subjective judgment matrices," *Journal of Mathematical Psychology*, vol. 29**,** 1985, pp. 387-405.

[2] R. E. Jensen, "An alternative scaling method for priorities in hierarchical structures," *Journal of Mathematical Psychology*, vol. 28, 1984, pp. 317-332.

[3] B. L. Golden, Q. Wang, "An alternate measure of consistency," in B. L. Golden, E. A. Wasil, and P. T. Harker (Eds), The Analytic Hierarchy Process: Application and Studies, 1989, pp. 68-81.

[4] T. L. Saaty, "Eigenvalue and logarithmic least squares," *European Journal of Operational Research*, vol. 48, 1990, pp. 156-160.

[5] K. Tone, "A logarithmic least squares method for incomplete pairwise comparisons in the analytic hierarchy process," *Institute for Policy Science Research Report*, 1993, pp. 94-B-2.

[6] T. L. Saaty, L. G. Vargas, "Comparison of eigenvalue, logarithmic least square and least square methods in estimating ratios," *Mathematical Modelling*, vol. 5, 1984, 309-324.

[7] T.L. Saaty, "A scaling method for priorities in hierarchy structures," *Journal of Mathematical Psychology*, vol. 15, 1977, pp. 234-281.

[8] S. Bozóki, "Solution of the least squares method problem of pairwise comparison matrices," *Central European Journal of Operational Research*, vol. 16, 2008, pp. 345-358.

[9] S. S. C. Lin, "Analytic hierarchy process by least square method revisit," *Mathematical Problems in Engineering*, vol. 2019, 2019, Article ID 2797515, 5 pages.

[10] S. Bozóki, "A method for solving LSM problems of small size in the AHP," *Central European Journal of Operational Research*, vol. 11, 2003, pp. 17-33.

[11] A. Farkas, A. Rózsa, "On the non-uniqueness of the solution to the least-squares optimization of pairwise comparison matrices," *Acta Polytechnica Hungarica, Journal of Applied Sciences*, vol. 1, 2004, pp. 1-20.
[12] S. Bozóki, R. H. Lewis, "Solving the least squares method problem in the AHP for 3 × 3 and 4 × 4 matrices," *Central European Journal of Operational Research*, vol. 13, no. 3, 2005, pp. 255-270.
[13] A. G. Kurosh, Lectures on General Algebra. (in Hungarian, translator Pollák György), 1971, Tankönyvkiadó, Budapest.
[14] T. Y. Li, "Numerical solution of multivariate polynomial systems by homotopy continuation methods," *Acta Numerica*, vol. 6, 1997, pp. 399-436.
[15] T. Gao, T. Y. Li, X. Wang, "Finding isolated zeros of polynomial systems in C^n with stable mixed volumes," *Journal of Symbolic Computation*, vol. 28, 1999, pp. 187-211.
[16] M. Kurttila, M. Pesonen, J. Kangas, M. Kajanus, "Utilizing the analytic hierarchy process (AHP) in SWOT analysis—a hybrid method and its application to a forest-certification case* 1," *Forest Policy and Economics*, vol. 1, no. 1, 2000, pp. 41–52.
[17] Y. C. Erensal, T. Oncan, M. L. Demircan, "Determining key capabilities in technology management using fuzzy analytic hierarchy process: A case study of Turkey," *Information Sciences*, vol. 176, no. 18, 2006, pp. 2755–2770.
[18] C. W. Chang, C. R. Wu, H. L. Lin, "Applying fuzzy hierarchy multiple attributes to construct an expert decision making process," *Expert Systems with Applications*, vol. 36, no. 4, 2009, pp. 7363–7368.
[19] T. Larimian, Z. S. S. Zarabadi, A. Sadeghi, "Developing a fuzzy AHP model to evaluate environmental sustainability from the perspective of Secured by Design scheme—A case study," *Sustainable Cities and Society*, vol. 7, 2013, pp. 25–36.
[20] E. Tolga, M. L. Demircan, C. Kahraman, "Operating system selection using fuzzy replacement analysis and analytic hierarchy process," *International Journal of Production Economics*, vol. 97, no. 1, 2005, pp. 89–117.
[21] C. Kahraman, T. Ertay, G. Buyukozkan, "A fuzzy optimization model for QFD planning process using analytic network approach," *European Journal of Operational Research*, vol. 171, no. 2, 2006, pp. 390–411.
[22] O. S. Vaidya, S. Kumar, "Analytic hierarchy process: An overview of applications," *European Journal of Operational Research*, vol. 169, no. 1, 2006, pp. 1–29.
[23] M. Dagdeviren, I. Yuksel, "Developing a fuzzy analytic hierarchy process (AHP) model for behavior-based safety management," *Information Sciences*, vol. 178, no. 6, 2008, pp. 1717–1733.
[24] C. W. Okonkwo, A. Ade-Ibijola, "Revision-Bot: A Chatbot for Studying Past Questions in Introductory Programming," *IAENG International Journal of Computer Science*, vol. 49, no. 3, 2022, pp. 644-652.
[25] G. Chen, S. Wang, Y. He, W. Shang, H. Long, "An Improved Chimp Optimization Algorithm for Short-term Hydrothermal Scheduling," *IAENG International Journal of Computer Science*, vol. 49, no. 3, 2022, pp. 666-682.
[26] P. Unyapoti, N. Pochai, "A Shoreline Evolution Model With the Wavelength Effect of Breaking Waves on Groin Structures," *IAENG International Journal of Computer Science*, vol. 49, no. 3, 2022, pp. 683-694.
[27] X. Ren, J. Ma, V. Y. Tsviatkou, V. K. Kanapelka, "A New Hard-decision Iterative Decoding Method for Hamming Product Codes," *Engineering Letters*, vol. 30, no. 3, 2022, pp, 948-954.
[28] P. D. Kusuma, R. A. Nugrahaeni, D. Adiputra, "Coordinated Ambulance Routing Problem for COVID-19 by Using Cloud-Theory-based Simulated Annealing to Minimize Number of Unserved Patients and Total Travel Distance," *Engineering Letters*, vol. 30, no. 3, 2022, pp. 955-963.
[29] L. Geng, J. Zhou, Y. Wang, J. Tang, "Existence of Generalized Augmented Lagrange Multipliers for Cone Constrained Optimization Problems," *Engineering Letters*, vol. 30, no. 3, 2022, pp. 964-971.
[30] H. Bustince, E. Barrenechea, M. Pagola, "Restricted equivalence functions," *Fuzzy Sets and Systems,* vol. 157, 2006, pp. 2333–2346.
[31] C. H. Chu, K. C. Hung, P. Julian, "A Complete Pattern Recognition Approach Under Atanassov's Intuitionistic Fuzzy Sets," *Knowledge-Based Systems*, vol. 66, 2014, pp. 36-45.
[32] P. C. P. Yen, K. C. Fan, H. C. J. Chao, "A New Method for Similarity Measures for Pattern Recognition," *Applied Mathematical Modelling*, vol. 37, 2013, pp. 5335-5342.

Scott Shu-Cheng Lin is an associate professor, at the Department of Hotel Management, Lee-Ming Institute of Technology. He received his Ph.D. degree from the Department of International Business Administration, Chinese Cultural University, in 1999. His research interest includes Analytic Hierarchy Process, Inventory Models, Fuzzy Sets, and Communities in Networks.