Parameter Estimation for Ornstein-Uhlenbeck Process Driven by Sub-fractional Brownian Processes

Chao Wei and Fang Xu

Abstract—**This paper is concerned with least squares estimation for Ornstein-Uhlenbeck process driven by sub-fractional Brownian processes from discrete observations. The contrast function is given to obtain the least squares estimators. The consistency and asymptotic distribution of two estimators are derived. Some numerical simulations are made to verify the effectiveness of the estimators.**

Index Terms—**Least squares estimation; Ornstein-Uhlenbeck process; sub-fractional Brownian processes; consistency; asymptotic distribution.**

I. Introduction

Almost all systems are affected by noise and exhibit certain random characteristics ([6], [7], [9]). Therefore, it is reasonable and interesting to use random systems to model actual systems. When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, parameters of the system are unknown. It is necessary to use theoretical tools to estimate the parameters of the system. In the past few decades, many authors studied the parameter estimation problem for stochastic models driven by Brownian processes ([17], [19]). For example, Hildebrandt and Trabs ([3]) studied parameter estimation for stochastic partial differential equations based on discrete observations in time and space. Liu ([10]) used generalized moment method to estimation the parameter for uncertain differential equations. Wei and Shu ([15]) discussed the existence, consistency and asymptotic normality of the maximum likelihood estimator for the nonlinear stochastic differential equations. When the system is observed partially, Botha et al. ([1]) applied particle methods for stochastic differential equation mixed effects models. Wei ([16]) analyzed state and parameter estimation for nonlinear stochastic systems by extended Kalman filtering. The parameter estimation for stochastic models driven by fractional Brownian processes is developed as well ([4], [12], [13]). Sometimes, the stochastic models are driven by some more general fractional Gaussian processes such as sub-fractional Brownian motion. Sub-fractional Brownian process has non stationary increments, the increments over non overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate, which makes the sub-fractional Brownian process a possible candidate for models involving long-range dependence, self-similarity and non-stationary. However, there are few literature about parameter estimation for sub-fractional Brownian processes. Li and Dong ([8]) investigated parametric estimation in the Vasicek model driven by sub-fractional Brownian motion. Prakasa Rao ([11]) investigated the asymptotic properties of the maximum likelihood estimator and Bayes estimator of the drift parameter for linear stochastic differential equations. Xiao et al. ([20]) provided least squares estimators for Vasicek processes, derived the strong consistency and asymptotic distribution of estimators.

The Ornstein-Uhlenbeck process is extensively used in finance during the past few decades as the one-factor short-term interest rate model. Therefore, statistical inference for Ornstein-Uhlenbeck processes has been studied by many authors. For example, Chen et al. ([2]) showed the Berry-Esseen bound of the least squares estimator for fractional Ornstein-Uhlenbeck processes based on continuous-time observation. Hu et al. ([4]) studied parameter estimation for fractional Ornstein-Uhlenbeck processes of general hurst parameter. Voutilainen et al. ([14]) considered estimation of the unknown model parameter in the multidimensional version of the Langevin equation. Wei et al. ([18]) analyzed the estimation for squared radial Ornstein-Uhlenbeck process from discrete observation. However, the Ornstein-Uhlenbeck processes discussed in above literature are not driven by sub-fractional Brownian processes and a common denominator in all these works is assumed that the equation admits only one unknown parameter. In this paper, we consider the parameter estimation problem for Ornstein-Uhlenbeck process with two unknown parameters driven by sub-fractional Brownian processes from discrete observations. The contrast function is introduced to obtain the least squares estimators. The consistency and asymptotic distribution of the estimators are derived by Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality.

This paper is organized as follows. In Section 2, we give the contrast function to obtain the least squares estimators. In Section 3, we obtain the consistency and asymptotic distribution of the estimators. In Section 4, some numerical simulations are provided. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

In this paper, we study the parametric estimation problem for Ornstein-Uhlenbeck process driven by sub-fractional Brownian processes described by the following stochastic

Manuscript received November 15, 2022; revised February 27, 2023.

This work was supported in part by the Key Research Projects of Universities under Grant 22A110001.

Chao Wei is a Professor of School of Mathematics and Statistics, An'yang Normal University, An'yang, 455000, China (email: chaowei0806@aliyun.com).

