Designing the Performance of EWMA Control Chart for Seasonal Moving Average Process with Exogenous Variables

Kanita Petcharat

Abstract— **The average run length (ARL) is deployed to measure control charts' effectiveness. This article provides a new exact analytical ARL solution for the exponentially weighted moving average (EWMA) control chart when the process is $SMAX(Q,r)_L$. The explicit ARL solution for the $SMAX(Q,r)_L$ process will be analyzed utilizing the Fredholm integral equation method. The fixed point theorem of Banach guarantees the solution's existence and uniqueness. In addition, ARL values are computed using numerical integral equations based on midpoint and Gaussian principles. The simulation's outcome revealed that the ARL values derived from the exact solution and numerical integral equation are identical. Regarding computational time, the result indicates that the exact analytical ARL solution outperforms the numerical integral equations. Therefore, the ARL values on EWMA control chart are evaluated using either the exact analytical ARL solution or the numerical integral equation.**

Index Terms— **Control chart, EWMA, seasonal moving average, Average run length, Exogenous variable.**

I. INTRODUCTION

Controlling quality is necessary to reduce the number of failures that occur during the manufacturing process. Control charts are a high-quality control tool that monitors and controls production to maintain a continuous production process in real-time and over time. In 1924, Shewhart [1] proposed the concept of the control chart as a means of enhancing the quality of production operations while also cutting down on waste. The present Shewhart control charts, cumulative sum control charts (CUSUM), and exponentially weighted average (EWWA) control charts are all employed for tracking and tracing changes in the quality of sample products and processes, including the manufacturing sector, medical and public health, financial and economics, and additional fields. Page [2] and Robert [3] discovered the CUSUM and EWMA control charts, respectively. Both charts were superior to the Shewhart control charts regarding the ability to discern minor fluctuations in the process's mean. Typically, the average run length (ARL) judges the effectiveness of control charts. The value of ARL_0 value indicates suggests that the process is still under control, whereas the value of ARL_1 indicates that the process is not under control and should be minimized. Multiple approaches are used to estimate the ARL's value, including Monte Carlo simulation [4], Markov chain [5], and Numerical integral equation (NIE) [6]-[8]. Traditionally, manufacturing processes data that is normally and independently distributed. In some instances, data may exhibit automatically correlated behavior, affecting the control chart's change detection performance. For example, the research's examination of the average change detection efficacy of the cumulative sum control chart when the data are the autoregressive order one and moving average order one processes reveals that the change detection performance is negatively affected. [9]. Accordingly, some researchers evaluate (ARL) value when the process involves autocorrelation. Lu and Reynolds [10] investigated the ability of the EWMA control chart to track the process when the data were autoregressive AR and autoregressive-moving average (ARMA) processes by employing the integral equation method. In the subsequent research, Petcharat [11] constructed an explicit solution of ARL to handle the EWMA control chart where the data set was a seasonal MA with exponential white noise. According to the study's findings, the EWMA control chart detected a change process with an enhanced level of sensitivity than the CUSUM control chart. In addition, numerous researchers have constructed explicit solutions of ARL on control charts for time series model, such as the explicit solution of ARL on EWMA for ARFIMA model [12], the explicit solution of ARL on CUSUM for MA and seasonal MA models [13], the explicit solution of ARL on CUSUM for seasonal AR with trend model [14], and the explicit solution of ARL on CUSUM c for ARMA models [15]. Recent research by Suriyakat and Petcharat [16] constructed explicit solutions of ARL for the EWMA control chart under MA model with exogenous variable with exponential random walk. They compared their findings to the utilization of numerical integration methods. There is no published work on the performance design on EWMA control chart with seasonal moving average with exogenous variables $(SMAX(Q,r)_L)$ process. Therefore, the intent of this paper aims to construct

Manuscript received September 2, 2022; revised May 21, 2023.

The Research on by King Mongkut's University of Technology North has received funding support from the National Science, Research and Innovation Fund (NSRF) (Grant No. 60544).

Kanita Petcharat is an assistant professor of Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bang Sue, Bangkok, 10800, THAILAND (e-mail: kanita.p@sci. kmutnb.ac.th).

the explicit solution of ARL on EWMA control chart for SMAX(Q,r)_L process with exponential random walk. Section 2 describes the EWMA and CUSUM control charts for the SMAX(Q,r)_L process. The exact formula of ARL on EWMA control chart under SMAX(Q,r)_L observations is proven in section 3. Section 4 describes the NIE methodology. Section 5 discusses simulation values derived from exact solutions and the NIE principle. Comparing EWMA and CUSUM control charts are the subject of Section 6. Finally, Section 7 contains the conclusion.

II. The EWMA and CUSUM Control Charts for SMAX (Q,r)_L Process

The features of EWMA and CUSUM control charts on the SMAX (Q,r)_L process addressed in this section. Both control charts effective at detecting minor changes in the process. For the EWMA control chart, assume Z_t is sequence of the stationary seasonal moving average process with exogenous variables (SMAX (Q,r)_L). The following equation describes the SMAX (Q,r)_L process subject to exponential white noise:

$$Z_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-L} - \theta_2\varepsilon_{t-2L} - \dots - \theta_Q\varepsilon_{t-QL} + \sum_{i=1}^{r}\beta_i X_{it},\ \varepsilon_t \sim Exp(\alpha) \quad (1)$$

where

μ is the mean of process,

α is the mean of exponential distribution,

$\theta_1, \theta_2, ..., \theta_Q$ are coefficient of seasonal moving average, $|\theta_Q| < 1$

L is the number of seasonal periods,

X_{it} is the exogenous variable,

ε_t is the error term with white noise exponential distribution,

β_i is the coefficient of X_{it}.

