Further Study for the Ordered Weighted Averaging Operator

Yu-Lan Wang, Te-Yuan Chiang

***Abstract*—In this article, an ordered weighted averaging operator with the degree of orness and the measure of entropy was examined by us. The purpose of this paper is twofold. We first prove that the maximum value of the degree of orness is attained by the weight with a uniform distribution. Next, we provide a reasonable explanation to clarify why the induced ordered weighted averaging operator can perform better than the regression model. Our further examination will help researchers understand an ordered weighted averaging method and an induced ordered weighted averaging approach in group decision-making.**

***Index Terms*—Ordered weighted averaging method, Group decision-making, Degree of orness, Induced ordered weighted averaging approach**

I. Introduction

THE ordered weighted averaging operator with the degree of orness and the measure of dispersion (or entropy) had been examined by many papers. We just list a few of them in the following: Bonissone and Decker [1], O'Hagan [2], Carbonell *et al.* [3], Filev and Yager [4], Fuller and Majlender [5, 6], Abbasbandy and Hajjari [7], Mendel *et al.* [8], Rodríguez *et al.* [9], Mendel *et al.* [10], Rodríguez *et al.* [11], Wan *et al.* [12], and Liao *et al.* [13], to show that it is an interesting issue among researchers. In this paper, we concentrate on Filev and Yager [4] to provide some discussions and then we present several improvements. In the literature, researchers explained why must apply the ordered weighted averaging operator, to replace the weighted averaging operator as follows. During a war, estimating the number of enemy planes is an important task to decide the defense strategy. For this kind of estimation, underestimating the force of the enemy planes would be more dangerous than overestimating it. Applying the ordered weighted averaging operator can give more weight to those priority targets and then those unimportant factors will not influence the estimated results significantly such that underestimating those important targets will not happen. We will first review the related published findings and then provide our comments and new theorems to improve those results. Several other related papers with ordered weighted averaging such as Torra [14], Tahayori and Sadeghian [15], Rodríguez and Martínez [16], Wang [17], Hu *et al.* [18], Song and Hu [19], Gao *et al.* [20], Zheng *et al.* [21], Cao *et al.* [22], Li *et al.* [23], and Qin *et al.* [24] that are important reference articles for this research trend.

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Yu-Lan Wang is a Professor of the College of Teacher Education, Weifang University of Science and Technology, Shandong, China (e-mail: yulan.duker@gmail.com).

Te- Yuan Chiang is an Associate Professor in the School of Intelligent Manufacturing, Weifang University of Science and Technology, Shandong, China (e-mail: kelly4921@gmail.com).

II. Review of Previous Results

We recall the definitions of an ordered weighted averaging method with the degree of orness and the measure of dispersion (or entropy) as follows.

The alpha value of *W*, expressed as $\alpha(W)$, is assumed in the following,

$$\alpha(W)=\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \ , \tag{2.1}$$

to calculate the level of maxness of the aggregation, as that is similar to a max measure.

The second assessment assumed by Filev and Yager [4]

$$H(W)=-\sum_{j=1}^{n}w_j \ln w_j \ , \tag{2.2}$$

is calculated as the measure of entropy (or dispersion) and was defined to assess the level to which W considers how much the given data is in the composition.

Filev and Yager [4] provided an example with the following three weighting vectors:

$$W_1=(0.2,0.2,0.2,0.2,0.2), \tag{2.3}$$

$$W_2=(0.5,0,0,0,0.5), \tag{2.4}$$

and

$$W_3=(0,0,1,0,0), \tag{2.5}$$

such that they derived the following results,

$$H(W_1)=\ln 5\,, \tag{2.6}$$

$$H(W_2)=\ln 2\,, \tag{2.7}$$

and

$$H(W_3)=0\,. \tag{2.8}$$

They saw that the higher entropy indicates the homogeneous distribution among weights and then they did not provide further explanation.

III. Our Improvement

We will prepare an analytical explanation to show that the maximum value of dispersion will happen at W_1.

