Eigenvalues of Discrete Second-order Coupled Boundary Value Problems with Sign-changing Weight

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***Abstract*—This paper is concerned with discrete second-order coupled boundary value problems with sign-changing weight. We find that these problems have T real eigenvalues (including the multiplicity). Specifically, the numbers of positive eigenvalues are equal to the number of positive elements in the weight function, and the numbers of negative eigenvalues are equal to the number of negative elements in the weight function. Furthermore, the relationships between the eigenvalues under three different coupled boundary conditions are established. These results extend the relevant existing results of periodic and anti-periodic boundary value problems with sign-changing weight and the coupled boundary value problems with definite weight.**

***Index Terms*—eigenvalues, second-order difference equations, coupled boundary condition, sign-changing weight.**

I. Introduction

LET $T > 2$ be an integer, $\mathbb{T} = \{1, 2, \ldots, T\}$. In this paper, we consider the second-order difference equation

$$Lu := \Delta[p(t-1)\Delta u(t-1)] - q(t)u(t) + \lambda a(t)u(t) = 0, t \in \mathbb{T} \tag{1}$$

with the coupled boundary conditions

$$\begin{pmatrix} u(T) \\ \Delta u(T) \end{pmatrix} = e^{i\alpha} K \begin{pmatrix} u(0) \\ \Delta u(0) \end{pmatrix}, \tag{2}$$

where $\Delta u(t) = u(t+1) - u(t)$, α is a constant parameter, $-\pi < \alpha \le \pi$,

$$K = \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix}, k_1, k_2, k_3 \in \mathbb{R}, \text{ with } k_1 k_3 = 1,$$

$q : \mathbb{T} \to [0, +\infty)$, $p : \{0, 1, \ldots, T\} \to (0, +\infty)$ satisfies $p(0) = p(T)$, and the weight function $a : \mathbb{T} \to \mathbb{R}$ satisfies the following condition:
(H0) $a(t)$ changes sign on $\mathbb{T}$, i.e., there exists a proper subset $\mathbb{T}_+$ of $\mathbb{T}$ such that

$$a(t) > 0 \text{ for } t \in \mathbb{T}_+, \text{ and } a(t) < 0 \text{ for } t \in \mathbb{T}\backslash\mathbb{T}_+.$$

Let n be the number of elements in $\mathbb{T}_+$. Then $T - n$ is the number of elements in $\mathbb{T}\backslash\mathbb{T}_+$.

When the weight function $a(t)$ in the equation (1) is not sign-changing, Atkinson [1], Jirari [2], Kelley and Peterson [3] studied the boundary condition

$$u(0) - hu(1) = 0, \; u(T+1) - lu(T) = 0, \tag{3}$$

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they obtained that this problem has T real eigenvalues, which can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_T$. Sun and Shi [4] discussed the boundary condition (2), where $p(t)$, $q(t)$, $a(t)$ are real functions with $p(t) > 0$ for $t \in \{0, 1, \ldots, T\}$, $a(t) > 0$ for $t \in \mathbb{T}$, $p(0) = p(T) = 1$, and

$$K = \begin{pmatrix} k_1 & 0 \\ k_2 & k_3 \end{pmatrix}, \; k_j \in \mathbb{R}, \; j = 1, 2, 3 \text{ with } k_1 k_3 = 1.$$

They obtained the following result.

Theorem 1 (Theorem A). *Assume $k_3 > 0$. Then, for every $\alpha \in (-\pi, 0) \cup (0, \pi)$, the eigenvalues $\eta_i (1 \le i \le T)$ of (1)-(2) satisfy the following inequalities:*

$$\begin{aligned} & \eta_1(K) < \eta_1(e^{i\alpha}K) < \eta_1(-K) \le \eta_2(-K) < \eta_2(e^{i\alpha}K) \\ < \; & \eta_2(K) \le \eta_3(K) < \eta_3(e^{i\alpha}K) < \eta_3(-K) \le \eta_4(-K) \\ < \; & \eta_4(e^{i\alpha}K) < \eta_4(K) \le \cdots \le \eta_{T-1}(-K) < \eta_{T-1}(e^{i\alpha}K) \\ < \; & \eta_{T-1}(K) \le \eta_T(K) < \eta_T(e^{i\alpha}K) < \eta_T(-K) \end{aligned}$$

if T is odd, and

$$\begin{aligned} & \eta_1(K) < \eta_1(e^{i\alpha}K) < \eta_1(-K) \le \eta_2(-K) < \eta_2(e^{i\alpha}K) \\ < \; & \eta_2(K) \le \eta_3(K) < \eta_3(e^{i\alpha}K) < \eta_3(-K) \le \eta_4(-K) \\ < \; & \eta_4(e^{i\alpha}K) < \eta_4(K) \le \cdots \le \eta_{T-1}(K) < \eta_{T-1}(e^{i\alpha}K) \\ < \; & \eta_{T-1}(-K) \le \eta_T(-K) < \eta_T(e^{i\alpha}K) < \eta_T(K) \end{aligned}$$

if T is even.