The EWMA statistic for SMAX(Q,r)_L process is represented as

$$V_t = (1-\lambda)V_{t-1} + \lambda Z_t\ ,\quad t = 1,2,... \quad (2)$$

where Z_t is the SMAX(Q,r)_L process, and λ represent the exponential smoothing parameter, $(0,1]$.

The EWMA control chart's control limits are identified as follows:

Upper control limit (UCL)

$$\text{UCL} = \mu + h_1\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}.$$

Center line (CL)

$$\text{CL} = \mu.$$

Lower control limit (LCL)

$$\text{LCL} = \mu - h_1\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}.$$

Where μ and σ are mean and standard deviation of EWMA statistic. Let h_1 be the width of the EWMA control limits and $V_0 = \mu$.

By substituting (1) into (2), then (2) can be express as follow:

$$V_t = (1-\lambda)V_{t-L} + \lambda(\mu + \varepsilon_t - \theta_1\varepsilon_{t-L} - \dots - \theta_Q\varepsilon_{t-QL} + \sum_{i=1}^{r}\beta_i X_{it}),\quad t = 1,2,...$$

$$V_t = (1-\lambda)V_{t-L} + \lambda\mu + \lambda\varepsilon_t - \lambda\theta_1\varepsilon_{t-L} - \dots - \lambda\theta_Q\varepsilon_{t-QL} + \lambda\sum_{i=1}^{r}\beta_i X_{it},\ \text{where } t = 1,2,...,\ V_0 = Z_0 = u.$$

The appropriate stopping time for (2) state as

$$\tau = \inf\{t > 0; V_t > h_1\}\quad ,V_0 = u,\ h_1 > v. \quad (3)$$

where τ be the stopping time, h_1 be upper control limit and V_0 is default values.

Let the notation $\mathbb{E}_\omega(.)$ represents the expectation that the change-point occurs at point π where $\pi < \infty$. Then, the ARL for the SMAX (Q,r)_L process with default values of $V_0 = u$ is defined as

$$ARL = l(u) = \mathbb{E}_\infty(\pi_{h_1}) < \infty. \quad (4)$$

The CUSUM control chart is typically designed to detect changes in the random variable's mean. Assuming $M_1, M_2, ...$ be sequence of random variable, the recursive CUSUM defined as follows:

$$M_t = \max\{M_{t-1} + Z_t - a, 0\},\quad t = 1,2,....$$

When $M_t > k$, k is the upper control limit, and a is the reference value, the CUSUM control chart depicts an out-of-control range.

III. The Average Length (ARL) for SMAX (Q,r)_L Process on EWMA Control Chart

The explicit solutions of ARL on EWMA control chart for SMAX (Q,r)_L process is proven in this part. Let $l(u)$ be the average run length of EWMA chart. Let $C_0 = u$ is the initial process in-control. The integral equation expresses in $l(u)$ as follows;

$$l(u) = 1 + \frac{1}{\lambda}\int_0^{h_1} j(z) f\left(\frac{z-(1-\lambda)u}{\lambda} + \begin{pmatrix}\mu + \varepsilon_t + \theta_1\varepsilon_{t-L} + \dots \\ +\theta_Q\varepsilon_{t-QL} - \sum_{i=1}^{r}\beta_i X_{it}\end{pmatrix}\right) d(z), \quad (5)$$

Then $$l(u) = 1 + \frac{1}{\lambda\alpha}\int_0^{h_1} l(z) e^{-\frac{z}{\lambda\alpha}} e^{\begin{pmatrix}\frac{(1-\lambda)u}{\lambda\alpha} + \frac{1}{\alpha}\mu + \varepsilon_t + \theta_1\varepsilon_{t-L} \\ + \dots + \theta_Q\varepsilon_{t-QL} - \sum_{i=1}^{r}\beta_i X_{it}\end{pmatrix}} d(z). \quad (6)$$

Suppose $C(u)=e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}$ then (6) can written as

$$l(u)=1+\frac{C(u)}{\lambda\alpha}\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}d(z),\ 0\le u\le h_1. \qquad (7)$$

Next, we employ Banach's fixed point theorem to confirm the existence and uniqueness of the ARL solution for the EWMA control chart.

Theorem 3.1 (Banach fixed point theorem) *Let* (S,d) *represent a complete metric space with a contraction mapping* $T:S\to S$. *Thus,* T *admits a single fixed-point* $s^*\in S$, $T(s^*)=s^*$. s^* *can defined as follows: start with random element* $s_0\in S$ *and defines a sequence* $\{s_n\}$ *by* $s_n=T(s_{n-1})$, *then* $s_n\to s^*$.

Proof

Hence, the right-hand side of (7) is continuous function and (7)'s solution is also continuous. Examine, complete metric space $(S(I),\|\,\|_\infty)$ where $S(I)$ be space of all continuous functions on compact interval I and the norm $\|l\|_\infty=Sup_{u\in I}|l(u)|$. Observed that the operator T is the contraction, if there exists a real constant $0\le\vartheta<1$ such that $\|T(l_1)-T(l_2)\|\le\vartheta\|l_1-l_2\|$; $\forall l_1,\forall l_2\in C(I)$, where $I=[0,h_1)$ and define the operator T by

$$T(l(u))=\ l(u)=1+\frac{1}{\lambda\alpha}\times\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}d(z). \qquad (8)$$

Thus, the (8) can be expressed as $T(l(u))=l(u)$. As stated by theorem 3.1, fixed point equations there exist a single solution if the operator T is a contraction. We will demonstrate this in the subsequent theorem.

