Secure Domination Parameters in Sierpiński Graphs

Gisha Saraswathy, Manju K. Menon

Abstract—**We have obtained exact values for some special kinds of secure domination parameters - secure domination number, co-secure domination number, complete co-secure domination number and perfect secure domination number in the Sierpiński graphs and in the Hanoi graphs.**

Index Terms—**Secure domination number; Co-secure domination number; Complete co-secure domination number; Perfect secure domination number; Sierpiński graphs.**

I. Introduction

Over the past 25 years, there has been a lot of research done on graphs whose drawings can be considered as approximate representations of the well-known Sierpiński triangle. The topological study of Lipscomb's space, which showed that this space is a generalisation of the Sierpiński triangular curve (Sierpiński gasket), served as the inspiration for the development of Sierpiński graphs $S(n, t)$. Sierpiński graphs are well studied graphs of fractal nature with applications in topology, computer science, and mathematics of Tower of Hanoi [11]. Switching the Tower of Hanoi for n pegs and t discs is a variant of the Tower of Hanoi that Klavžar and Milutinović introduced in [10], and they called its state graph $S(n, t)$, the Sierpiński graph.

A well-known puzzle called the Tower of Hanoi is said to have originated from the legend of theTower of Brahma. This legend describes three diamond needles and a tower of 64 golden discs that must be transferred, one disc at a time, to another needle by a group of Brahmin monks. Since the discs all have different diameters, no larger disc may ever be stacked on a smaller one. The world will come to an end after the monks finish their task. Fortunately, even moving a disc per second and executing the right strategy would still take 40 times as much time as the universe has been known to exist.[14].

All graphs considered in this paper are finite, simple, and undirected. For notations, definitions, and terminologies in graph theory that are not explained in this paper, see [2]. By G in this paper, we mean $G = (V, E)$, where $V = V(G)$, the set of vertices and $E = E(G)$, the set of edges.

A vertex u is a neighbour of v in G if uv is an edge of G. The set of all neighbours of v is the open neighbourhood of v, and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v in G. The number of edges incident at $v \in G$ is called the degree of the vertex v in G, and is denoted by $d(v)$. Let $S \subseteq V$ and $v \in S$, a vertex $u \in V$ is an S-private neighbour of v if $N(u) \cap S = \{v\}$. The set of all S-private neighbours of v is denoted by $PN(v, S)$. If $u \in V \setminus S$, then u is called an S-external private neighbour of v. The set of all S-external private neighbours of v is denoted by $EPN(v, S)$. A set $S \subseteq V$ is a dominating set of G if every vertex $u \in V \setminus S$ has at least one neighbour $v \in S$. The minimum cardinality of a dominating set is the domination number of G, $\gamma(G)$.

A dominating set $S \subseteq V$ is a secure dominating set (SDS) of G if for each $u \in V \setminus S$, there exists a vertex v such that $v \in N(u) \cap S$ and $(S \setminus \{v\}) \bigcup \{u\}$ is a dominating set of G (in which case v S-defends u). The minimum cardinality of an SDS of G is the secure domination number of G, $\gamma_s(G)$ and the corresponding set is γ_s-set [4]. The following circumstance serves as inspiration for the idea of a secure domination number. In the case of a graph $G = (V, E)$, we want to position a guard at each vertex of a set $S \subseteq V$ such that S is a dominating set of G and that, if a guard at v moves down an edge to defend an unguarded vertex u, the new arrangement of guards likewise forms a dominating set [12]. This idea was introduced by E. J. Cockayne, P. J. P. Grobler, W. R. Gründlingh, J. Munganga, and J. H. Van Vuuren in [5] and has been investigated by several authors [3], [4], [6].

A set S is a perfect secure dominating set (PSDS) of G if for each vertex $v \in V \setminus S$, there exists a unique vertex $u \in S$ such that u and v are adjacent and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. The minimum cardinality of a PSDS of G is the perfect secure domination number of G, $\gamma_{ps}(G)$ [13].