For the case when $a(t)$ is not sign-changing, further important results in linear Hamiltonian difference systems, including the oscillation properties of solutions, can be seen in Shi and Chen [5], Bohner [6], Agarwal et al. [7] and the references therein.

However, there are few results on the spectra the weight function $a(t)$ in the equation (1) changes its sign on $\mathbb{T}$. In 2007, Ji and Yang [8], [9] studied the structure of the eigenvalues of (1) and (3), and they obtained that the numbers of positive eigenvalues are equal to the number of positive elements in the weight function, and the numbers of negative eigenvalues are equal to the number of negative elements in the weight function. In 2018, Ma et al. [10] considered the general separate boundary condition

$$\alpha u(0) - \beta \Delta u(0) = 0, \; \gamma u(T+1) + \delta \Delta u(T) = 0, \tag{4}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfy $\alpha\beta \ge 0, \gamma\delta \ge 0$ with $\alpha^2 + \beta^2 \ne 0, \gamma^2 + \beta^2 \ne 0, p(t) > 0, t \in \{0, 1, \ldots, T\}, q : \mathbb{T} \to [0, \infty)$, and $a(t)$ satisfies (H0). They obtained that if $q(t) \not\equiv 0, t \in \mathbb{T}$ or $\alpha^2 + \gamma^2 \ne 0$, the problem (1)and (4) has T real eigenvalues, which can be ordered as $\lambda_{T-n,-} < \cdots <$

$\lambda_{1,-} < 0 < \lambda_{1,+} < \cdots < \lambda_{n,+}$. In 2015, Gao and Ma [12] discussed the periodic boundary condition

$$u(0) = u(T),\ u(1) = u(T+1) \tag{5}$$

and the antiperiodic boundary condition

$$u(0) = -u(T),\ u(1) = -u(T+1), \tag{6}$$

where $p(t) > 0$ for $t \in \{0, 1, \ldots, T\}$, $p(0) = p(T)$, $q(t) \geq 0$ and $a(t)$ satisfies (H0). They found out the following very beautiful results (see [12, Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6]): the periodic and antiperiodic boundary value problems with sign-changing weight respectively have exactly T real eigenvalues, $\{\lambda_{j,\nu}\}$ and $\{\widetilde{\lambda}_{j,\nu}\}, \nu \in \{+,-\}$, which satisfy

$$\begin{aligned} & \lambda_{T-n,-} < \widetilde{\lambda}_{T-n,-} \leq \widetilde{\lambda}_{T-n-1,-} < \lambda_{T-n-1,-} \\ < \ & \cdots < \lambda_{3,-} \leq \lambda_{2,-} < \widetilde{\lambda}_{2,-} \leq \widetilde{\lambda}_{1,-} < \lambda_{1,-} \leq 0 \\ \leq \ & \lambda_{1,+} < \widetilde{\lambda}_{1,+} \leq \widetilde{\lambda}_{2,+} < \lambda_{2,+} \leq \lambda_{3,+} < \cdots < \lambda_{n-1,+} \\ < \ & \widetilde{\lambda}_{n-1,+} \leq \widetilde{\lambda}_{n,+} < \lambda_{n,+} \text{ if } T \text{ is even and } n \text{ is even;} \end{aligned}$$

$$\begin{aligned} & \widetilde{\lambda}_{T-n,-} < \lambda_{T-n,-} \leq \lambda_{T-n-1,-} < \widetilde{\lambda}_{T-n-1,-} \\ < \ & \cdots < \lambda_{3,-} \leq \lambda_{2,-} < \widetilde{\lambda}_{2,-} \leq \widetilde{\lambda}_{1,-} < \lambda_{1,-} \leq 0 \\ \leq \ & \lambda_{1,+} < \widetilde{\lambda}_{1,+} \leq \widetilde{\lambda}_{2,+} < \lambda_{2,+} \leq \lambda_{3,+} < \cdots < \widetilde{\lambda}_{n-1,+} \\ < \ & \lambda_{n-1,+} \leq \lambda_{n,+} < \widetilde{\lambda}_{n,+} \text{ if } T \text{ is even and } n \text{ is odd;} \end{aligned}$$