Theorem 3.2 *On the metric space* $(S(I),\|\,\|_\infty)$ *with the norm* $\|l\|_\infty=\sup_{u\in I}|l(u)|$ *the operator* T *is a contraction.*

Proof

Start by proving T is a contraction $\forall u\in I$, and $l_1,l_2\in S(I)$. The inequality $\|T(l_1)-T(l_2)\|\le\vartheta\|l_1-l_2\|$, with $0\le\vartheta<1$. Following (8), then

$$\|T(l_1)-T(l_2)\|_\infty\le\sup_{u\hat{I}[0,h_1)}|l_1(0)-l_2(0)|\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}d(z)\times\frac{1}{\lambda\alpha}e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}$$

$$\le\sup_{u\in[0,h_1)}\left|\|l_1-l_2\|\frac{1}{\lambda\alpha}e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}(-\lambda\alpha)\right|$$

$$\le\|l_1-l_2\|_\infty\sup_{u\in[0,b)}\left|\frac{1}{\lambda\alpha}e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}(-\lambda\alpha)\right|$$

$$=\|l_1-l_2\|_\infty\left|1-e^{-\frac{h_1}{\lambda\alpha}}\right|\sup_{u\in[0,h_1)}\left|\frac{1}{\lambda\alpha}e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-Q:}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}\right|$$

$\le\vartheta\|l_1-l_2\|_\infty$, where

$$\vartheta=\left|1-e^{-\frac{h_1}{\lambda\alpha}}\right|\sup_{u\in[0,h_1)}\left|\frac{1}{\lambda\alpha}e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}\right|.$$

Since $0\le\vartheta<1$, the triangular inequality has can be deployed, as well as the fact that

$$|l_1(0)-l_2(0)|\le\sup_{u\in[0,h_1)}|l_1(u)-l_2(u)|=\|l_1-l_2\|_\infty.$$

Both Theorem 3.1 and Theorem 3.2 ensure that no other solutions exist. The following theorem utilizes the Fredholm integral of the second kind to prove the existence of an ARL solution for the SMAX(Q,r)$_L$ process.

Theorem 3.3 *The explicit solution of the integral equation as follow:*

$$l(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}e^{-\frac{h_1}{\lambda\alpha}}-1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}+e^{-\frac{h_1}{\alpha}}-1},\ u\ge 0.$$

Proof

From (6) $l(u)=1+\frac{C(u)}{\lambda\alpha}\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}d(z)$

where $\varepsilon_t\sim Exp(\alpha)$, and set b be constant as $b=\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}dz$. The function $l(u)$ can be written as

$$l(u)=1+\frac{C(u)}{\lambda\alpha}b \qquad (9)$$

Consider $b=\int_0^{h_1} l(z)e^{-\frac{z}{\lambda\alpha}}dz$

$$=\int_0^{h_1}\left(1+\frac{C(z)}{\lambda\alpha}b\right)e^{-\frac{z}{\lambda\alpha}}dz$$

$$=\int_0^{h_1} e^{-\frac{z}{\lambda\alpha}}dz+b\int_0^{h_1}\frac{C(z)}{\lambda\alpha}e^{-\frac{z}{\lambda\alpha}}dz$$

$$=-\lambda\alpha\left(e^{-\frac{h_1}{\lambda\alpha}}-1\right)+\int_0^{h_1}\frac{C(y)}{\lambda\alpha}be^{-\frac{y}{\lambda\alpha}}dy$$

$$=-\alpha\lambda\left(e^{-\frac{h_1}{\lambda\alpha}}-1\right)$$

$$+\frac{b}{\alpha\lambda}e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+..+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}\int_0^{h_1}e^{-\frac{z}{\lambda\alpha}+\left(\frac{(1-\lambda)z}{\lambda\alpha}\right)}dz$$

$$=\alpha\lambda\left(1-e^{-\frac{h_1}{\lambda\alpha}}\right)+\frac{b}{\lambda}\left[\left(1-e^{-\frac{h_1}{\alpha}}\right)\right]\times e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}$$

$$b=\frac{\lambda\alpha\left(1-e^{-\frac{h_1}{\lambda\alpha}}\right)}{1-e^{-\frac{\mu+\varepsilon_t-\theta_1\varepsilon_{t-L}-...-\theta_Q\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}\left(\frac{1-e^{-\frac{h_1}{\alpha}}}{\lambda}\right)} \tag{10}$$

Since (10) is substituted into (9), then (9) may be rewritten as:

$$l(u)=1+\frac{1}{\lambda\alpha}e^{\frac{(1-\lambda)u}{\lambda\alpha}}e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}\times\frac{\lambda\alpha\left(1-e^{-\frac{h_1}{\lambda\alpha}}\right)}{1-\left(\frac{1-e^{-\frac{h_1}{\alpha}}}{\lambda}\right)e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}},$$

$$=1+e^{\frac{(1-\lambda)u}{\lambda\alpha}}e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}\times\frac{\lambda\left(1-e^{-\frac{h_1}{\lambda\alpha}}\right)}{\lambda-\left(1-e^{-\frac{h_1}{\alpha}}\right)e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}},$$

$$=1-\frac{1}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}\times\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}}\lambda\left(e^{-\frac{h_1}{\lambda\alpha}}-1\right)}{\lambda+\frac{\left(e^{-\frac{h}{\alpha}}-1\right)}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}},$$

$$=1-\frac{1}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}\times\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}}\lambda\left(e^{-\frac{h_1}{\lambda\alpha}}-1\right)}{\frac{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}+\left(e^{-\frac{h_1}{\alpha}}-1\right)}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}}.$$

Then

$$l(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}e^{-\frac{h_1}{\lambda\alpha}}-1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}+e^{-\frac{h_1}{\alpha}}-1} \tag{11}$$