Fang Xu is a Doctor of School of clinic, An'yang Normal University, An'yang, 455000, China (email: xysxxyy@163.com).

differential equation:

$$\begin{cases} dX_t = (\alpha - \beta X_t)dt + \varepsilon dB_t^H, \quad t \geq 0, \\ X_0 = x_0, \end{cases} \tag{1}$$

where α and β are unknown parameters, $\varepsilon \in (0,1]$, B_t^H is a sub-fractional Brownian process with $H \in (\frac{1}{2}, 1)$. It is assumed that $\{X_t, t \geq 0\}$ is observed at n regular time intervals $\{t_i = \frac{i}{n}, i = 1, 2, \cdots, n\}$.

The sub-fractional Brownian process B_t^H is a mean zero Gaussian process with $B_0^H = 0$ and the covariance

$$\mathbb{E}(B_t^H B_s^H) = s^{2H} + t^{2H} - \frac{1}{2}\{|t-s|^{2H} + |t+s|^{2H}\},$$

where $s, t \geq 0$. When $H = \frac{1}{2}$, B_t^H is the standard Brownian motion.

Moreover, for all $s \leq t$,

$$\mathbb{E}(|B_t^H - B_s^H|^2) = -2^{2H-1}(t^{2H}+s^{2H}) + (t+s)^{2H} - (t-s)^{2H},$$

and for $m \leq n \leq s \leq t$,

$$\begin{aligned} &\mathbb{E}(B_t^H - B_s^H)(B_n^H - B_m^H) \\ &= \frac{1}{2}[(t+m)^{2H} + (t-m)^{2H} + (s+n)^{2H} + (s-n)^{2H} \\ &-(t+n)^{2H} - (t-n)^{2H} - (s+m)^{2H} - (s-m)^{2H}]. \end{aligned}$$

Consider the following contrast function

$$\rho_n(\alpha, \beta) = \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the least square estimators

$$\begin{cases} \widehat{\alpha}_n = \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \widehat{\beta}_n = \dfrac{n^2\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2}. \end{cases} \tag{3}$$

III. Main Results and Proofs

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameters:

$$dX_t^0 = (\alpha - \beta X_t^0)dt, \quad X_0^0 = x_0. \tag{4}$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\alpha - \beta\int_{t_{i-1}}^{t_i} X_s ds + \varepsilon\int_{t_{i-1}}^{t_i} dB_s^H. \tag{5}$$

Then, we can give a more explicit decomposition for $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ as follows

$$\begin{aligned} &\widehat{\alpha}_n - \alpha \\ &= \frac{n\beta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\beta\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n\varepsilon\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \frac{\beta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\beta\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}. \end{aligned}$$

$$\begin{aligned} &\widehat{\beta}_n \\ &= \frac{n\beta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n^2\beta\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n^2\varepsilon\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \frac{\beta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\beta\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}. \end{aligned}$$

Before giving the theorems, we need to establish some preliminary results.

Lemma 1: When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Observe that

$$X_t - X_t^0 = -\beta\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t dB_s^H. \tag{6}$$

By using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} & |X_t - X_t^0|^2 \\ \leq\ & 2\beta^2 |\int_0^t (X_s - X_s^0)ds|^2 + 2\varepsilon^2 |\int_0^t dB_s^H|^2 \\ \leq\ & 2\beta^2 t \int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 \sup_{0\leq t\leq 1} |\int_0^t dB_s^H|^2. \end{aligned}$$

According to the Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{2\beta^2 t^2} \sup_{0\leq t\leq 1} |\int_0^t dB_s^H|^2. \tag{7}$$

Since $B_0^H = 0$, it follows that

$$\begin{aligned} & \sup_{0\leq t\leq 1} |X_t - X_t^0| \leq \sqrt{2}\varepsilon e^{\beta^2} \sup_{0\leq t\leq 1} |\int_0^t dB_s^H| \\ & = \sqrt{2}\varepsilon e^{\beta^2} \sup_{0\leq t\leq 1} |B_t^H|. \end{aligned} \tag{8}$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, it is easy to check that

$$\sup_{0\leq t\leq 1} |X_t - X_t^0| \xrightarrow{P} 0. \tag{9}$$

The proof is complete. ■

Lemma 2: When $\varepsilon \to 0$ and $n \to \infty$, we have,

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt.$$

Proof: Since

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 + \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2). \tag{10}$$

It is easy to check that

$$\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \tag{11}$$

According to Lemma 3.1, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\begin{aligned} & |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)| \\ & = |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}} + X_{t_{i-1}}^0)(X_{t_{i-1}} - X_{t_{i-1}}^0)| \\ & \leq \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|(|X_{t_{i-1}}| + |X_{t_{i-1}}^0|) \\ & \leq \frac{1}{n}\sum_{i=1}^n (|X_{t_{i-1}} - X_{t_{i-1}}^0|^2 + 2|X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0|) \\ & = \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ & +2\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0| \\ & \leq (\sup_{0\leq t\leq 1} |X_t - X_t^0|)^2 \\ & +2\sup_{0\leq t\leq 1} |X_t - X_t^0| \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \\ & \xrightarrow{P} 0. \end{aligned}$$

Therefore, we obtain

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \tag{12}$$

The proof is complete. ■

In the following theorem, the consistency of the least squares estimators are proved.