There are a number of real-world scenarios where one guard must be replaced by another. In other sense, the group of guards must constitute a dominating set S with the condition that for every guard $u \in S$, there exists $v \in V \setminus S$ such that $v \in N(u)$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. So, every guard u can be swapped out with another guard, and the new group of guards will still be able to keep G safe. This observation resulted in the definition of the co-secure domination number [1]. A dominating set S is called a co-secure dominating set (CSDS) if for each $u \in S$ there exists $v \in V \setminus S$ such that v is adjacent to u and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a CSDS in G is the co-secure domination number $\gamma_{cs}(G)$ of G and the corresponding set is called γ_{cs}-set [1]. For $u \in S$, if there exists $v \in V \setminus S$ such that u and v are adjacent and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set, we say that u protects v or v replaces u.

As we can easily see, $\gamma_s(P_5)$ = 3 and $\gamma_{cs}(P_5)$ = 2.

A guard in the aforementioned domination parameters can only guarantee the safety of one of its neighbouring unguarded vertex. This inspired us to define a new domination

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parameter called the complete co-secure domination number, which allows a guard to move to any of its neighbouring unguarded vertices without compromising the safety of G. A co-secure dominating set S is a complete co-secure dominating set (CCSDS) if for every $u \in S$ and for every $v \in V \setminus S$ such that u is adjacent to v and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. The minimum cardinality of a CCSDS is the complete co-secure domination number of G, $\gamma_{ccs}(G)$. For any graph G, a CCSDS will be a CSDS. Hence, $\gamma_{cs}(G) \leq \gamma_{ccs}(G)$.

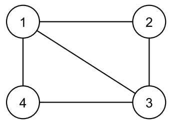

Fig. 1. G

In Fig 1., $\{1, 3\}$ forms a minimal CCSDS whereas $\{1, 2, 3, 4\}$ forms a minimal PSDS of G. Hence, $\gamma_{ccs}(G) = 2$, and $\gamma_{ps}(G) = 4$.

S. Klavžar, and U. Milutinović [10] introduced the Sierpiński graph of dimension t, $S(K_n, t)$, $t \geq 1$, as the graph with vertex set the set of t-tuples on $\{1, 2, \ldots, n\}$, $V^t = \{1, 2, \ldots, n\}^t$ and the edge set defined as follows: two vertices $(i_1 i_2 \ldots i_t)$ and $(j_1 j_2 \ldots j_t)$ are adjacent if and only if there exists $h \in \{1, 2, \ldots, t\}$ such that

1) $i_k = j_k$ for all $k < h$
2) $i_h \neq j_h$
3) $i_k = j_h$ and $j_k = i_h$ for all $k > h$.

When h is equal to t, the condition (3) is trivially true being empty. The graphs $S(3, t)$ are isomorphic to the graphs of the Tower of Hanoi problem.

Fig. 2. shows $S(4, 3)$ as an example.

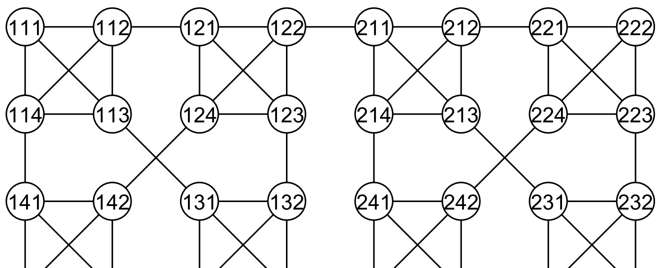

Fig. 2. $S(4, 3)$

Clearly, $|S(n, t)| = n^t$. Let $i \epsilon \{1, 2, \ldots, n\}$, then the vertices of the form $ii \ldots i$ of $S(n, t)$ are called the extreme vertices. Clearly, there are n extreme vertices and they are of degree $n - 1$, while all the other $n(n^{t-1} - 1)$ vertices are of degree n. Let $r \epsilon \{1, 2, \ldots, t\}$ and let $i_1, i_2, \ldots, i_r \epsilon \{1, 2, \ldots, n\}$. Then the subgraph of $S(n, t)$ induced by vertices whose first r co-ordinates are $i_1 i_2 \ldots i_r$ will be denoted by $S(n, t, i_1 i_2 \ldots i_r)$. Note that $S(n, t, i_1 i_2 \ldots i_r)$ is isomorphic to $S(n, t-r)$. A subgraph $S(n, t, i)$ contains one extreme vertex of $S(n, t)$ namely $ii \ldots i$ and $n - 1$ vertices of the form $ijj \ldots j$, $j \neq i$ that are respectively adjacent to vertices $jii \ldots i$ of subgraphs $S(n, t, j)$. Hence, in $S(n, t)$, there is exactly one edge between each pair of n subgraphs $S(n, t, i)$, for $i = 1, 2, \ldots, n$. In $S(n, t)$, the vertices which are either of the form $i^{t-1} j$ or ij^{t-1}, where $i \neq j$ are called almost extreme vertices of $S(n, t)$. In general, for any subgraph $S(n, t, i_1 i_2 \ldots i_r)$ of $S(n, t)$, $r = 1, 2, \ldots, (t-1)$, a vertex of the form $i_1 i_2 \ldots i_r jj \ldots j$ where j is taken $t - r$ times, is an extreme vertex of $S(n, t, i_1 i_2 \ldots i_r)$. Note that each $S(n, t)$ contains n^r copies of $S(n, t - r)$ [9].