Assuming $\alpha=\alpha_0$ verified the process is in the controlled state. The explicit ARL_0 solution for $SMAX(Q,r)_L$ process of EWMA control chart can be rewritten as follow:

$$ARL_0=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_0}}e^{-\frac{h_1}{\lambda\alpha_0}}-1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha_0}}+e^{-\frac{h_1}{\alpha_0}}-1}. \tag{12}$$

Alternatively, let $\alpha=\alpha_1$ verified the process is uncontrolled state, $\alpha_1=\alpha_0(1+\delta)$ where δ is the shift size. Then, the explicit formula of ARL_1 for the $SMAX(Q,r)_L$ process of EWMA control chart can be rewritten as follow:

$$ARL_1=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_1}}e^{-\frac{h_1}{\lambda\alpha_1}}-1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha_1}}+e^{-\frac{h_1}{\alpha_1}}-1}. \tag{13}$$

IV. THE NUMERICAL INTEGRAL EQUATION (NIE) OF ARL FOR $SMAX(Q,r)_L$ PROCESS ON EWMA CONTROL CHART

The following section will present the numerical method for calculating the ARL value of the EWMA control chart for $SMAX(Q,r)_L$ process. The cumulative distribution function and probability distribution function of exponential distribution with mean α are states as follows:

$$F(y)=1-e^{-\alpha y}\text{ and }f(y)=\frac{dF(y)}{du}=\alpha e^{-\alpha y}.$$

Two approaches are mentioned in this paper; the first is the midpoint rule, and the other is the Gaussian rule. The fundamental method of approximating numerical integration, with the following differences in approximation:

Midpoint Rule

Assume an integral function $g(y)$ is evaluated on interval $[0,h_1)$, a finite interval. The weight function $W(y)$ with a value of 1 is chosen, and a set of evenly distributed elements is employed. The interval $[0,h_1)$ has been divided into m subintervals $\{[y_{j-1},y_j], j=1,2,...,m\}$ of equal length $H=\frac{h_1-0}{m}$ by using spaced point $y_j=y_0+jH$ for $j=1,2,...,m$ then $y_0=0$ and $y_j=h_1$. The mid-point are given by $m_j=\frac{1}{2}\left(y_{j-1}+y_j\right)=\left(j-\frac{1}{2}\right)h_1$. The selected value for weight ϖ_j for each midpoint was 1. The midpoint rule precisely combines a constant function into each subinterval. The composite midpoint rule for subintervals is obtained by combining the rules for m subintervals and is as follows:

$$M(g,H)=H\sum_{j=1}^{m}g\left(\left(j-\frac{1}{2}\right)H\right)$$

The integral's approximate value is provided by

$$\int_0^{h_1}g(s)ds\approx H\sum_{j=1}^{m}g\left(\left(j-\frac{1}{2}\right)H\right).$$

Gaussian Rule

The integral function $g(y)$ is calculated on interval $[0,h_1)$, infinite interval. Given $W(y)$ is a weight function, the set of points $\{p_j, j=1,...,m\}$ may not be evenly spaced, and $W(y)$ may not equal 1. The Gaussian rule evaluates an integral approximation as follows:

$$\int_0^{h_1}W(y)g(y)dy\approx\sum_{j=1}^{m}\varpi_j g(p_j).$$

Since p_j be a set of point, $0\le p_1\le p_2\le...\le p_m\le h_1$ and $\varpi_j=h_1/m\ge 0$; $j=1,2,...,m$ be a set of constant weights. The ARL notation using the NIE approach is represented by $l_{NIE}(u)$. The numerical approximation of the integral equation can essentially express as follows:

$$l(p_j)=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j l(p_j)f\left(\frac{p_j-(1-\lambda)p_j}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right), j=1,2,...,m. \quad (14)$$

The m linear equations $l(p_1),l(p_2),..,l(p_m)$, as

$$l(p_1)=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j l(p_j)f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right)$$

$$l(p_2)=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j l(p_j)f\left(\frac{p_j-(1-\lambda)p_2}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right)$$

.
.
.

$$l(p_m)=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j l(p_j)f\left(\frac{p_j-(1-\lambda)p_m}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right).$$

In matrix form, the following is how it can be demonstrated:

$$l_{m\times1}=\mathbf{1}_{m\times1}+\mathbf{R}_{m\times m}l_{m\times1},$$

where $l_{m\times1}=\begin{pmatrix}l(p_1)\\ l(p_2)\\ \vdots\\ l(p_m)\end{pmatrix}$, $\mathbf{1}_{m\times1}=\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$,

$$[R]_{ij}=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j f\left(\frac{p_j-(1-\lambda)p_i}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right), i,j=1,2,...,m,$$

and $\mathbf{I}_m=\mathrm{diag}(1,1,...,1)$. If $(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}$ there exist, then $l_{m\times1}=(\mathbf{1}_m+\mathbf{R}_{m\times m})^{-1}l_{m\times1}$,

After that, p_j is replaced by u in (14), then $l_{NIE}(u)$ can be rewritten as:

$$l_{NIE}(u)=1+\frac{1}{\lambda}\sum_{j=1}^{m}\varpi_j l(p_j)f\left(\frac{p_j-(1-\lambda)u}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right), \quad (15)$$

where $p_j=\frac{h_1}{m}\left(j-\frac{1}{2}\right)$ and $\varpi_j=\frac{h_1}{m}$; $j=1,2,...,m$.