$$\begin{aligned} & \widetilde{\lambda}_{T-n,-} < \lambda_{T-n,-} \leq \lambda_{T-n-1,-} < \widetilde{\lambda}_{T-n-1,-} \\ < \ & \cdots < \lambda_{3,-} \leq \lambda_{2,-} < \widetilde{\lambda}_{2,-} \leq \widetilde{\lambda}_{1,-} < \lambda_{1,-} \leq 0 \\ \leq \ & \lambda_{1,+} < \widetilde{\lambda}_{1,+} \leq \widetilde{\lambda}_{2,+} < \lambda_{2,+} \leq \lambda_{3,+} < \cdots < \lambda_{n-1,+} \\ < \ & \widetilde{\lambda}_{n-1,+} \leq \widetilde{\lambda}_{n,+} < \lambda_{n,+} \text{ if } T \text{ is odd and } n \text{ is even;} \end{aligned}$$

$$\begin{aligned} & \lambda_{T-n,-} < \widetilde{\lambda}_{T-n,-} \leq \widetilde{\lambda}_{T-n-1,-} < \lambda_{T-n-1,-} \\ < \ & \cdots < \lambda_{3,-} \leq \lambda_{2,-} < \widetilde{\lambda}_{2,-} \leq \widetilde{\lambda}_{1,-} < \lambda_{1,-} \leq 0 \\ \leq \ & \lambda_{1,+} < \widetilde{\lambda}_{1,+} \leq \widetilde{\lambda}_{2,+} < \lambda_{2,+} \leq \lambda_{3,+} < \cdots < \widetilde{\lambda}_{n-1,+} \\ < \ & \lambda_{n-1,+} \leq \lambda_{n,+} < \widetilde{\lambda}_{n,+} \text{ if } T \text{ is odd and } n \text{ is odd.} \end{aligned}$$

Motivated by [4] and [12], we apply some oscillation results obtained by [13] to prove the existence, the number of eigenvalues of (1)-(2) with sign-changing weight and to compare these eigenvalues as α varies. These results extend above results obtained in [12].

This paper is organized as follows. Section 2 gives some properties of eigenvalues of Neumann boundary value problem with sign-changing weight, which will be used in Section 3. Section 3 pays attention to comparison between the eigenvalues of problem (1) and (2) as α varies.

II. Preliminaries

Equation (1) can be rewritten as the recurrence formula

$$p(t)u(t+1) = [p(t)+p(t-1)+q(t)-\lambda a(t)]u(t)-p(t-1)u(t-1) \tag{7}$$

for $t \in \mathbb{T}$. Clearly, $u(t)$ is a polynomial in λ with real coefficients since $p(t), q(t)$ and $a(t)$ are all real. Especially, for $t \leq T+1$, the degree of $u(t)$ is t if $u(1) \neq 0$, and $t-1$ if $u(0) \neq 0$ and $u(1) = 0$.

Let $x(t, \lambda)$ be a solution of $Lx = 0$ with the initial condition

$$x(0, \lambda) = 1,\ \Delta x(0, \lambda) = 0 \tag{8}$$

and $y(t, \lambda)$ be a solution of $Ly = 0$ under the initial condition

$$y(0, \lambda) = 0,\ \Delta y(0, \lambda) = 1. \tag{9}$$

Then $x(t, \lambda)$ and $y(t, \lambda)$ are two independent solutions of (1), and they are all polynomials of degree t of λ.

Now, multiplying both sides of $Lx = 0$ and $Ly = 0$ by $y(t, \lambda)$ and $x(t, \lambda)$ separately, summing from $t = 1$ to $t = T$, then subtracting these two equations, we get

$$x(T, \lambda)y(T+1, \lambda) - x(T+1, \lambda)y(T, \lambda) = 1. \tag{10}$$

Ma et al. [13] discussed the spectra of the problem (1) with the Neumann boundary condition

$$\Delta u(0) = \Delta u(T) = 0. \tag{11}$$

They obtained the following result.

Lemma 1. *Suppose $p : \{0, 1, \ldots, T\} \to (0, +\infty)$, $q(t) \equiv 0$ on $\mathbb{T}$ and (H0) hold. Then the problem (1) and (11) has T real eigenvalues $\eta_{j,\nu}, j \in \mathbb{T}$, $\nu \in \{+,-\}$, which satisfy*

$$\eta_{T-n,-} < \cdots < \eta_{1,-} \leq 0 \leq \eta_{1,+} < \cdots < \eta_{n,+}. \tag{12}$$

The eigenfunction $\psi_{j,\nu}$, which corresponds to $\eta_{j,\nu}$, exhibits $j-1$ changes of sign on the integral $[0, T]$.