Theorem 1: When $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-H} \to 0$, the least squares estimators $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ are consistent, namely

$$\widehat{\alpha}_n \xrightarrow{P} \alpha, \quad \widehat{\beta}_n \xrightarrow{P} \beta.$$

Proof: According to Lemmas 3.1 and 3.2, we have

$$(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} (\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt. \tag{13}$$

With the results that $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt$ and $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt$, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \beta \int_0^1 X_t dt \int_0^1 (X_t^0)^2 dt, \tag{14}$$

and

$$\begin{aligned} & \beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \\ & \xrightarrow{P} \beta \int_0^1 X_t X_t^0 dt \int_0^1 X_t^0 dt. \end{aligned} \tag{15}$$

Then, we have

$$\begin{aligned} & \beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \\ & -\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} 0. \end{aligned} \tag{16}$$

Since

$$\begin{aligned} & |\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H| \\ & \leq \varepsilon \sum_{i=1}^n |X_{t_{i-1}}|| \int_{t_{i-1}}^{t_i} dB_s^H| \\ & \leq \varepsilon \sum_{i=1}^n (|X_{t_{i-1}}^0| + |X_{t_{i-1}} - X_{t_{i-1}}^0|)| \int_{t_{i-1}}^{t_i} dB_s^H| \\ & \leq \varepsilon \sum_{i=1}^n |X_{t_{i-1}}^0|| \int_{t_{i-1}}^{t_i} dB_s^H| \\ & +\varepsilon \sup_{0\leq t\leq 1} |X_t - X_t^0|| \int_{t_{i-1}}^{t_i} dB_s^H|. \end{aligned}$$

By the Markov inequality, when $\varepsilon \to 0$, $n \to \infty$ and

$\varepsilon n^{1-H} \to 0$, we obtain

$$\begin{aligned}
&P(|\varepsilon \sum_{i=1}^{n} |X_{t_{i-1}}^0|| \int_{t_{i-1}}^{t_i} dB_s^H|| > \delta) \\
&\le \delta^{-1}\varepsilon \sum_{i=1}^{n} (\mathbb{E}|X_{t_{i-1}}^0|^2)^{\frac{1}{2}} (\mathbb{E}| \int_{t_{i-1}}^{t_i} dB_s^H|^2)^{\frac{1}{2}} \\
&= \delta^{-1}\varepsilon \sum_{i=1}^{n} (\mathbb{E}|X_{t_{i-1}}^0|^2)^{\frac{1}{2}} (\mathbb{E}|B_{t_i}^H - B_{t_{i-1}}^H|^2)^{\frac{1}{2}} \\
&\le \delta^{-1}\varepsilon \sum_{i=1}^{n} \sup_{0\le t\le 1} |X_t^0| (|t_i - t_{i-1}|^{2H})^{\frac{1}{2}} \\
&= \delta^{-1}\varepsilon n^{1-H} \sup_{0\le t\le 1} |X_t^0| \\
&\to 0.
\end{aligned}$$

Thus, when $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-H} \to 0$, we have

$$\varepsilon \sum_{i=1}^{n} |X_{t_{i-1}}^0|| \int_{t_{i-1}}^{t_i} dB_s^H| \xrightarrow{P} 0. \tag{17}$$

When $\varepsilon \to 0$ and $n \to \infty$, it is obvious that

$$\varepsilon \sup_{0\le t\le 1} |X_t - X_t^0|| \int_{t_{i-1}}^{t_i} dB_s^H| \xrightarrow{P} 0. \tag{18}$$

Then, we obtain

$$\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \xrightarrow{P} 0. \tag{19}$$