V. Numerical Results

The performance of the control chart indicates by ARL. This section compares the ARL values obtained from explicit ARL solutions in-control (12) and out-of-control (13) with the numerical integral equation (NIE) method employing the midpoint rule (14) and Gaussian rule (15) on $m=500$ subintervals for the $\mathrm{SMAX(Q,r)_L}$ process with exponential white noise on the EWMA control chart. Let

$l(u)$ be ARL from the explicit solution, $l_M(u)$ be ARL from the NIE method using the midpoint rule, and $l_G(u)$ be ARL from the NIE method using the Gaussian rule. We also compare computational time between three methods. The computational time are evaluated by central processing unit (CPU) time (Operating system: Window 8 OEM, intel(R) core(TM) i8265-5U CPU@1.60GHz 1.80GHz Ram 8.00 GB (7.89 GB usable)) in seconds.

In Table I and Table II, the parameter value of h_1 for EWMA control chart was selected by setting λ = (0.05, 0.10, 0.15), ARL_0= 500 and $\alpha = \alpha_0$ =1. In Table I, in the case of SMAX(2,2)$_4$ with parameter $\theta_1 = 0.25$, $\theta_2 = (0.35, 0.45, 0.55, 0.65)$, $\beta_1 = 1.50$ and $\beta_2 = 0.70$, respectively. In Table II, in the case of SMAX(3,2)$_{12}$ with parameter $\theta_1 = 0.10, \theta_2 = 0.10$, θ_3 = (0.10, 0.20, 0.30, 0.40), $\beta_1 = 0.80$, and $\beta_2 = 0.70$, respectively .

TABLE I

ARLS VALUES FOR IN-CONTROL STATE FOR SMAX(2,2)$_4$ USING EXPLICIT SOLUTION AGAINST NIE METHOD GIVEN θ_1 =0.25, β_1 = 1.50, β_2 = 0.70 FOR ARL_0 = 500.

λ	θ_2	h_1	Explicit	NIE	
			$l(u)$	$l_M(u)$	$l_G(u)$
0.05	0.35	0.01012757	500.143 (0.012)	500.143 (1.793)	500.143 (7.986)
	0.45	0.01119888	500.121 (0.012)	500.121 (1.824)	500.121 (8.156)
	0.55	0.01238424	500.138 (0.012)	500.138 (1.855)	500.138 (8.247)
	0.65	0.01369595	500.129 (0.012)	500.129 (1.831)	500.129 (8.665)
0.10	0.35	0.02035858	500.072 (0.012)	500.012 (1.786)	500.012 (8.994)
	0.45	0.02252438	500.007 (0.012)	500.007 (1.858)	500.007 (8.247)
	0.55	0.02492352	500.069 (0.012)	500.069 (1.779)	500.069 (8.224)
	0.65	0.0275818	500.232 (0.012)	500.232 (1.824)	500.232 (8.103)
0.15	0.35	0.0306952	500.321 (0.012)	500.321 (1.799)	500.321 (8.387)
	0.45	0.03397942	500.100 (0.012)	500.100 (1.844)	500.100 (8.079)
	0.55	0.03762177	500.011 (0.012)	500.011 (1.794)	500.011 (7.995)
	0.65	0.04166285	500.092 (0.012)	500.092 (1.853)	500.092 (8.117)

The parentheses are CPU times in seconds.

TABLE II

ARLS VALUES FOR IN-CONTROL STATE FOR SMAX (3, 2)$_{12}$ USING EXPLICIT SOLUTION AGAINST NIE METHOD GIVEN θ_1 =0.10, θ_2 = 0.10, β_1 = 0.80, β_2 = 0.70 FOR ARL_0 = 500.

λ	θ_3	h_1	Explicit	NIE	
			$l(u)$	$l_M(u)$	$l_G(u)$
0.05	0.10	0.01514770	500.556 (0.012)	500.556 (1.76)	500.556 (8.828)
	0.20	0.01675461	500.136 (0.012)	500.136 (1.805)	500.136 (8.646)
	0.30	0.01853363	500.023 (0.012)	500.023 (1.787)	500.023 (8.734)
	0.40	0.02050356	500.228 (0.012)	500.228 (1.834)	500.228 (8.876)
0.10	0.10	0.03052800	500.258 (0.012)	500.258 (1.812)	500.258 (8.679)
	0.20	0.03379429	500.007 (0.012)	500.007 (1.782)	500.007 (8.667)
	0.30	0.03741678	500.244 (0.012)	500.244 (1.813)	500.244 (8.640)
	0.40	0.04143575	500.305 (0.012)	500.305 (1.784)	500.305 (8.713)
0.15	0.10	0.0461482	500.259 (0.012)	500.259 (1.828)	500.259 (8.807)
	0.20	0.0511290	500.409 (0.012)	500.409 (1.785)	500.409 (8.803)
	0.30	0.0566628	500.067 (0.012)	500.067 (1.818)	500.067 (8.797)
	0.40	0.0628148	500.083 (0.012)	500.083 (1.834)	500.083 (8.692)

The parentheses are CPU time in seconds.

Tables I and II illustrate that ARL_0 values from the explicit formulae and NIE method on m =500 subintervals are identical. As demonstrated in Tables III and IV, the ARL values derived from the explicit formulae and NIE method with m=500 subintervals for tracking shifts in the process mean are equivalent. The ARL_1 values are plotted in Fig.1 and Fig.2, respectively. However, the CPU timings for the explicit formulae are significantly lower than the NIE method. In addition, the midpoint rule-based NIE method requires less CPU time than the Gaussian rule. Fig. 3 and Fig.4 depict the CPU timings, respectively.