The dispersion of $W=(w_1,...,w_n)$ is $-\sum_{i=1}^{n}w_i \ln w_i$

with $\sum_{i=1}^{n} w_i = 1$, with $w_i \geq 0$, for $i = 1,...,n$. We assume that

$$G(w_1,\ldots,w_{n-1}) = -\sum_{k=1}^{n-1} w_k \ln(w_k)$$
$$-(1-\textstyle\sum_{k=1}^{n-1} w_k)\ln\left((1-\sum_{k=1}^{n-1} w_k)\right), \quad (3.1)$$

and then

$$\frac{\partial G}{\partial w_i} = \ln\left(1-\sum_{k=1}^{n-1} w_k\right) - \ln w_i. \quad (3.2)$$

If we solve the system for the roots of the first partial derivative, from $\frac{\partial G}{\partial w_i} = 0$, it follows that

$$w_n = 1-\sum_{k=1}^{n-1} w_k = w_i, \quad (3.3)$$

for $i = 1,...,n-1$. Hence, $G(w_1,...,w_{n-1})$ has one critical point at $(1/n,...,1/n)$.

Next, we will show that $(1/n,...,1/n)$ is the global maximum point.

We obtain that

$$\frac{\partial^2 G}{\partial w_j \partial w_i} = \frac{-1}{1-\sum_{k=1}^{n-1} w_k} = \frac{-1}{w_n}, \quad (3.4)$$

and

$$\frac{\partial^2 G}{\partial w_i^2} = \frac{-1}{w_n} + \frac{-1}{w_i}. \quad (3.5)$$

Therefore, we need to prove that a matrix, say

$$M = (a_{ij})_{n\times n}, \quad (3.6)$$

for $i \neq j$,

$$a_{ij} = \frac{-1}{w_n}, \quad (3.7)$$

and

$$a_{ii} = \frac{-1}{w_n} + \frac{-1}{w_i}. \quad (3.8)$$

To simplify the computation, we consider another matrix, say N, with

$$N = -w_n M. \quad (3.9)$$

such that if $N = (b_{ij})_{n\times n}$ then $b_{ij} = 1$ for $i \neq j$, and $b_{ii} = 1 + c_i$ with $c_i = w_n / w_i$. Our goal is to verify that N is a positive definite matrix.

We compute the determinant of *N* as follows.

$$\det N = \det\begin{bmatrix} 1+c_1 & 1 & 1 & 1 \\ 1 & \ldots & 1 & \vdots \\ \vdots & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1+c_n \end{bmatrix}$$

$$= \prod_{j=1}^{n-1} c_j + c_n \det\begin{bmatrix} 1+c_1 & 1 & 1 & 1 \\ 1 & \ldots & 1 & \vdots \\ \vdots & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1+c_{n-1} \end{bmatrix}. \quad (3.10)$$

If we use row operation $-R_1 + R_j$ for $j = 2,...,n$, then use the last column to compute the determinant then

$$\det N = c_n \det\begin{bmatrix} 1+c_1 & 1 & 1 & 1 \\ -c_n & c_2 & 0 & \vdots \\ \vdots & 1 & \ldots & 0 \\ -c_n & 1 & 0 & c_{n-1} \end{bmatrix}$$
$$+(-1)^{n+1}\det\begin{bmatrix} -c_1 & c_2 & 0 & \cdots \\ -c_1 & \cdots & 0 & 0 \\ \vdots & 0 & \ldots & c_{n-1} \\ -c_1 & 0 & \ldots & 0 \end{bmatrix}. \quad (3.11)$$

We use the row operation $-R_1 + R_j$, for $j = 2,...,n-1$, for the first matrix in Equation (3.11) and use the second-row expansion repeatedly for $n-2$ times for the second matrix, then it yields

$$\det N = c_n \det\begin{bmatrix} 1+c_1 & 1 & 1 & 1 \\ 1 & 1+c_2 & 1 & \vdots \\ \vdots & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1+c_{n-1} \end{bmatrix}$$
$$+(-1)^{n+1}(-1)^{3(n-2)}\prod_{j=2}^{n-1} c_j \det[-c_1]. \quad (3.12)$$

After we further simplify Equation (3.12), then Equation (3.11) is verified.

The rest proof is dependent on the mathematical induction to show that $\det N > 0$.