In [9], the existence of 1-perfect code in the Sierpiński graphs have been studied. The authors of [8] deduced explicit formulae to calculate the distance in Sierpiński graphs between an arbitrary vertex and an almost-extreme vertex. They also provide a formula of the metric dimension of $S(n, t)$. It was shown in [10] that $S(3, t)$ is isomorphic to H_3^t for any t, where H_n^t denotes the Hanoi graphs. Although for any n, $t > 1$, the graphs $S(n, t)$ and H_n^t are defined on the same vertex set, they are not isomorphic for $n > 3$ and $t > 1$ [7]. In this paper, we have evaluated the above-mentioned parameters in the Sierpiński graphs and in the Hanoi graphs. We have obtained exact values for all the four parameters. Even though the choice of vertices is different, we have proved that all the four secure domination parameters have the same value in the celebrated Sierpiński networks and in the Hanoi graphs.

II. Preliminary Results

***Theorem** 2.1:* [1] $\gamma_{cs}(K_n) = 1$.

***Theorem** 2.2:* [5] $\gamma_s(K_n) = 1$.

***Theorem** 2.3:* [13] Let G be any graph of order n, then $\gamma_{ps}(G) = 1$ if and only if $G = K_n$.

***Theorem** 2.4:* [1] Let S be a CSDS of G. A vertex $v \in V \setminus S$ replaces (refer introduction) $u \in S$ if and only if $v \in N(u)$ and $EPN(u, S) \subseteq N[v]$.

***Theorem** 2.5:* [1] For any graph G with $\delta(G) \geq 2$, $\gamma_{cs}(G) \leq \gamma_s(G)$, where $\delta(G)$ denotes the minimum degree of G.

***Remark** 2.6:* For any graph G, a PSDS will be an SDS. Hence, $\gamma_s(G) \leq \gamma_{ps}(G)$.

III. Co-Secure Domination number of Sierpiński Graphs

***Theorem** 3.1:* For $n \geq 3$, $\gamma_{cs}\{S(n, t)\} = n^{t-1}$.

Proof: For $t = 1$, $S(n, 1) = K_n$. Hence, by Theorem 2.1, $\gamma_{cs}\{S(n, 1)\} = 1$.

For $t = 2$, let $S_\omega = \{i\omega / i \epsilon \{1, 2, \ldots, n\}\}$, for a fixed $\omega \epsilon \{1, 2, \ldots, n\}$. Clearly, S_ω forms a dominating set of $S(n, 2)$. For any $u = i\omega \epsilon S_\omega$, there exists $v = ij \epsilon V \setminus S_\omega$, where u and v belongs to the same K_n, such that $(S_\omega \setminus \{u\}) \cup \{v\}$ is a dominating set of $S(n, 2)$. Hence, S_ω forms a CSDS of $S(n, 2)$. Hence,

$$\gamma_{cs}(S(n, 2)) \leq n \tag{1}$$

Clearly, any copy of $K_n \epsilon S(n,2)$ contains an extreme vertex ii where $i \epsilon \{1,2,\ldots,n\}$, which is not adjacent to any other copy of K_n. In order to dominate these extreme vertices, we need to take at least one vertex from each K_n of $S(n,t)$. Hence,