Table III and Table IV illustrate ARL values that indicate the performance for detecting the mean shift of processes between explicit formulae and numerical integration method using the midpoint rule and the Gaussian rule on m = 500 subintervals. In in-control state, the value of parameter α_0 = 1 and out-of-control state parameter values $\alpha_1 = \alpha_0(1+\delta)$ where shift size (δ) = 0.000, 0.001, 0.005 0.010, 0.050, 0.100, 0.300, and 0.500. In Table III, the initial ARL_0 = 500 for SMAX(2,2)$_4$ with parameter λ = 0.05, θ_1 = 0.25, θ_2 = 0.45, β_1 = 1.50, β_2 = 0.70, and h_1 = 0.01119888. In Table IV, the comparison ARL values using explicit formulae between EWMA and CUSUM control chart with the initial ARL_0 = 500 for SMAX(3,2)$_{12}$ with parameter λ =0.15, θ_1 =0.10, θ_2 =0.10, θ_2 =0.20, β_1 =0.80, β_2 =0.70, and h_1 =0.0511290.

TABLE III
ARLs VALUES FOR SMAX(2,2)$_4$ USING EXPLICIT SOLUTION AGAINST NIE METHOD GIVEN λ = 0.05, θ_1 = 0.25, θ_2 = 0.45, β_1 = 0.15, β_2 = 0.70 AND h_1 = 0.01119888 .

shift (δ)	Explicit	NIE	
	$l(u)$	$l_M(u)$	$l_G(u)$
0.000	500.305 (0.012)	500.305 (1.784)	500.305 (8.713)
0.001	210.296 (0.012)	210.296 (1.785)	210.296 (8.723)
0.005	63.9141 (0.012)	63.914 (1.894)	63.914 (8.718)
0.010	34.520 (0.012)	34.520 (1.835)	34.520 (8.734)
0.050	7.992 (0.012)	7.992 (1.857)	7.992 (8.720)
0.100	4.466 (0.012)	4.466 (1.848)	4.466 (8.766)
0.300	2.100 (0.01)	2.100 (1.862)	2.100 (8.750)
0.500	1.633 (0.012)	1.633 (1.796)	1.633 (8.796)

The parentheses are CPU times in seconds.

TABLE IV
ARLs VALUES FOR SMAX(3,2)$_{12}$ USING EXPLICIT SOLUTION AGAINST NIE METHOD GIVEN λ = 0.10, θ_1 = 0.10, θ_2 = 0.10, θ_3 = 0.40, β_1 = 0.80, β_2 = 0.70, AND h_1 = 0.04143575.

shift (δ)	Explicit	NIE	
	$l(u)$	$l_M(u)$	$l_G(u)$
0.000	500.121 (0.012)	500.121 (1.844)	500.121 (8.079)
0.001	236.019 (0.012)	210.296 (1.826)	210.296 (8.701)
0.005	76.385 (0.012)	76.385 (1.794)	76.385 (8.714)
0.010	41.764 (0.012)	41.764 (1.855)	41.7640 (8.798)
0.050	9.711 (0.012)	9.711 (1.887)	9.711 (8.725)
0.100	5.387 (0.012)	5.387 (1.844)	5.387 (8.757)
0.300	2.461 (0.01)	2.461 (1.875)	2.461 (8.732)
0.500	1.873 (0.012)	1.873 (1.821)	1.873 (8.796)

The parentheses are CPU times in seconds.

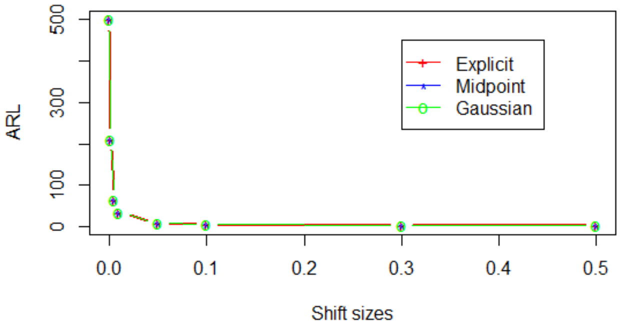

Fig.1. Comparison of ARL$_1$ values of EWMA control chart using explicit solution and NIE methods on the SMAX(2,2)$_4$.

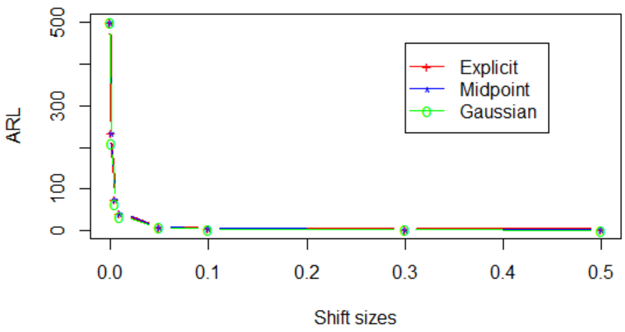

Fig.2 Comparison of ARL$_1$ values of EWMA control chart using explicit solution and NIE methods on the SMAX(3,2)$_{12}$.