IV. FURTHER DISCUSSION FOR THE ORDERED WEIGHTED AVERAGING OPERATOR

We recall how to decide the relative weights for the ordered weighted averaging operator. The approach to deciding weights is a crucial factor that was developed in the literature.

A traditional approach to deciding the weights is related to an ordered weighted averaging method with a verbal language expressed as a fuzzy subset *Q*. Following the above-mentioned procedure, the weighting vector with n components is assumed as follows

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right), \quad (4.1)$$

for $j = 1,2,...,n$ with $Q(0) = 0$ and $Q(1) = 1$.

Another approach, suggested by O'Hagan [2], is to derive a weighting vector by the alpha value and the dispersion of *W*. For a given alpha value α to solve

$$Max - \sum_{j=1}^{n} w_j \ln w_j, \quad (4.2)$$

subject to

$$\text{(i)}\ \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad (4.3)$$

$$\text{(ii)}\ \sum_{i=1}^{n} w_i = 1, \quad (4.4)$$

and

$$\text{(iii)}\ 0 \leq w_j \leq 1, \quad (4.5)$$

for $j = 1,...,n$.

A process to derive the weights was proposed by Filev and Yager [4] related to the ordered weighted averaging aggregation from observational data with a collection of m samples each comprised of an n-tuple of values $(a_{k1},...,a_{kn})$, for $k = 1,...,m$, called the arguments, and an assorted single value called the aggregated value, which is denoted as d_k. The reordered objects, b_{kj}, is the jth largest element of the argument collection $\{a_{k1},...,a_{kn}\}$. Our purpose is to solve the weights for the ordered weighted averaging operator $W = (w_1,...,w_n)$ such that to minimize the instantaneous error e_k, with

$$e_k = \frac{1}{2}(b_{k1}w_1 + ... + b_{kn}w_n - d_k)^2, \quad (4.6)$$

under the constraints $\sum_{i=1}^{n} w_i = 1$ and $0 \le w_j \le 1$, for $j = 1,...,n$.

In a neural society with backpropagation, the gradient descent algorithm was adopted with the initial values of the ordered weighted averaging operator $W = \left(\frac{1}{n},...,\frac{1}{n}\right)$, and the estimate of the aggregated value d_k, with

$$\widetilde{d}_k = b_{k1}w_1 + ... + b_{kn}w_n \ . \quad (4.7)$$

To overcome the constraints on w_i, we present each the weight of the ordered weighted averaging operator as follows

$$w_i = \frac{e^{\lambda_i}}{\sum_{k=1}^{n} e^{\lambda_k}} \quad (4.8)$$

and then updating the parameters λ_i as follows

$$\lambda_i(l+1) = \lambda_i(l) - \beta\, w_i (b_{ki} - \widetilde{d}_k)(\widetilde{d}_k - d_k), \quad (4.9)$$

that β indicates the learning coefficient $(0 \le \beta \le 1)$ and after the *l*th iteration, $\lambda_i(l)$ denotes the approximation of λ_i.

In the experiment, the estimated values of λ_i after 350 iterations were recorded.

An induced ordered weighted averaging method is expressed as

$$F_W(\langle u_1, a_1\rangle,...,\langle u_n, a_n\rangle) = \sum_{i=1}^{n} w_i b_i, \quad (4.10)$$

where a two-tuple $\langle u_i, a_i\rangle$ is presented where b_j is the *a* value of the pair having the *j*th largest u value, and then u_i is the ordering inducing variable (locator; descriptor) and a_i is the argument variable (prescribed value).

Assuming g that $\langle u_i, a_i\rangle$ and $\langle u_i, a_i'\rangle$ satisfying $a_i \ge a_i'$ for $i = 1,...,n$, then the monotonicity property is assumed

$$F_W(\langle u_i, a_i\rangle) \ge F_W(\langle u_i, a_i'\rangle). \quad (4.11)$$

We recall that Filev and Yager [4] assumed that if the monotonicity is not held then the inducing variables are preserved in their ordering. On the other hand, Filev and Yager [4] pointed out that when a tie occurs then (i) the induced ordered weighted averaging approach, and (ii) the ordered weighted averaging method must be paid attention to their distinction.