Furthermore, Ma et al. [10], [12] indicated that $\eta_{1,-}$ and $\eta_{1,+}$ are not zero when $q(t) \not\equiv 0, t \in \mathbb{T}$, that is,

$$\eta_{T-n,-} < \cdots < \eta_{1,-} < 0 < \eta_{1,+} < \cdots < \eta_{n,+}. \tag{13}$$

Lemma 2. *Let $\eta_{j,\nu}, j \in \mathbb{T}, \nu \in \{+,-\}$, be the eigenvalues of (1) and (11). Then $x(t, \eta_{j,\nu})$ is the eigenfunction with respect to $\eta_{j,\nu}$, that is, $x(t, \eta_{j,\nu})$ is a nontrivial solution of (1) satisfying*

$$\Delta x(0, \eta_{j,\nu}) = \Delta x(T, \eta_{j,\nu}) = 0. \tag{14}$$

Lemma 3. *If j is odd, $x(T, \eta_{j,\nu}) > 0$ and if j is even, $x(T, \eta_{j,\nu}) < 0$.*

Proof: Since $x(0, \eta_{j,\nu}) = 1$, $\Delta x(T, \eta_{j,\nu}) = \Delta x(0, \eta_{j,\nu}) = 0$, then $x(T, \eta_{j,\nu}) > 0$ if $x(T, \eta_{j,\nu})$ has an even number of sign changes in the interval $[0, T)$, and $x(T, \eta_{j,\nu}) < 0$ if $x(T, \eta_{j,\nu})$ has an odd number of sign changes in $[0, T)$. This can be obtained directly from Lemma 1. ■

III. Main results

Let $x(t, \lambda)$ and $y(t, \lambda)$ be defined as in Section 2, and let $\lambda_{j,\nu}(e^{i\alpha}K)$, $j \in \mathbb{T}, \nu \in \{+,-\}$, be the eigenvalues of the coupled boundary value problem (1)-(2). We now present the main results of this paper.

Theorem 2. *Assume that $k_3 > 0$, $k_1 > k_2$ and (H0) holds. Then (1)-(2) has T real eigenvalues $\lambda_{j,\nu}(e^{i\alpha}K)$, among which n non-negative eigenvalues and $T-n$ non-positive eigenvalues.*

(i) If T is an even number and n is an even number, then

$$\begin{aligned}
&\lambda_{T-n,-}(K) < \lambda_{T-n,-}(e^{i\alpha}K) < \lambda_{T-n,-}(-K)\\
\leq\ & \eta_{T-n,-}\\
\leq\ & \lambda_{T-n-1,-}(-K) < \lambda_{T-n-1,-}(e^{i\alpha}K) < \lambda_{T-n-1,-}(K)\\
\leq\ & \cdots \leq \lambda_{2,-}(K) < \lambda_{2,-}(e^{i\alpha}K) < \lambda_{2,-}(-K)\\
\leq\ & \eta_{2,-} \leq \lambda_{1,-}(-K) < \lambda_{1,-}(e^{i\alpha}K) < \lambda_{1,-}(K)\\
\leq\ & \eta_{1,-} \leq 0 \leq \eta_{1,+}\\
\leq\ & \lambda_{1,+}(K) < \lambda_{1,+}(e^{i\alpha}K) < \lambda_{1,-}(-K) \leq \eta_{2,+}\\
\leq\ & \lambda_{2,+}(-K) < \lambda_{2,+}(e^{i\alpha}K) < \lambda_{2,+}(K)\\
\leq\ & \cdots \leq \eta_{n-1,+}\\
\leq\ & \lambda_{n-1,+}(K) < \lambda_{n-1,+}(e^{i\alpha}K) < \lambda_{n-1,+}(-K)\\
\leq\ & \eta_{n,+} \leq \lambda_{n,+}(-K) < \lambda_{n,+}(e^{i\alpha}K) < \lambda_{n,+}(K).
\end{aligned}$$

(ii) If T is an odd number and n is an even number, then

$$\begin{aligned}
&\lambda_{T-n,-}(-K) < \lambda_{T-n,-}(e^{i\alpha}K) < \lambda_{T-n,-}(K)\\
\leq\ & \eta_{T-n,-}\\
\leq\ & \lambda_{T-n-1,-}(K) < \lambda_{T-n-1,-}(e^{i\alpha}K) < \lambda_{T-n-1,-}(-K)\\
\leq\ & \cdots \leq \lambda_{2,-}(K) < \lambda_{2,-}(e^{i\alpha}K) < \lambda_{2,-}(-K)\\
\leq\ & \eta_{2,-} \leq \lambda_{1,-}(-K) < \lambda_{1,-}(e^{i\alpha}K) < \lambda_{1,-}(K)\\
\leq\ & \eta_{1,-} \leq 0 \leq \eta_{1,+}\\
\leq\ & \lambda_{1,+}(K) < \lambda_{1,+}(e^{i\alpha}K) < \lambda_{1,-}(-K) \leq \eta_{2,+}\\
\leq\ & \lambda_{2,+}(-K) < \lambda_{2,+}(e^{i\alpha}K) < \lambda_{2,+}(K)\\
\leq\ & \cdots \leq \eta_{n-1,+}\\
\leq\ & \lambda_{n-1,+}(K) < \lambda_{n-1,+}(e^{i\alpha}K) < \lambda_{n-1,+}(-K)\\
\leq\ & \eta_{n,+} \leq \lambda_{n,+}(-K) < \lambda_{n,+}(e^{i\alpha}K) < \lambda_{n,+}(K).
\end{aligned}$$