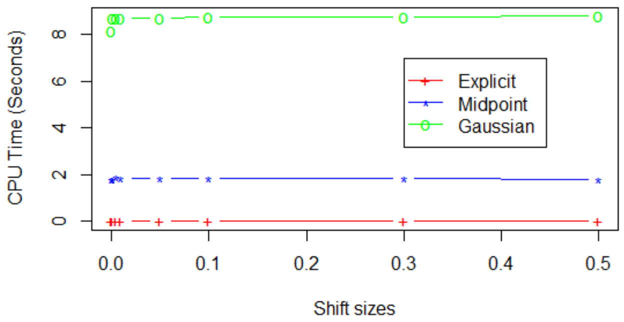

Fig. 3. The CPU times for computing ARLs values for SMAX(2,2)$_4$ on EWMA control chart

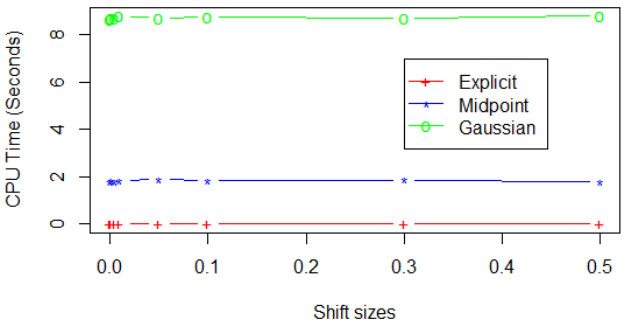

Fig. 4. The CPU times for computing ARLs values for on SMAX(3,2)$_{12}$ on EWMA control chart

Moreover, Table V illustrates ARL values that indicate the performance for detecting the mean shift of processes from explicit solution with various smoothing parameters (λ) of the EWMA control chart. In this case, the process is

SMAX(2,2)$_4$ with parameter θ_1 =0.25, θ_2 =0.45, β_1 =1.50, β_2 =0.70, λ =0.05, 0.10, 0.15, 0.25,0.35 and 0.45. The shift size (δ) = 0.000, 0.001, 0.005, 0.010, 0.050, 0.100, 0.300, and 0.500. The initial ARL_0 = 500.

TABLE V
COMPARISON ARL VALUES FOR USING EXPLICIT SOLUTION GIVEN $\theta_1 = 0.25$, $\theta_2 = 0.45$, $\beta_1 = 0.15$, $\beta_2 = 0.70$ ON SMAX(2,2)$_4$

shift (δ)	λ =0.05 h_1 = 0.01012757	λ =0.10 h_1 = 0.02035858	λ =0.15 h_1 = 0.0306952	λ =0.25 h_1 = 0.0516940
0.000	500.143	500.072	500.321	500.271
0.001	206.586	207.469	208.312	209.927
0.005	62.284	62.635	62.996	63.725
0.010	33.586	33.785	33.989	34.402
0.050	7.773	7.817	7.860	7.950
0.100	4.349	4.370	4.392	4.436
0.300	2.054	2.061	2.067	2.080
0.500	1.602	1.606	1.610	1.617

The numerical results from Table V found that values of λ affect control chart detection performance.

VI. A COMPARISON OF PERFORMANCE FOR EWMA AND CUSUM CONTROL CHARTS

The comparison of the performance control chart between the proposed explicit solution of EWMA and explicit solution of CUSUM control charts [13] is present in this section. The explicit solution of the CUSUM control chart for SMAX(Q,r)$_L$ process is as follows;

$$ARL_0 = \alpha_0 k\left(1+e^{\alpha_0(a-\mu+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}-\alpha_0 k\right)-e^{\alpha_0 u}, u \geq 0. \quad (16)$$

$$ARL_1 = \alpha_1 k\left(1+e^{\alpha_1(a-\mu+\theta_1\varepsilon_{t-L}+...+\theta_Q\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}-\alpha_1 k\right)-e^{\alpha_1 u}, u \geq 0. \quad (17)$$

Table VI illustrates the performance comparison between EWMA and CUSUM control charts investigated by setting ARL_0=500. The numerical results for ARL_0 and ARL_1 were calculated by using the explicit solution of the EWMA control chart from equations (12) to (13) and using the explicit solution of the CUSUM control chart from equations (16) to (17), respectively. Setting parameters of EWMA and CUSUM control charts are λ = 0.10, θ_1 = 0.10, θ_2 = 0.10, and β_1 = 0.10 for SMAX(2,1)$_4$ and λ = 0.10, θ_1 = 0.10, θ_2 = 0.10, θ_3 = 0.20, β_1 = 0.80, and β_2 = 0.70 for SMAX(3,2)$_{12}$. In in-control state, the value of parameter α_0 = 1 and out of control state parameter values α_1 = $\alpha_0(1+\delta)$ where shift size (δ) = 0.000, 0.001, 0.005 0.010, 0.050, 0.100, 0.300, and 0.500.

The numerical results from Table VI found that ARL values from the EWMA control chart are significantly less than ARL values from the CUSUM control chart. We can summarize that the EWMA control chart outperforms the CUSUM control chart for detecting all the magnitude shifts, as plotted in Fig.5 and Fig.6, respectively.

TABLE VI
COMPARISON ARL VALUES BASED ON EXPLICIT SOLUTION BETWEEN EWMA AND CUSUM CONTROL CHARTS

shift (δ)	SMAX(2,1)$_4$		SMAX(3,2)$_{12}$	
	EWMA h_1 = 0.11695971	CUSUM a =5, k=2.3477	EWMA h_1 = 0.03379429	CUSUM a =4, k=2.136
0.000	500.184	500.194	500.007	500.006
0.001	299.281	497.061	227.151	496.850
0.005	115.431	484.780	71.915	484.486
0.010	65.734	469.986	39.145	469.593
0.050	15.661	370.664	9.083	369.673
0.100	8.571	282.205	5.049	280.802
0.300	3.702	116.812	2.327	115.17
0.500	2.698	61.140	1.783	59.750

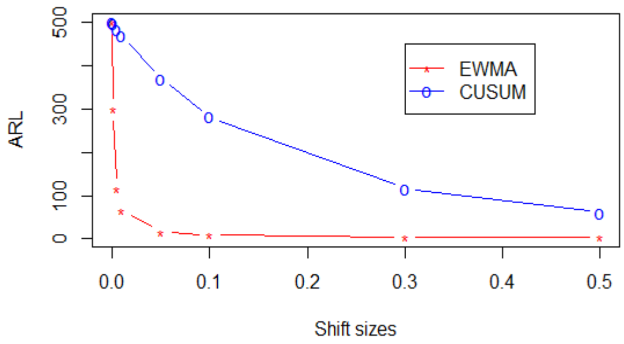

Fig.5. Comparison ARL values between EWMA and CUSUM control charts on the SMAX(2,1)$_4$.