Sometimes, there is no number but linear order. In such conditions, researchers applied an implicit lexicographic ordering to express as u_i which is similar to the ordering of words in dictionaries. We list several examples of the ordered weighted averaging operators:

(i) The nearest neighbor rule: $W = (1,0,...,0)$;

(ii) The classic exponential smoothing: $w_n = (1-\alpha)^{n-1}$, $w_j = \alpha(1-\alpha)^{j-1}$ for $j = 1,...,n-1$.

(iii) the moving average of the last *m* readings: $w_j = \frac{1}{m}$ for $j = 1,...,m$ and $w_j = 0$ for all others j.

V. Modeling using Induced Ordered Weighted Averaging Operators

The development of a model needs some background about the field in which we are trying to build a system. Background about a field can obtain through several different approaches. One of them is the original raw observational data, other kinds of background contain expertise and experience. The process of developing a system is partitioned into two stages, (i) formation identification, and (ii) variable estimation. During data collection, the values of u_j are not obtained but they play an important factor to decide the weights related to the value of a_j.

We recall the Best Yesterday Model of Filev and Yager [4]. First, we quote the prediction data for four experts, A, B, C, and D for five days and the actual opening price in Table 1.

Table 1. Expert data

	Expert predictions				Actual
Day#	A	B	C	D	OP
1	101	99	82	116	100
2	97	76	90	121	92
3	96	75	91	107	100
4	104	95	90	118	99
5	105	89	91	112	105

Reproduction from Filev and Yager [4]. OP: Opening Price.

Filev and Yager [4] first used a linear regression model to predict that

$$\begin{aligned} P_k &= w_A X_A(k) + w_B X_B(k) \\ &+ w_C X_C(k) + w_D X_D(k) \end{aligned} \quad (5.1)$$

where P_k is the predicted opening price on day *k*, w_A is the weight assigned to expert *A*, and $X_A(k)$ is the estimate provided by expert *A* for day *k*. They found that

$$w_A = 0.84,\ w_B = 0.16,$$
$$w_C = 0.55 \text{ and } w_D = -0.42 \quad (5.2)$$

and then the forecasts for the opening price on day *k*

$$P_1 = 107.25,\ P_2 = 122.45,\ P_3 = 88.37,$$
$$P_4 = 116.18 \text{ and } P_5 = 94.75 \quad (5.3)$$

such that the residue-mean sum error of our estimation is 5.11 apart from the opening price.

Secondly, they used the induced ordered weighted averaging operator model to solve the problem where the ordered weighted averaging operator pair is assumed

$$u_A(k) = -\left|TP(k-1) - X_A(k-1)\right| \quad (5.4)$$

and

$$a_A(k) = X_A(k), \quad (5.5)$$

similar to other experts, *B*, *C*, and *D*. Filev and Yager [4] tried to use the induced ordered weighted averaging operator pair $(\langle u,a \rangle)$ with the learning model to calculate the prediction for day *k* as

$$\hat{P}(k) = F_W\left(\langle u_A(k), a_A(k)\rangle, \ldots, \langle u_D(k), a_D(k)\rangle\right) \quad (5.6)$$

under this structure to get the predicted value

$$\hat{P}(k) = \sum_{j=1}^{4} w_j y_j(k) \quad (5.7)$$

where $y_j(k)$ is today's prediction of yesterday's *j*th best expert. Therefore, the findings of w_j are not related to an expert as the linear model but related to the location of yesterday's data. Filev and Yager [4] found that

$$w_1 = 0.20,\ w_2 = 0.12,$$
$$w_3 = 0.08 \text{ and } w_4 = 0.60 \quad (5.8)$$

and then estimations of the opening price on day *k* are expressed as

$$P_1 = 100.07,\ P_2 = 92.14,\ P_3 = 100,$$
$$P_4 = 99 \text{ and } P_5 = 105 \quad (5.9)$$

such that the residue-mean sum error of our estimation is 0.19 apart from the opening price. They concluded that in this case, the induced ordered weighted averaging operator model performs well.