(iii) If T is an even number and n is an odd number, then

$$\begin{aligned}
&\lambda_{T-n,-}(-K) < \lambda_{T-n,-}(e^{i\alpha}K) < \lambda_{T-n,-}(K)\\
\leq\ & \eta_{T-n,-}\\
\leq\ & \lambda_{T-n-1,-}(K) < \lambda_{T-n-1,-}(e^{i\alpha}K) < \lambda_{T-n-1,-}(-K)\\
\leq\ & \cdots \leq \lambda_{2,-}(K) < \lambda_{2,-}(e^{i\alpha}K) < \lambda_{2,-}(-K)\\
\leq\ & \eta_{2,-} \leq \lambda_{1,-}(-K) < \lambda_{1,-}(e^{i\alpha}K) < \lambda_{1,-}(K)\\
\leq\ & \eta_{1,-} \leq 0 \leq \eta_{1,+}\\
\leq\ & \lambda_{1,+}(K) < \lambda_{1,+}(e^{i\alpha}K) < \lambda_{1,-}(-K) \leq \eta_{2,+}\\
\leq\ & \lambda_{2,+}(-K) < \lambda_{2,+}(e^{i\alpha}K) < \lambda_{2,+}(K)\\
\leq\ & \cdots \leq \lambda_{n-1,+}(-K) < \lambda_{n-1,+}(e^{i\alpha}K) < \lambda_{n-1,+}(K)\\
\leq\ & \eta_{n,+} \leq \lambda_{n,+}(K) < \lambda_{n,+}(e^{i\alpha}K) < \lambda_{n,+}(-K).
\end{aligned}$$

(iv) If T is an odd number and n is an odd number, then

$$\begin{aligned}
&\lambda_{T-n,-}(K) < \lambda_{T-n,-}(e^{i\alpha}K) < \lambda_{T-n,-}(-K)\\
\leq\ & \eta_{T-n,-}\\
\leq\ & \lambda_{T-n-1,-}(-K) < \lambda_{T-n-1,-}(e^{i\alpha}K) < \lambda_{T-n-1,-}(K)\\
\leq\ & \cdots \leq \lambda_{2,-}(K) < \lambda_{2,-}(e^{i\alpha}K) < \lambda_{2,-}(-K)\\
\leq\ & \eta_{2,-} \leq \lambda_{1,-}(-K) < \lambda_{1,-}(e^{i\alpha}K) < \lambda_{1,-}(K)\\
\leq\ & \eta_{1,-} \leq 0 \leq \eta_{1,+}\\
\leq\ & \lambda_{1,+}(K) < \lambda_{1,+}(e^{i\alpha}K) < \lambda_{1,-}(-K) \leq \eta_{2,+}\\
\leq\ & \lambda_{2,+}(-K) < \lambda_{2,+}(e^{i\alpha}K) < \lambda_{2,+}(K)\\
\leq\ & \cdots \leq \lambda_{n-1,+}(-K) < \lambda_{n-1,+}(e^{i\alpha}K) < \lambda_{n-1,+}(K)\\
\leq\ & \eta_{n,+} \leq \lambda_{n,+}(K) < \lambda_{n,+}(e^{i\alpha}K) < \lambda_{n,+}(-K).
\end{aligned}$$

Corollary 1. *For every fixed $\alpha \neq 0$, $-\pi < \alpha < \pi$, $\lambda_{j,\nu}(e^{i\alpha}K)$ is a simple eigenvalue of (1)-(2).*

Corollary 2. *If $T-n$ is even, then $\lambda_{T-n,-}(K)$ is simple, otherwise $\lambda_{T-n,-}(-K)$ is simple.*

Corollary 3. *If n is even, then $\lambda_{n,+}(K)$ is simple, otherwise $\lambda_{n,+}(-K)$ is simple.*