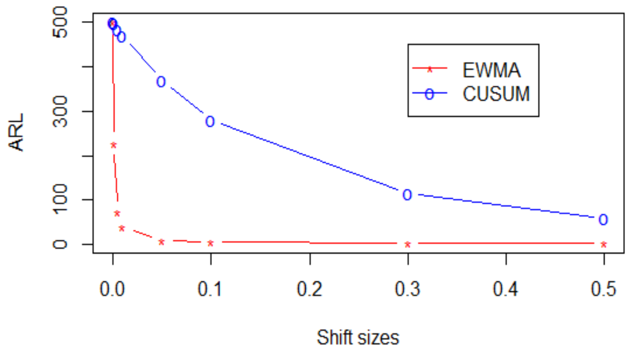

Fig.6. Comparison ARL values between EWMA and CUSUM control charts on the SMAX(3,2)$_{12}$.

VII. CONCLUSION

This paper demonstrates explicit solution and NIE principle based on midpoint and Gaussian rules for ARL on the EWMA control chart when dataset are seasonal moving average models with exogenous variables SMAX(Q,r)$_L$. Moreover, the uniqueness and existence of explicit ARL solution have been confirmed. The outcomes show that, values of ARL derived from the explicit solution presented are equivalent to the NIE principle based on midpoint and Gaussian rules. The time required to compute explicit solutions is less than one second. In contrast, the midpoint rule takes less than 2 seconds, and the Gaussian rule takes less than 9 seconds. Accordingly, the explicit solution can reduce computing times faster than the numerical integration equation method based on midpoint and Gaussian principles on EWMA control chart. Moreover, a comparison of the effectiveness of EWMA and CUSUM control charts in tracking the variation in the mean for the $SMAX(Q,r)_L$ process revealed that the EWMA control chart outperforms the CUSUM control chart in identifying the mean change for all magnitudes of shift.

REFERENCES

[1] W. A. Shewhart, "*Economic control of quality of manufactured product*," Van Nostrand, New York, 1931.
[2] E.P. Page "Continuous inspection schemes," *Biometrika*, vol. 41, pp100–115, 1954.
[3] S. W. Robert, "Control chart tests based on geometric moving average," *Technometrics*, vol.1 no.3, pp239-250, Aug. 1959.
[4] K. Petcharat, and S. Sukparungsee, "Development of a new MEWMA-Wilcoxon sign rank chart for detection of change in mean parameter," *Applied Science and Engineering Progress*, vol. 16, no.2, pp1-11, 2023.
[5] S. Sukparungsee, "Average run length of cumulative sum control chart by Markov chain approach for zero-inflated Poisson processes," *Thailand Statistician*, vol. 16, no.1, pp6-13, 2018.
[6] J. M. Lucas, and M.S. Saccucci, "Exponentially weighted moving average control schemes: properties and enhancements," *Technometrics*, vol. 32 no.1, pp1-12, 1990.
[7] C. Chananet, and S. Phanyaem, "Improving CUSUM control chart for monitoring a change in processes based on seasonal ARX model," *IAENG International Journal of Applied Mathematics*, vol. 52, no.3, pp589-596, 2022.
[8] K. Karoon, Y. Areepong, and S. Sukparungsee, "Exact run length evaluation on extended EWMA control chart for seasonal autoregressive process," *Engineering Letters*, vol. 30, no.4, pp1377-1390, 2022.
[9] R. A. Johnson, and M. Bagshaw, "The effect of serial correlation on the performance of CUSUM tests II," *Technometrics*, vol.17, no.1, pp73-80, 1974.
[10] C. W. Lu, and M.R. Reynolds, "EWMA control charts for monitoring the mean of autocorrelated processes," *Journal of Quality Technology*, vol. 31, no.2 pp166-188, 1999.
[11] K. Petcharat, "Explicit formula of ARL for $SMA(Q)_L$ with exponential white noise on EWMA chart," *International Journal of Applied Physics and Mathematics*, vol. 6, no.4, pp218-225, 2016.
[12] R. Sunthornwat, Y. Areepong, and S. Sukparungsee, "Average run length with a practical investigation of estimating parameters of the EWMA control chart on the long memory AFRIMA process. *Thailand Statistician*, vol. 16, no.2, pp190-202, 2018.
[13] R. Sunthornwat, and Y. Areepong, "Average run length on CUSUM control chart for seasonal and non-seasonal moving average processes with exogenous variables," *Symmetry*, vol. 12, no.1, pp1-15, 2020.
[14] K. Petcharat, "The effectiveness of CUSUM control chart for trend stationary seasonal autocorrelated data," *Thailand Statistician*, vol. 20, no.2, pp475-488, 2022.
[15] W. Peerajit, and Y. Areepong, "The performance of CUSUM control chart for monitoring process mean for autoregressive moving average with exogenous variable model," *Applied Science and Engineering Progress*, vol. 15, no.1, pp1-10, 2022.
[16] W. Suriyakat and K. Petcharat. "Exact run length computation on EWMA control chart for stationary moving average process with exogenous variables," *Mathematics and Statistics*, vol. 10 no.3, pp 624-635, 2022.

P. Kanita is a Lecturer of Applied Statistics Department, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Thailand.