$$\gamma_{cs}(S(n,2)) \geq n \tag{2}$$

From (1) and (2) we get,

$$\gamma_{cs}\{S(n,2)\} = n.$$

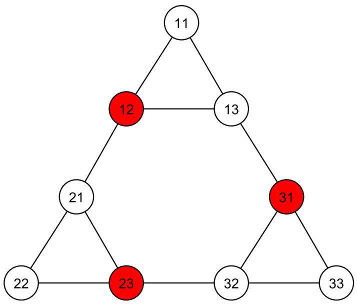

Fig. 3. $S(3,2)$

As we can see in Fig 3., the red vertices indicate the minimum CSDS for $S(3,2)$.
Now consider $S(n,t)$ for $t > 2$. Let $S_\omega = \{i_1 i_2 \ldots i_{t-1}\omega / i_j \epsilon \{1,2,\ldots,n\}$, for $j = 1,2,\ldots,(t-1)\}$, for a fixed $\omega \epsilon \{1,2,\ldots,n\}$. S_ω contains one vertex from each copy of K_n in $S(n,t)$. Hence, $S\omega$ forms a dominating set of $S(n,t)$. Also for any $u = i_1 i_2 \ldots i_{t-1}\omega \epsilon S_\omega$, there exists $v = i_1 i_2 \ldots i_{t-1} j \epsilon V \setminus S_\omega$, for $j \epsilon \{1,2,\ldots,n\}$, $j \neq \omega$, where u and v belongs to the same K_n, such that $(S_\omega \setminus \{u\}) \cup \{v\}$ is a dominating set of $S(n,t)$. Hence, S_ω forms a co-secure dominating set of $S(n,t)$. Thus,

$$\gamma_{cs}\{S(n,t)\} \leq n^{t-1}. \tag{3}$$

Let S be the minimal CSDS of $S(n,t)$. We have, $S(n,t,i_1 i_2 \ldots i_{t-2})$ denotes the subgraph of $S(n,t)$, which is isomorphic to $S(n,2)$. Take any arbitrary K_n (without containing extreme vertices) from $S(n,t)$. We can view each copy of K_n in $S(n,t)$ as a subgraph of $S(n,t,i_1 i_2 \ldots i_{t-2})$, by changing the choice of $i_1, i_2, \ldots, i_{t-2}$. Let the copies of K_n in $S(n,t,i_1 i_2 \ldots i_{t-2})$ be denoted by $1K_n$, $2K_n$, $\ldots$, nK_n, where the vertex set of iK_n is given by $\{i_1 i_2 \ldots i_{t-2} i1, i_1 i_2 \ldots i_{t-2} i2, \ldots, i_1 i_2 \ldots i_{t-2} in\}$, $i \epsilon \{1,2,\ldots,n\}$. Without loss of generality, let the arbitrary K_n which we have already taken be $1K_n$. Suppose $|V(1K_n) \cap S| = \phi$. Then the vertices of $1K_n$ are dominated by the vertices of other copies of K_n's which are adjacent to $1K_n$. The vertices of $1K_n$ is given by $i_1 i_2 \ldots i_{t-2}11$, $i_1 i_2 \ldots i_{t-2}12$, $\ldots$, $i_1 i_2 \ldots i_{t-2}1n$, where $i_1 i_2 \ldots i_{t-2}12$ is adjacent to $i_1 i_2 \ldots i_{t-2}21$ of $2K_n$, $i_1 i_2 \ldots i_{t-2}13$ is adjacent to $i_1 i_2 \ldots i_{t-2}31$ of $3K_n$ $\ldots$, $i_1 i_2 \ldots i_{t-2}1n$ is adjacent to $i_1 i_2 \ldots i_{t-2}n1$ of nK_n. Also $i_1 i_2 \ldots i_{t-2}11$ is adjacent to a vertex of K_n of $S(n,t,j_1 j_2 \ldots j_{t-2})$ for some $j_1, j_2, \ldots, j_{t-2}$. Let that vertex be x.

Fig 4. shows a portion of $S(4,3)$, in which the copy of K_n in green colour is the $1K_n$ defined in the proof. The vertices of other copies of K_n that are adjacent to the vertices of $1K_n$ can be easily understood from the picture.