Remark 1. *If $k_3 < 0$, $k_1 < k_2$, a similar results can be obtained by applying Theorem 2 to $-K$. In fact, $e^{i\alpha}K = e^{i(\pi+\alpha)}(-K)$ for $\alpha \in (-\pi, 0)$ and $e^{i\alpha}K = e^{i(-\pi+\alpha)}(-K)$ for $\alpha \in (0, \pi)$. Hence, the boundary condition (2) in the case of $k_3 < 0$, $k_1 < k_2$ and $\alpha \neq 0$, $-\pi < \alpha < \pi$, can be written as condition (2) replaced by $\pi + \alpha$ for $\alpha \in (-\pi, 0)$ and $-\pi + \alpha$ for $\alpha \in (0, \pi)$, and K is replaced by $-K$.*

Remark 2. *Theorem 2 extends [4, Theorem 3.1] and [12, Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6].*

Before proving Theorem 2, we prove the following lemmas.

Lemma 4. *λ is an eigenvalue of (1)-(2) if and only if*

$$f(\lambda) = 2\cos\alpha, \tag{15}$$

where

$$f(\lambda) := k_3 x(T, \lambda) + k_1 \Delta y(T, \lambda) - k_2 \Delta x(T, \lambda). \tag{16}$$

Proof: If the general solution of equation (1) $u(t, \lambda) = C_1 x(t, \lambda) + C_2 y(t, \lambda)$ satisfies (2), then

$$\begin{pmatrix} x(T,\lambda) - e^{i\alpha}k_1 & y(T,\lambda) - e^{i\alpha}k_2 \\ \Delta x(T,\lambda) & \Delta y(T,\lambda) - e^{i\alpha}k_3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

It is evident that $\lambda \in \mathbb{C}$ is an eigenvalue of (1)-(2) if and only if (17) has a nontrivial solution (C_1, C_2), i.e.,

$$\det \begin{pmatrix} x(T,\lambda) - e^{i\alpha}k_1 & y(T,\lambda) - e^{i\alpha}k_2 \\ \Delta x(T,\lambda) & \Delta y(T,\lambda) - e^{i\alpha}k_3 \end{pmatrix} = 0,$$

which, together with (10) and $k_1 k_3 = 1$, implies that

$$1 + e^{2i\alpha} - e^{i\alpha} f(\lambda) = 0.$$

Then (15) follows from the above relation and the fact $e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$. This completes the proof. ∎

Lemma 5. *Assume that $k_3 > 0$ and (H0) holds. Let $\eta_{j,\nu}$, $\nu \in \{+, -\}$, be the eigenvalues of (1) and (11), and they are arranged as (12). Then*

(i) $f(\eta_{j,\nu}) \geq 2$ when j is an odd number;

(ii) $f(\eta_{j,\nu}) \leq -2$ when j is an even number.

Proof: From (14) and (10), we have

$$x(T, \eta_{j,\nu}) \Delta y(T, \eta_{j,\nu}) = 1. \tag{18}$$

By (18), (16) and the fact that $k_1 k_3 = 1$, we obtain

$$f(\eta_{j,\nu}) = k_3 x(T, \eta_{j,\nu}) + \frac{1}{k_3 x(T, \eta_{j,\nu})}.$$

Hence, noting $k_3 > 0$, and by Lemma 3, we have that if k is odd, then $f(\eta_{j,\nu}) \geq 2$ and if k is even, then $f(\eta_{j,\nu}) \leq -2$. This completes the proof. ∎

Lemma 6. *Assume that $k_3 > 0$, $k_1 > k_2$ and (H0) holds. Let $\eta_{T-n,-}$ and $\eta_{n,+}$ be the minimum and the maximum eigenvalue of (1) and (11), respectively. Then for every* *fixed* $\alpha \in (-\pi, \pi]$, *there exist two roots,* $\lambda_{T-n,-}(e^{i\alpha}K)$, $\lambda_{n,+}(e^{i\alpha}K)$, *of* $f(\lambda) = 2\cos\alpha$ *such that*

$$\lambda_{T-n,-}(e^{i\alpha}K) \leq \eta_{T-n,-} \tag{19}$$

and

$$\lambda_{n,+}(e^{i\alpha}K) \geq \eta_{n,+}. \tag{20}$$

Proof: By the discussion in the first paragraph in Section 2, $x(t,\lambda)$ and $y(t,\lambda)$ are both polynomials of degree t of λ for $t \leq T+1$. By (7)-(9), it is not difficult to see that

$$\begin{aligned} f(\lambda) &= (-1)^T(k_1-k_2)\frac{a(1)a(2)\cdots a(T)}{p(1)p(2)\cdots p(T)}\lambda^T + Q_{T-1}(\lambda) \\ &= (-1)^n(k_1-k_2)\frac{|a(1)||a(2)|\cdots|a(T)|}{p(1)p(2)\cdots p(T)}\lambda^T \\ &\quad +Q_{T-1}(\lambda), \end{aligned}$$

where $Q_{T-1}(\lambda)$ is a $T-1$ degree polynomial of λ.