Filev and Yager [4] mentioned that implicit in their solution method is related to the iterative hypothesis, which is a hypothesis saying "any model found to provide a good approximation to the observations for a large training set will also provide a good approximation over other unobserved examples." Filev and Yager [4] also claimed that this hypothesis may not hold. It is trivial that we can check human history.

VI. Our Explanation

Here, we try to provide an insightful explanation of why the induced ordered weighted averaging operator model may provide a better explanation.

Motivated by the construction of $u_A(k)$ and $a_A(k)$ of the equation, we derive the following Table 2 for the ordered weighted averaging operator pairs with the expression, $\langle j, c_j \rangle$ to denote the *j*th largest $u(k)$ with the corresponding distance of expert prediction with the actual opening price.

Table 2. The OWA pairs for experts

	Expert			
Day#	A	B	C	D
1	⟨1.5,1⟩	⟨1.5,1⟩	⟨4,18⟩	⟨3,16⟩
2	⟨2,5⟩	⟨3,16⟩	⟨1,2⟩	⟨4,31⟩
3	⟨1,4⟩	⟨4,25⟩	⟨3,9⟩	⟨2,7⟩
4	⟨2,5⟩	⟨1,4⟩	⟨3,9⟩	⟨4,19⟩
5	⟨1,0⟩	⟨4,16⟩	⟨3,14⟩	⟨2,7⟩

From Table 2, we compute the sum of ordinal numbers for the *j*th best prediction from day *k* to day k+1.

Table 3. The ordinal sum for the *j*th best prediction

	From day k to day k+1				
	k=1	k=2	k=3	k=4	SOV
j=1	2.5	3	2	4	11.5
j=2	2.5	1	4	1	8.5
j=3	4	4	3	3	14
j=4	1	2	1	2	6

SOV: sum of ordinal values

Table 3 reveals that the special property of the information data that the worst prediction for day *k* will imply the best or the second best prediction for day k+1. If we compare with Equation (5.8), then there is a trend that the small sum of ordinal value will imply the large weight, but there is one exception of $j = 1$ and $j = 2$. Hence, we further compute the sum for the cardinal value from day *k* to day k+1, for j=1,...,4, in next Table 4.

Table 4. The cardinal sum for the *j*th best prediction

	From day k to day k+1				
	k=1	k=2	k=3	k=4	SCV
j=1	5	9	5	16	35
j=2	16	4	19	0	39
j=3	31	25	9	14	79
j=4	2	7	4	7	20

SCV: sum of cardinal values

Table 4 illustrates that the sum of cardinal values is consistent with the results of Filev and Yager [4] with Equation (5.8). Our work provides a further explanation to discover the distinct characteristic of information data. Studying the weights from the data is a reasonable approach.

Induced ordered weighted averaging operators evaluated information data in pairs, an ordering for the second components was decided by the first component, and then they are synthesized. There are many possible application areas for induced ordered weighted averaging operators. For example, social welfare, distributed detection, sensor fusion, and decision-making.

VII. A Related Open Problem

We study the paper of Mandal *et al.* [25] published in Fuzzy Sets and Systems. Mandal *et al.* [25] developed inventory models under a fuzzy environment, and then they applied fuzzy theory and a geometric programming approach to derive the optimal solution under fuzzy constraints. There are about one hundred papers that have cited Mandal *et al.* [25] in their references. We just list a few of them in the following; Yaghin *et al.* [26], Bean *et al.* [27], Srivastav and Agrawal [28], Shekarian *et al.* [29], Jafarian *et al.* [30], Lechuga and Sanchez [31], Nobil *et al.* [32], Pramanik and Maiti [33], Moghdani *et al.* [34], and Taleizadeh *et al.* [35].

We have run a literature review to find out that Mandal et al. [25] and those referred papers did not examine the optimal solution when their fuzzy model reducing to a crisp model. To fulfill this research gap, we will study the fuzzy inventory model proposed by Mandal *et al.* [25] under a crisp environment.

VIII. Notation and assumptions

To be compatible with Mandal *et al.* [25], we adopt the following notation and assumptions.

S: shortage level (decision variable).

Q: lot size (decision variable).