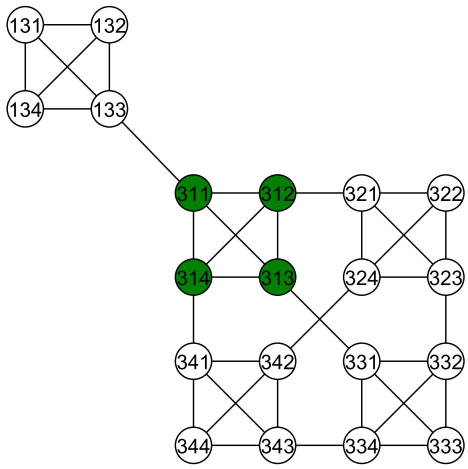

Fig. 4.

According to Fig 4., the vertex x in the proof is 133.
Claim: Each vertex of $1K_n$ replaces the corresponding adjacent vertex in other K_n's.
Clearly, $\{i_1 i_2 \ldots i_{t-2}21, i_1 i_2 \ldots i_{t-2}31, i_1 i_2 \ldots i_{t-2}n1, x\} \subseteq S$. Hence, they dominate their corresponding adjacent vertices in $1K_n$. Here, $i_1 i_2 \ldots i_{t-2}12$ replaces $i_1 i_2 \ldots i_{t-2}21$, since there does not exists any vertex $v \epsilon V \setminus S$ other than $i_1 i_2 \ldots i_{t-2}12$, such that $EPN(i_1 i_2 \ldots i_{t-2}21, S) \subseteq N[v]$. Similarly $i_1 i_2 \ldots i_{t-2}13$ replaces $i_1 i_2 \ldots i_{t-2}31$, $\ldots$, $i_1 i_2 \ldots i_{t-2}1n$ replaces $i_1 i_2 \ldots i_{t-2}n1$, $i_1 i_2 \ldots i_{t-2}11$ replaces x.
If $i_1 i_2 \ldots i_{t-2}12$ replaces $i_1 i_2 \ldots i_{t-2}21$ then there exists at least one vertex to dominate $i_1 i_2 \ldots i_{t-2}22$, $i_1 i_2 \ldots i_{t-2}23, \ldots, i_1 i_2 \ldots i_{t-2}2n$. Then, either $S \cap \{i_1 i_2 \ldots i_{t-2}22, i_1 i_2 \ldots i_{t-2}23, \ldots, i_1 i_2 \ldots i_{t-2}2n\} \neq \emptyset$, or the vertices adjacent to $i_1 i_2 \ldots i_{t-2}22$, $i_1 i_2 \ldots i_{t-2}23$, $\ldots$, $i_1 i_2 \ldots i_{t-2}2n$ belongs to S. Similar argument holds for the vertices $i_1 i_2 \ldots i_{t-2}13$, $\ldots$, $i_1 i_2 \ldots i_{t-2}1n$. In any of the cases, we get $\left|S \cap V(S(n,t,i_1 i_2 \ldots i_{t-2}))\right| \geq n$. Thus, $|S| \geq n^{t-1}$. ie,

$$\gamma_{cs}(S(n,t)) \geq n^{t-1}. \tag{4}$$

Now (3) and (4) implies

$$\gamma_{cs}(S(n,t)) = n^{t-1}.$$

Corollary *3.2:* In Hanoi graphs, $\gamma_{cs}\{H_3^t\} = 3^{t-1}$.

IV. Secure Domination number of Sierpiński Graphs

Theorem *4.1:* For $n \geq 3$, $\gamma_s\{S(n,t)\} = n^{t-1}$.
Proof: By Theorem 2.5, $\gamma_{cs}\{S(n,t)\} \leq \gamma_s\{S(n,t)\}$. Hence, by Theorem 3.1,

$$\gamma_s\{S(n,t)\} \geq n^{t-1}. \tag{5}$$

For $t = 1$, $S(n,1) = K_n$. Since $\gamma_s(K_n) = 1$, we get $\gamma_s\{S[n,1]\} = 1$.
Consider $S(n,t)$ for $t > 1$. Let $S_\omega = \{i_1 i_2 \ldots i_{t-1}\omega / i_j \epsilon \{1,2,\ldots,n\}$, $j = 1,2,\ldots,(t-1)\}$ for a fixed $\omega \epsilon \{1,2,\ldots,n\}$. S_ω contains one vertex from each copy of K_n in $S(n,t)$. Clearly, S_ω forms a dominating set of $S(n,t)$. For any $v = i_1 i_2 \ldots i_{t-1} i_t \epsilon V \setminus S_\omega$, there exists