Thus, we have

$$\frac{\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \tag{20}$$

Moreover, when $\varepsilon \to 0$ and $n \to \infty$, it is easy to check that

$$\frac{\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \tag{21}$$

Therefore, when $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-H} \to 0$, we obtain

$$\widehat{\alpha}_n \xrightarrow{P} \alpha.$$

Using the same methods, it can be easily to check that

$$\widehat{\beta}_n \xrightarrow{P} \beta.$$

The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$, $n \to \infty$, $\varepsilon n^{1-H} \to 0$ and $n\varepsilon \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} \to \frac{\int_0^1 X_t^0 dt \int_0^1 X_t^0 dB_t^H - \int_0^1 (X_t^0)^2 dt B_1^H}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt},$$

$$\varepsilon^{-1}(\widehat{\beta}_n - \beta) \xrightarrow{d} \frac{\int_0^1 X_t^0 dB_t^H - B_1^H \int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}.$$

Proof: According to the explicit decomposition for $\widehat{\alpha}_n$, it is obvious that

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\alpha}_n - \alpha) \\
&= \frac{\varepsilon^{-1}\beta \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\varepsilon^{-1}\beta \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&+ \frac{\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}.
\end{aligned}$$

From Lemma 3.1, when $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$,

$$\begin{aligned}
&|\varepsilon^{-1}\beta \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds| \\
&\le \varepsilon^{-1}\beta \sum_{i=1}^{n} |X_{t_{i-1}}|| \int_{t_{i-1}}^{t_i} X_s ds| \\
&\le \varepsilon^{-1} n^{-1}\beta \sum_{i=1}^{n} (|X_{t_{i-1}} - X_{t_{i-1}}^0| + |X_{t_{i-1}}^0|) \\
&\sup_{t_{i-1}\le t\le t_i} |X_t| \\
&\xrightarrow{P} 0.
\end{aligned}$$

Then, it is easy to check that

$$\varepsilon^{-1}\beta \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} 0.$$

Thus, we have

$$\frac{\varepsilon^{-1}\beta \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0, \tag{22}$$

and

$$\frac{\varepsilon^{-1}\beta \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \tag{23}$$

Since

$$\begin{aligned}
&\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \\
&= \sum_{i=1}^{n} (X_{t_{i-1}} - X_{t_{i-1}}^0 + X_{t_{i-1}}^0) \int_{t_{i-1}}^{t_i} dB_s^H \\
&= \sum_{i=1}^{n} (X_{t_{i-1}} - X_{t_{i-1}}^0) \int_{t_{i-1}}^{t_i} dB_s^H \\
&+ \sum_{i=1}^{n} X_{t_{i-1}}^0 \int_{t_{i-1}}^{t_i} dB_s^H.
\end{aligned}$$

According to Theorem 3.3, we have

$$\sum_{i=1}^{n} (X_{t_{i-1}} - X_{t_{i-1}}^0) \int_{t_{i-1}}^{t_i} dB_s^H \xrightarrow{P} 0. \tag{24}$$

Moreover,

$$\begin{aligned}&|\sum_{i=1}^{n} X_{t_{i-1}}^0 \int_{t_{i-1}}^{t_i} dB_s^H - \int_0^1 X_s^0 dB_s^H| \\ &\leq \int_0^1 |X_{n,\varepsilon}^0 - X_s^0| dB_s^H \\ &\leq \sup_{0\leq s\leq 1} |X_{n,\varepsilon}^0 - X_s^0|| \int_0^1 dB_s^H| \\ &\xrightarrow{P} 0.\end{aligned}$$

We have

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H \xrightarrow{P} \int_0^1 X_s^0 dB_s^H. \tag{25}$$

Then, we obtain

$$\varepsilon^{-1}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} \frac{\int_0^1 X_t^0 dt \int_0^1 X_t^0 dB_t^H - \int_0^1 (X_t^0)^2 dt B_1^H}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}. \tag{26}$$

As

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\beta}_n - \beta) \\ &= \frac{\varepsilon^{-1}\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon^{-1}\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} - \varepsilon^{-1}\beta \\ &+ \frac{\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}.\end{aligned}$$

It is obvious that

$$\begin{aligned}&\frac{\varepsilon^{-1}\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon^{-1}\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \varepsilon^{-1}\beta \xrightarrow{P} 0,\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\frac{\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dB_s^H}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} dB_s^H \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &\xrightarrow{d} \frac{\int_0^1 X_t^0 dB_t^H - B_1^H \int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}.\end{aligned} \tag{28}$$

Then, we have

$$\varepsilon^{-1}(\widehat{\beta}_n - \beta) \xrightarrow{d} \frac{\int_0^1 X_t^0 dB_t^H - B_1^H \int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}. \tag{29}$$