D: demand per unit item (decision variable).

c_3: set up cost.

c_2: shortage cost per unit item.

c_1: holding cost per unit item.

β: degree of economies of scale, and $\beta > 1$.

α: scaling constant and $\alpha > 0$.

A deterministic inventory model with an economic ordering quantity and shortages allowed, where the demand is an additional decision variable was developed by Mandal *et al.* [25].

IX. Our improvement

Based on our previous discussion, we will try to solve the minimum solution for the following inventory model,

$$TC(S,Q,D)=\frac{S^2c_2}{2Q},$$

$$+c_1\frac{(Q-S)^2}{2Q}+c_3\frac{D}{Q}+\alpha D^{1-\beta}, \tag{9.1}$$

under the restriction $\beta>1$.

We compute the first partial derivatives with respect to D, Q, and S, respectively, then it follows that

$$\frac{\partial}{\partial D}TC(S,Q,D)=\alpha(1-\beta)D^{-\beta}+\frac{c_3}{Q} \tag{9.2}$$

$$\frac{\partial}{\partial Q}TC(S,Q,D)=-c_3\frac{D}{Q^2}$$

$$+\frac{c_1}{2}\left(1-\frac{S^2}{Q^2}\right)-\frac{c_2S^2}{2Q}, \tag{9.3}$$

and

$$\frac{\partial}{\partial S}TC(S,Q,D)=c_1\left(\frac{S}{Q}-1\right)+c_2\frac{S}{Q}. \tag{9.4}$$

Solving the system of first partial derivatives equals to zero, by $\frac{\partial}{\partial S}TC(D,Q,S)=0$, based on Equation (9.4), it implies that

$$c_1Q=(c_1+c_2)S. \tag{9.5}$$

From $\frac{\partial}{\partial D}TC(D,Q,S)=0$, based on Equation (9.2), it yields that

$$D^{-\beta}=\frac{c_3}{\alpha(\beta-1)Q}. \tag{9.6}$$

Using $\frac{\partial}{\partial Q}TC(D,Q,S)=0$, based on Equation (9.3), it follows that

$$c_1Q^2=2c_3D+(c_1+c_2)S^2. \tag{9.7}$$

If we plug Equations (9.5) and (9.6) into Equation (9.7) to rewrite as a function in Q only, then we obtain that

$$c_1Q^2=2c_3\left(\frac{c_3}{\alpha(\beta-1)Q}\right)^{\frac{-1}{\beta}}$$

$$+(c_1+c_2)\left(\frac{Qc_1}{c_1+c_2}\right)^2. \tag{9.8}$$

We multiply (c_1+c_2) on both sides of Equation (9.8) to derive that

$$(c_1+c_2)\,c_1Q^2=(Qc_1)^2$$

$$+2c_3\,(c_1+c_2)\left(\frac{c_3}{\alpha(\beta-1)Q}\right)^{\frac{-1}{\beta}}. \tag{9.9}$$

We cancel out $(Qc_1)^2$ from the right-hand side and the left-hand side of Equation (9.9) to find that

$$c_2\,c_1Q^2=2c_3\,(c_1+c_2)\left(\frac{c_3}{\alpha(\beta-1)Q}\right)^{\frac{-1}{\beta}}. \tag{9.10}$$

We take $(-\beta)$ power on both sides of Equation (9.10), and then we arrange the variable Q on the left-hand side, and the rest constant terns are on the right-hand side. Therefore, we show that

$$Q^{1-2\beta}=\left[\left(\frac{2c_3(c_1+c_2)}{c_1c_2}\right)^{\beta}\frac{\alpha(\beta-1)}{c_3}\right]^{-1}. \tag{9.11}$$

Owing to the restriction of $\beta>1$, we know that

$$2\beta-1>0. \tag{9.12}$$

Motivated by Equation (9.12), finally, we solve the optimal solution,

$$Q=\left[\left(\frac{2c_3(c_1+c_2)}{c_1c_2}\right)^{\beta}\frac{\alpha(\beta-1)}{c_3}\right]^{\frac{1}{2\beta-1}}. \tag{9.13}$$