$u = i_1 i_2 \dots i_{t-1}\omega \; \epsilon \; S_\omega$, for $i_t \neq \omega$, where u and v belongs to the same K_n, such that $(S_\omega \setminus \{u\}) \cup \{v\}$ is a dominating set of $S(n,t)$. Hence, S_ω forms an SDS of $S(n,t)$, where $|S| = n^{t-1}$.

$$\gamma_s\{S(n,t)\} \leq n^{t-1}. \tag{6}$$

From (5) and (6), we get

$$\gamma_s\{S(n,t)\} = n^{t-1}.$$

***Corollary** 4.2:* In Hanoi graphs, $\gamma_s\{H_3^t\} = 3^{t-1}$.

V. Complete Co-Secure Domination number of Sierpiński Graphs

***Theorem** 5.1:* $\gamma_{ccs}(G) = 1$ if and only if $G = K_n$.

Proof: Let $\gamma_{ccs}(G) = 1$. Then there is a CCSDS $S = \{u\}$ such that u dominates every vertex in $V \setminus S$, which implies $d(u) = n - 1$.
Since S is a CCSDS, for every $v \; \epsilon \; V \setminus S$, $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. Hence, $d(v) = n - 1$ for all $v \; \epsilon \; V(G)$. Thus, $G = K_n$.
Let $G = K_n$. Then by Theorem 2.1, $S = \{u\}$ forms a CSDS, and for all $v \; \epsilon \; V \setminus S$, $(S \setminus \{u\}) \cup \{v\}$ forms a dominating set of K_n, which implies $\gamma_{ccs}(G) = 1$.

***Theorem** 5.2:* For $n \geq 3$, $\gamma_{ccs}\{S(n,t)\} = n^{t-1}$.

Proof: We have, $\gamma_{cs}(G) \leq \gamma_{ccs}(G)$, for any graph G. By Theorem 3.1,

$$\gamma_{ccs}\{S(n,t)\} \geq n^{t-1}. \tag{7}$$

For $t = 1$, the result is obvious, since $S(n,1) = K_n$ and Hence, by Theorem 5.1, $\gamma_{ccs}\{S(n,1)\} = \gamma_{ccs}(K_n) = 1$.
For $t = 2$, define $S = \{ii/i \; \epsilon \; \{1,2,\dots,n\}\}$, the set of extreme vertices of $S(n,2)$. S is clearly a dominating set of $S(n,2)$. Moreover, by Theorem 3.1, S is a CSDS of $S(n,2)$. From the definition of S, for every $u \; \epsilon \; S$, and for every $v \; \epsilon \; V \setminus S$, such that u and v are adjacent, the set $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. Hence, S forms a CCSDS of $S(n,2)$. Hence, $\gamma_{ccs}\{S(n,2)\} \leq n$. Using (7) (for t =2) we get,

$$\gamma_{ccs}\{S(n,2)\} = n.$$

Consider $S(n,t)$, $t > 2$. We are going to find a CCSDS having cardinality n^{t-1}, such that, $\gamma_{ccs}\{S(n,t)\} \leq n^{t-1}$. Here, we will select one vertex from each K_n in a specific way to form a CCSDS.