If T and n are both even number, it is easy to see that, for $k_1 > k_2$,

$$f(\lambda) \to +\infty \text{ if } \lambda \to \pm\infty.$$

On the other hand, from Lemma 5, we know that $f(\eta_{T-n,-}) \leq -2$ and $f(\eta_{n,+}) \leq -2$. So, there exist two numbers $\lambda_{T-n,-}(e^{i\alpha}K)$, $\lambda_{n,+}(e^{i\alpha}K)$, such that

$$-\infty < \lambda_{T-n,-}(e^{i\alpha}K) \leq \eta_{T-n,-},$$

$$f(\lambda_{T-n,-}(e^{i\alpha}K)) = 2\cos\alpha$$

and

$$\eta_{n,+} \leq \lambda_{n,+}(e^{i\alpha}K) < +\infty,\ f(\lambda_{n,+}(e^{i\alpha}K)) = 2\cos\alpha.$$

Similarly, for the cases of (i) T and n are both odd number, (ii) T is an even number and n is an odd number, (iii) T is an odd number and n is an even number, we have the same conclusion. This completes the proof. ■

Lemma 7. *Assume that* $k_3 > 0$, $k_1 > k_2$ *and (H0) holds. Let* $\eta_{j,\nu}$, $\nu \in \{+,-\}$, *be the eigenvalues of (1) and (11) and be arranged as (12). Then there exists unique root,* $\lambda_{j,\nu}(e^{i\alpha}K)$, *of* $f(\lambda) = 2\cos\alpha$ *such that*
(i) for $\alpha \in (-\pi,\pi)$ *and* $\alpha \neq 0$,

$$|\eta_{j,\nu}| < |\lambda_{j,\nu}(e^{i\alpha}K)| < |\eta_{i+1,\nu}|; \tag{21}$$

(ii) for $\alpha = 0$,

$$\begin{aligned} |\eta_{2j-1,\nu}| &\leq |\lambda_{2j-1,\nu}(K)| < |\eta_{2j,\nu}| \\ &< |\lambda_{2j,\nu}(K)| \leq |\eta_{2j+1,\nu}|; \end{aligned} \tag{22}$$

(iii) for $\alpha = \pi$,

$$\begin{aligned} |\eta_{2j-1,\nu}| &< |\lambda_{2j-1,\nu}(-K)| \leq |\eta_{2j,\nu}| \\ &\leq |\lambda_{2j,\nu}(-K)| < |\eta_{2j+1,\nu}|. \end{aligned} \tag{23}$$

Proof: From the proof of Lemma 6, we get that $f(\lambda)$ is a polynomial of degree T of λ with real coefficients. By Lemma 5, $f(\eta_{2j-1,+}) \geq 2$ and $f(\eta_{2j,+}) \leq -2$. The existence of the eigenvalues of (1) and (2) which satisfy the inequations (21)-(23) can be obtained from the intermediate value theorem of continuous functions and the fact that $f(\lambda_{j,\nu}(K)) = 2$, $f(\lambda_{j,\nu}(-K)) = -2$, $f\left(\lambda_{j,\nu}(e^{i\alpha}K)\right) \in (-2,2)$ for $\alpha \in (-\pi,\pi)$, $\alpha \neq 0$. The uniqueness of such eigenvalues can be obtained by the fact that the equation $f(\lambda) = 2\cos\alpha$ has and only has T zeros. ■

Lemma 8. *Let* $\lambda_{j,\nu}(e^{i\alpha}K)$, $\nu \in \{+,-\}$, *be the eigenvalues of (1) and (2). Then for* $\alpha \in (-\pi,\pi)$ *and* $\alpha \neq 0$,

$$|\lambda_{2j-1,\nu}(K)| < |\lambda_{2j-1,\nu}(e^{i\alpha}K)| < |\lambda_{2j-1,\nu}(-K)| \tag{24}$$

$$|\lambda_{2j,\nu}(-K)| < |\lambda_{2j,\nu}(e^{i\alpha}K)| < |\lambda_{2j,\nu}(K)|. \tag{25}$$

Proof: We only prove (24) for the case $\nu = +$. The inequality relations (24) for the case $\nu = -$ and the inequality relations (25) can be obtained similarly.

Firstly, let us prove that for $j = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil$,

$$\lambda_{2j-1,+}(K) < \lambda_{2j-1,+}(e^{i\alpha}K), \tag{26}$$

where $\left\lceil \frac{n}{2} \right\rceil$ is the largest integer less than or equal to $\frac{n}{2}$.