Consequently, We plug our findings of Equation (9.13) into Equation (9.6), and then we derive

$$D=\left[\frac{c_1c_2c_3}{2\alpha^2(\beta-1)^2(c_1+c_2)}\right]^{\frac{1}{2\beta-1}}, \qquad (9.14)$$

and by the same argument, we plug our results of Equation (9.13) into Equation (9.5) to show that

$$S=\left[\left(\frac{2c_1c_3}{c_2(c_1+c_2)}\right)^{\beta}\left(\frac{c_2}{2\alpha(\beta-1)}\right)\right]^{\frac{1}{2\beta-1}}. \qquad (9.15)$$

Hence, we obtain the optimal solutions that are expressed in Equations (9.13-9.15).

X. Managerial meaning of our revisions

We derive the optimal solution under the crisp environment. Owing to those fuzzy parameters containing the crisp parameters such that a reasonable fuzzy solution derived by Mandal *et al.* [25] should contain the crisp optimal solution developed by our paper. Therefore, our findings provide a check condition for those fuzzy solutions obtained by various fuzzy and de-fuzzy procedures.

Several important papers are recently published. We cite them for readers for further study: Shoukralla *et al.* [36], Liao and Tang [37], Zhong *et al.* [38], Hussain *et al.* [39], Jiang *et al.* [40], and Timpitak and Pochai [41].

XI. Our improvement for a derivative

In this section, we point out a small computation error that were mentioned in four related paper: Moon and Gallego [42], Paknejad et al. [43], Wu and Ouyang [43], and Tung et al. [44]. They claimed that the equation (7) related to Moon and Gallego [42], Paknejad et al. [43], Wu and Ouyang [43], and Tung et al. [44], is expressed as follows,

$$\frac{\partial}{\partial k}EAC(Q,k,L_i)=h\sigma L_i^H\frac{k}{\sqrt{k^2+1}}>0. \qquad (11.1)$$

However, we examined of Equation (11.1) to find it should be revised such that we evaluated

$$\frac{\partial}{\partial k}EAC(Q,k,L_i)=h\sigma L_i^H\left[1+\frac{1-M_\beta}{2}\left(\frac{k}{\sqrt{k^2+1}}-1\right)\right]$$

$$=h\sigma L_i^H\left[\frac{1+M_\beta}{2}+\frac{1-M_\beta}{2}\frac{k}{\sqrt{k^2+1}}\right]$$

$$>h\sigma L_i^H\left[\frac{1+M_\beta}{2}\frac{k}{\sqrt{k^2+1}}+\frac{1-M_\beta}{2}\frac{k}{\sqrt{k^2+1}}\right]$$

$$=h\sigma L_i^H\frac{k}{\sqrt{k^2+1}}>0. \qquad (11.2)$$

Based on the above discussion, we provided an improvement for Moon and Gallego [42], Paknejad et al. [43], Wu and Ouyang [43], and Tung et al. [44]. Their conclusion of the partial derivative with respect to k is positive which is right. However, their derivation contained questionable results which is improved by Equation (11.2).

XII. Revision for Bustince

We study a pending problem of Bustince et al. [46]. We recall the following assertion on Page 506, Line 2, of Bustince et al. [46], they mentioned that the expected value of $\mu_{Q_t}(q)$ for $q=0,\ldots,t$ that satisfies

$$\mu_{Q_t}(q)=\frac{m_b(t)}{L-1}, \qquad (12.1)$$

under their assumption of

$$REF(x,y)=1-|x-y|, \qquad (12.2)$$

and

$$\varphi(x)=x. \qquad (12.3)$$

We evaluate that

$$E(\mu_{Q_t}(q))=1-\frac{\sum_{q=0}^{t}h(q)\left|\frac{q-m_b(t)}{L-1}\right|}{\sum_{q=0}^{t}h(q)}. \qquad (12.4)$$

Depending on the sign of $q-m_b(t)$, we rewrite Equation (12.4) as follows,

$$E(\mu_{Q_t}(q))=1-\frac{\sum_{q=0}^{t_0}h(q)(q-m_b(t))}{(L-1)\sum_{q=0}^{t}h(q)},$$

$$-\frac{\sum_{q=t_0}^{t}h(q)(m_b(t)-q)}{(L-1)\sum_{q=0}^{t}h(q)}. \qquad (12.5)$$