Case 1: n is odd.
Define $S_1 = \{i_1 i_2 \dots i_{t-2} ij, i_1 i_2 \dots i_{t-2} ji / i \neq j$, and no two vertices belongs to the same K_n, $i, j, i_k \; \epsilon \; \{1,2,\dots,n\}$ for $k = 1,2,\dots,(t-2)$ and neither of them are adjacent to the extreme vertices of any $S(n, t-r)$, for $t - r \geq 3\}$. In S_1, whenever $i_1 i_2 \dots i_{t-2} ij$ belongs to S_1, the vertex $i_1 i_2 \dots i_{t-2} ji$ also belongs to S_1. There exists $(n^{t-1} - n^{t-2})$ vertices in S_1. Now, consider K_n's having vertices with $i = j = i_{t-2} \neq i_{t-3}$. Define $S_2 = \{i_1 i_2 \dots i_{t-3} iii, i_1 i_2 \dots i_{t-4} i i_{t-3} i_{t-3} i_{t-3} / i \neq i_{t-3}; i, i_{t-3} \; \epsilon \; \{1,2,\dots,n\}$ for $k = 1,2,\dots,(t-3)\}$. There exists $(n^{t-2} - n^{t-3})$ vertices in S_2. Similarly consider K_n's having vertices with $i = j = i_{t-2} = i_{t-3} \neq i_{t-4}$. Define $S_3 = \{i_1 i_2 \dots i_{t-4} iiii, i_1 i_2 \dots i_{t-5} i i_{t-4} i_{t-4} i_{t-4} i_{t-4} / i \neq i_{t-4}; i, i_k \; \epsilon \; \{1,2,\dots,n\}$ for $k = 1,2,\dots,(t-4)\}$. There exists $(n^{t-3} - n^{t-4})$ vertices in S_3. Proceeding like this, define $S_{t-2} = \{ijj \dots j, jii \dots i / i \neq j; i,j \; \epsilon \; \{1,2,\dots,n\}\}$, the set of almost extreme vertices. There exists $(n^2 - n)$ vertices in S_{t-2}. Finally, define $S_{t-1} = \{ii \dots i / i \; \epsilon \; \{1,2,\dots,n\}\}$, the set of extreme vertices of $S(n,t)$. There exists n vertices in S_{t-1}. Now, define $S = S_1 \cup S_2 \cup \dots \cup S_{t-1}$, where $|S| = n^{t-1}$. Since we have selected one vertex from each K_n in $S(n,t)$, S forms a dominating set of $S(n,t)$. Moreover, by Theorem 3.1, S forms a CSDS.
Take any $u \; \epsilon \; S$. Consider the set of vertices in $V \setminus S$, that are adjacent to u. By the definition of S, these vertices are those that belong to the same K_n that contains u. From a different K_n, no vertex in $V \setminus S$ is adjacent to u. Hence, for every $u \; \epsilon \; S$ and for every $v \; \epsilon \; V \setminus S$, $(S \setminus \{u\}) \cup \{v\}$ forms a dominating set. Hence, S forms a CCSDS of $S(n,t)$. Hence,

$$\gamma_{ccs}\{S(n,t)\} \leq n^{t-1} \tag{8}$$

From (7) and (8), we get

$$\gamma_{ccs}\{S(n,t)\} = n^{t-1}.$$

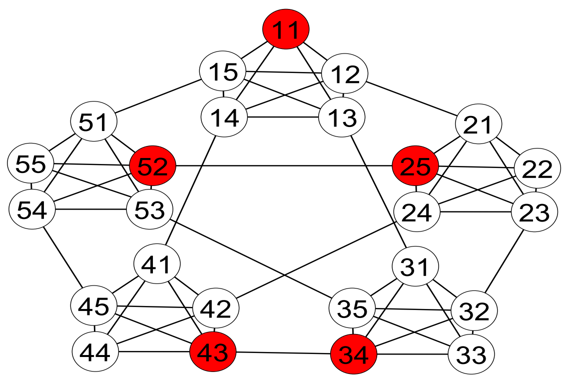

Fig. 5. $S(5,2)$

Case 2: n is even.
Since n is even, each $S(n,t,i_1 i_2 \dots i_{t-2})$ contains even number of K_n's. Define $S = \{i_1 i_2 \dots i_{t-2} ij, i_1 i_2 \dots i_{t-2} ji / i \neq j$, and no two vertices belongs to the same K_n; $i_k, i, j \; \epsilon \; \{1,2,\dots,n\}$, for $k = 1,2,\dots,(t-2)\}$. In S, whenever $i_1 i_2 \dots i_{t-2} ij$ belongs to S, the vertex $i_1 i_2 \dots i_{t-2} ji$ should also be in S. Here, S contains n vertices from each $S(n,t,i_1 i_2 \dots i_{t-2})$. ie, one vertex from each K_n. Evidently, S forms a dominating set of $S(n,t)$. Moreover, by Theorem 3.1, S forms a CSDS of $S(n,t)$. As in Case 1, for each $u \; \epsilon \; S$ and for every $v \; \epsilon \; V \setminus S$, $(S \setminus \{u\}) \cup \{v\}$ forms a dominating set. Hence, S forms a CCSDS of $S(n,t)$. Thus,

$$\gamma_{ccs}\{S(n,t)\} \leq n^{t-1} \tag{9}$$

From (7) and (9), we get

$$\gamma_{ccs}\{S(n,t)\} = n^{t-1}.$$

In Fig. 5., the set of red vertices indicates a CCSDS of minimum cardinality for $S(5,2)$.