On the contrary, suppose that

$$\lambda_{2j-1,+}(K) \geq \lambda_{2j-1,+}(e^{i\alpha}K).$$

If $\lambda_{2j-1,+}(K) = \lambda_{2j-1,+}(e^{i\alpha}K)$, then

$$f(\lambda_{2j-1,+}(e^{i\alpha}K)) = f\left(\lambda_{2j-1,+}(K)\right) = 2,$$

which contradicts $f(\lambda_{2j-1,+}(e^{i\alpha}K)) = \cos\alpha < 2$ for $\alpha \neq 0$. Therefore, $\lambda_{2j-1,+}(K) > \lambda_{2j-1,+}(e^{i\alpha}K)$. This combines (21), we have

$$\eta_{2j-1,+} < \lambda_{2j-1,+}(e^{i\alpha}K) < \lambda_{2j-1,+}(K).$$

By Lemma 5 and Lemma 7, $f(\eta_{2j-1,+}) \geq 2$, $\lambda_{2j-1,+}(K)$ is the unique root of $f(\lambda) = 2$ in the interval $[\eta_{2j-1,+}, \eta_{2j,+})$. Then we get that $f(\eta_{2j-1,+}) > 2$ and then $f(\lambda_{2j-1,+}(e^{i\alpha}K)) > 2$, which is a contradiction. Thus

$$\lambda_{2j-1,+}(K) < \lambda_{2j-1,+}(e^{i\alpha}K).$$

Secondly, we claim that for $j = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil$,

$$\lambda_{2j-1,+}(e^{i\alpha}K) < \lambda_{2j-1,+}(-K). \tag{27}$$

Similarly, from (21)-(23), we known $\lambda_{2j-1,+}(e^{i\alpha}K) \in (\eta_{2j-1,+}, \eta_{2j,+})$, and $\lambda_{2j-1,+}(-K)$ is the unique root of $f(\lambda) = \cos\pi = -2$ in the interval $(\eta_{2j-1,+}, \eta_{2j,+}]$. On the contrary, suppose $\lambda_{2j-1,+}(e^{i\alpha}K) \geq \lambda_{2j-1,\nu}(-K)$. If $\lambda_{2j-1,+}(e^{i\alpha}K) = \lambda_{2j-1,\nu}(-K)$, then

$$f(\lambda_{2j-1,+}(e^{i\alpha}K)) = f\left(\lambda_{2j-1,+}(-K)\right) = -2,$$

which contradicts $f(\lambda_{2j-1,+}(e^{i\alpha}K)) = \cos\alpha > -2$ for $\alpha \in (-\pi,\pi)$ and $\alpha \neq 0$. Therefore,

$$\lambda_{2j-1,+}(e^{i\alpha}K) > \lambda_{2j-1,\nu}(-K).$$

By Lemma 5, $f(\eta_{2j,+}) \leq -2$, this combines with that fact that $\lambda_{2j-1,\nu}(-K) < \lambda_{2j-1,+}(e^{i\alpha}K) < \eta_{2j,+}$ and $\lambda_{2j-1,+}(-K)$ is the unique root of $f(\lambda) = -2$ in the interval $(\eta_{2j-1,+}, \eta_{2j,+}]$, we get that $f(\eta_{2j,+}) < -2$ and then $f(\lambda_{2j-1,+}(e^{i\alpha}K)) < -2$, which is a contradiction. Thus

$$\lambda_{2j-1,+}(e^{i\alpha}K) < \lambda_{2j-1,+}(-K).$$

Thirdly, if n is an odd number, the inequality relations

$$\lambda_{n,+}(K) < \lambda_{n,+}(e^{i\alpha}K) < \lambda_{n,\nu}(-K) \tag{28}$$

can be proved in a similar way as we used in the two cases above. But we use the conditions $f(\eta_{n,+}) \geq 2$ and $f(\lambda) \to -\infty$ as $\lambda \to +\infty$ instead of the conditions $f(\eta_{2j-1,+}) \geq 2$ and $f(\eta_{2j,+}) \leq -2$. ■

Proof of Theorem 2: From Lemma 5, Lemma 6, Lemma 7 and Lemma 8, we can get the comparison theorem. ■

From (13) and the proof of Theorem 2, we immediately come to the following conclusion

Theorem 3. *Assume that* $k_3 > 0$, $k_1 > k_2$ *and (H0) holds. If* $q(t) \not\equiv 0$ *for* $t \in \{1, 2, \cdots, T\}$, *then*

$$\lambda_{1,-}(K) < 0 < \lambda_{1,+}(K),$$

i.e., $\lambda_{1,-}(K)$ *and* $\lambda_{1,+}(K)$ *are simple eigenvalues of (1)-(2) with* $\alpha = 0$.

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