The proof is complete. ■

TABLE I
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α,β)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	$\lvert\widehat{\alpha}_n-\alpha\rvert$	$\lvert\widehat{\beta}_n-\beta\rvert$
(1,0.7)	10000	1.0539	0.7615	0.0539	0.0615
	20000	1.0227	0.7329	0.0227	0.0329
	50000	1.0083	0.7071	0.0083	0.0071
(1.5,0.8)	10000	1.5603	0.8723	0.0603	0.0723
	20000	1.5368	0.8369	0.0368	0.0369
	50000	1.5075	0.8104	0.0075	0.0104
(2,1)	10000	2.0617	1.0308	0.0617	0.0308
	20000	2.0468	1.0176	0.0468	0.0176
	50000	2.0092	1.0085	0.0092	0.0085

IV. SIMULATION

In this experiment, we use iterative approach to generate a discrete sample $(X_{t_{i-1}})_{i=1,\ldots,n}$ and compute $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ from the sample. We let $x_0 = 0.3$. For every given true value of the parameters-(α, β), the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $H = 0.3$, $\varepsilon = 0.01$, the size is increasing from 10000 to 50000. In Table 2, $H = 0.65$, $\varepsilon = 0.01$, the size is increasing from 10000 to 50000. In Table 3, $H = 0.8$, $\varepsilon = 0.01$, the size is increasing from 10000 to 50000. In Table 4, $H = 1.5$, $\varepsilon = 0.01$, the size is increasing from 10000 to 50000. The tables list the value of least squares estimators "$\widehat{\alpha}_n$","$\widehat{\beta}_n$" and the absolute errors (AE) "$|\widehat{\alpha}_n - \alpha|$","$|\widehat{\beta}_n - \beta|$".

The tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. If we let n converge to the infinity and ε converge to zero, the estimator will converge to the true value.

V. CONCLUSIONS

The aim of this paper is to study the parameter estimation problem for Ornstein-Uhlenbeck process driven by sub-fractional Brownian processes from discrete observations. The contrast function has been introduced to obtain the explicit formula of two estimators. The consistency and asymptotic distribution of the estimators have been derived by Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality. Further research topics will include parameter estimation for partially observed Ornstein-Uhlenbeck process driven by sub-fractional Brownian processes.

TABLE II
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α, β)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	$\lvert\widehat{\alpha}_n - \alpha\rvert$	$\lvert\widehat{\beta}_n - \beta\rvert$
(1,0.7)	10000	1.0246	0.7328	0.0246	0.0328
	20000	1.0118	0.7185	0.0118	0.0185
	50000	1.0032	0.7029	0.0032	0.0029
(1.5,0.8)	10000	1.5261	0.8273	0.0261	0.0273
	20000	1.5127	0.8106	0.0127	0.0106
	50000	1.5039	0.8025	0.0039	0.0025
(2,1)	10000	2.0255	1.0308	0.0255	0.0308
	20000	2.0183	1.0176	0.0183	0.0176
	50000	2.0019	1.0021	0.0019	0.0021

TABLE III
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α, β)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	$\lvert\widehat{\alpha}_n - \alpha\rvert$	$\lvert\widehat{\beta}_n - \beta\rvert$
(1,0.7)	10000	1.0125	0.7143	0.0125	0.0143
	20000	1.0013	0.7016	0.0013	0.0016
	50000	1.0002	0.7003	0.0002	0.0003
(1.5,0.8)	10000	1.5118	0.8152	0.0118	0.0152
	20000	1.5016	0.8013	0.0016	0.0013
	50000	1.5004	0.8002	0.0004	0.0002
(2,1)	10000	2.0115	1.0136	0.0115	0.0136
	20000	2.0010	1.0017	0.0010	0.0017
	50000	2.0003	1.0005	0.0003	0.0005

TABLE IV
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α, β)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	$\lvert\widehat{\alpha}_n - \alpha\rvert$	$\lvert\widehat{\beta}_n - \beta\rvert$
(1,0.7)	10000	1.0214	0.7223	0.0214	0.0223
	20000	1.0169	0.7094	0.0169	0.0094
	50000	1.0015	0.7010	0.0015	0.0010
(1.5,0.8)	10000	1.5182	0.8207	0.0182	0.0207
	20000	1.5091	0.8113	0.0091	0.0113
	50000	1.5009	0.8012	0.0009	0.0012
(2,1)	10000	2.0126	1.0156	0.0126	0.0156
	20000	2.0035	1.0061	0.0035	0.0061
	50000	2.0007	1.0003	0.0007	0.0003

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