***Corollary** 5.3:* In Hanoi graphs, $\gamma_{ccs}\{H_3^t\} = 3^{t-1}$.

VI. Perfect Secure Domination number of Sierpiński Graphs

***Theorem** 6.1:* For $n \geq 3$, $\gamma_{ps}\{S(n,t)\} = n^{t-1}$.

Proof: Clearly, $\gamma_s(G) \le \gamma_{ps}(G)$, for any graph G. Then by Theorem 4.1,

$$\gamma_{ps}\{S(n,t)\} \ge n^{t-1}. \tag{10}$$

We will now look for a PSDS with cardinality n^{t-1} such that $\gamma_{ps}\{S(n,t)\} \le n^{t-1}$. Here, we are selecting one vertex from each K_n in the same way mentioned in the proof of Theorem 5.2.

Case 1: n is odd.

Here, the set $S = S_1 \cup S_2 \cup \ldots \cup S_{t-1}$ (as defined in Theorem 5.2) forms a dominating set of $S(n,t)$, since we have selected one vertex from each K_n. Moreover, by Theorem 4.1, S forms an SDS of $S(n,t)$.

Let $v \in V \setminus S$. Because S contains one vertex from each copy of K_n, for any vertex $v \in V \setminus S$ there exists a vertex $u \in S$ where both u and v belong to the same K_n, such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. And it is obvious from the definition of S that the vertex u is unique. Hence, S forms a PSDS of $S(n,t)$. Thus,

$$\gamma_{ps}\{S(n,t)\} \le n^{t-1}. \tag{11}$$

From (10) and (11) we get,

$$\gamma_{ps}\{S(n,t)\} = n^{t-1}.$$

Case 2: n is even.

$S = \{i_1 i_2 \ldots i_{t-2} ij, i_1 i_2 \ldots i_{t-2} ji / i \neq j$, and no two vertices belong to the same K_n; $i_k, i, j \in \{1, 2, \ldots, n\}$, for $k = 1, 2, \ldots, (t-2)\}$. If the vertex $i_1 i_2 \ldots i_{t-2} ij$ belongs to S, then the vertex $i_1 i_2 \ldots i_{t-2} ji$ does as well. This is the same set as defined in Case 2 of Theorem 5.2. Clearly, S forms a dominating set of $S(n,t)$, since we have selected one vertex from each K_n. Moreover, by Theorem 4.1, S forms a secure dominating set of $S(n,t)$.

Let $v \in V \setminus S$; then, as in Case 1, there exists a unique vertex $u \in S$ such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. Hence, S forms a PSDS of $S(n,t)$. Thus,

$$\gamma_{ps}\{S(n,t)\} \le n^{t-1}. \tag{12}$$

From (10) and (12) we get,

$$\gamma_{ps}\{S(n,t)\} = n^{t-1}.$$

In Fig. 5., the set of red vertices indicates a PSDS of minimum cardinality for $S(5,2)$.

Corollary *6.2:* In Hanoi graphs, $\gamma_{ps}\{H_3^t\} = 3^{t-1}$.

VII. Conclusion

In this paper, we have computed the exact values of the secure domination parameters in the Sierpiński graphs and in the Hanoi graphs. For $n \ge 3$, we have obtained that, $\gamma_{cs}\{S(n,t)\} = \gamma_s\{S(n,t)\} = \gamma_{ccs}\{S(n,t)\} = \gamma_{ps}\{S(n,t)\} = n^{t-1}$. Also, we have proved that in the Hanoi graphs, for $t \ge 1$; $\gamma_{cs}\{H_3^t\} = \gamma_s\{H_3^t\} = \gamma_{ccs}\{H_3^t\} = \gamma_{ps}\{H_3^t\} = 3^{t-1}$. In several classes of graphs, the secure domination number and the co-secure domination number have been examined, and the values vary frequently. The equality of all four parameters in the Sierpiński graphs thus surprised us. Exploring the conditions in which these factors are equivalent will be fascinating. It will be interesting to examine the secure domination number in some different well-known kinds of graphs and networks as the challenge of finding the secure domination number for bipartite graphs and split graphs is NP-complete.

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