We recall that

$$m_b(t)=\frac{\sum_{q=0}^{t_0}qh(q)+\sum_{q=t_0}^{t}qh(q)}{(L-1)\sum_{q=0}^{t}h(q)}. \qquad (12.6)$$

Based on Equation (12.6), we rewrite Equation (12.5) as follows

$$m_b(t)=1-\frac{\sum_{q=0}^{t_0}qh(q)+\sum_{q=t_0}^{t}qh(q)}{(L-1)\sum_{q=0}^{t}h(q)}$$

$$-\frac{2\sum_{q=t_0}^{t}qh(q)-m_b(t)\left[\sum_{q=t_0}^{t}h(q)-\sum_{q=0}^{t_0}h(q)\right]}{(L-1)\sum_{q=0}^{t}h(q)}. \qquad (12.7)$$

Consequently, we obtain that

$$m_b(t)=1-\frac{m_b(t)}{L-1}$$

$$-\frac{2\sum_{q=t_0}^{t} qh(q)-m_b(t)\left[\sum_{q=t_0}^{t} h(q)-\sum_{q=0}^{t_0} h(q)\right]}{(L-1)\sum_{q=0}^{t} h(q)}. \quad (12.8)$$

Based on our above derivations, it points out that simple and elegant results proposed by Bustince et al. [46] that contained severe questionable derivations.

XIII. Open question of Cárdenas-Barrón

We study Cárdenas-Barrón [47] to point out several questionable findings to help future researchers. First, we call that Cárdenas-Barrón [47] obtained that

$$B^* = \frac{hQ^* - \pi d}{h+\hat{\pi}}. \quad (13.1)$$

However, Cárdenas-Barrón [47] forgot to check the condition of $B^* \geq 0$, that is, there is a restriction of Q^* that satisfies that

$$Q^* \geq \pi d/h, \quad (13.2)$$

which is a lower bound for the ordering quantity to have an interior optimal solution.

On the other hand, Cárdenas-Barrón [47] derived that

$$Q^* = \sqrt{\frac{2Ad(h+\hat{\pi})-(\pi d)^2}{h\hat{\pi}}}. \quad (13.3)$$

However, Cárdenas-Barrón [47] forgot to check the condition of $Q^* > 0$, that is, there is a restriction among parameters that satisfies that

$$2Ad(h+\hat{\pi}) > (\pi d)^2, \quad (13.4)$$

to guarantee an interior optimal solution.

There are two inventory models in Cárdenas-Barrón [47], EPQ model and EOQ model.

We can claim that his solution procedure of EPQ model is repeated his method for EOQ model. We can predict that after changing of expressions, the solution of EOQ model can be directly applied to EPQ model.

Consequently, the repeated solution procedure proposed by Cárdenas-Barrón [47] for his EPQ model becomes unnecessary.

Last, but not least, Cárdenas-Barrón [47] completely neglected the cases when the optimal solution occurred on the two boundaries. Hence, the easy computation procedure proposed by Cárdenas-Barrón [47] for inventory models solving by algebraic methods is incomplete.

XIV. Conclusion

Our results provide insight observation for the induced ordered weighted averaging operator to show that we can find out the distinct characteristic of information data. On the other hand, we verify that the weighted vector with the uniform distribution will attain the maximum value of the orness. Our findings will help practitioners to realize interrelationships among various operators and different methods.

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Yu-Lan Wang received her Ph.D. degree from Tianjin Nankai University in 2014 and is currently a professor at the School of Teacher Education of Shandong Weifang University of Science and Technology. The main research directions are preschool education, creative flipped education, educational management, and educational psychology.

Te-Yuan Chiang is an Associate Professor, at the School of Intelligent Manufacturing, Weifang University of Science and Technology. Received his Ph.D. degree from the Department of Mechanical Engineering, Yuan Ze University, in 2015. His research interest includes Management Science, Mechanical and Materials Science, Management Information Systems, Artificial Intelligence, Pattern Recognition, and Image Processing.