Qualitative Analysis of k-order Rational Fuzzy Difference Equation

Qianhong ZHANG, Bairong PAN

Abstract—**In this paper, we study the positive solution to a k-order rational fuzzy difference equation with the form**

$$a_{n+1} = 1 + \frac{a_n}{T + a_{n-k}}, \quad n \in N,$$

here, $\{a_n\}$ is a sequence of positive fuzzy numbers, $T \in \Re_f^+$(positive fuzzy numbers) and the initial values $a_{-t} \in \Re_f^+(t = 0, 1, \cdots, k), k \in N$. The focus of this paper is to analyze the dynamic behavior of this fuzzy difference equation. Furthermore, numerical example emphasize the validity of theoretical results.

Index Terms—**Rational fuzzy difference equation, existence, boundedness, convergence, asymptotic stability**

I. Introduction

As we all know, through the continuous exploration of scholars, the theory of difference equations derived from equations has been gradually developed in the past. Difference equations or discrete-time dynamical systems are widely used to set up mathematical model in numerous fields, such as ecology, population dynamics, computer science, electronic networks, economics, demography, etc. Because of the extensive application of difference equations, the study of dynamical behavior of difference equations with time delay has grown in importance in practical mathematics ([1], [2], [3], [4], [5]).

Actually, another significant mathematical model named fuzzy difference equation (FDE) represents natural phenomenon and objective laws with fuzzy uncertainty in the real world, it has attracted much attention of numerous scholars, as well as its theory has evolved rapidly since 1990s(see,[6], [7], [8], [9], [10], [7], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], and the references therein). Give a few examples, Wang et al. [19], By combining the advantages of fuzzy neural network and genetic algorithm, put forward a fuzzy neural network temperature prediction method based on genetic algorithm. Hussain et al. [21] in order to improve the absolute error accuracy of experimental understanding, the second-order fuzzy ordinary differential equation is solved directly by using the derivative self-starting block method of two orders. Allahviranloo et al. [22] proposed a concrete application of FDE in foretelling a specific cardiovascular disturbance. In [7], Cecconello et al. evaluates a kind of linear difference equations, and then analyzes the existence and stability of solutions of this fuzzy difference equations under different precision. Li et al. [23] propose a fuzzy maximum scatter difference(MSD) method to improve the recognition performance of the traditional this model. Mondal et al. [24] conducted a qualitative analysis of an epidemic model that incorporates fuzzy transmission, among other factors.

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In 2002, based on Zadeh extension principle of fuzzy numbers, Papaschinopoulos et al. [10] researched a class of nonlinear FDE

$$x_{n+1} = A + \frac{B}{x_n}, \quad \mathrm{n} \in N,$$

in which the initial value x_0 and parameters A and B are positive fuzzy numbers.

In 2009, Zhang and Liu [15] discussed the dynamical properties of the first-order linear FDE

$$x_{n+1} = Ax_n + B, \quad \mathrm{n} \in N,$$

here, the initial value x_0 and parameters A and B are positive fuzzy numbers.

In 2014, Zhang et al. [16] consider that positive fuzzy solutions exist for the first-order Riccatti fuzzy difference equation

$$x_{n+1} = \frac{A + x_n}{B + x_n}, \quad \mathrm{n} \in N,$$

, and investigated their asymptotic behaviors. The initial value x_0 and parameters A and B are positive fuzzy numbers.

In 2022, Han et al. [25] consided the following k-order nonlinear FDE

$$x_{n+1} = \frac{x_n}{A + Bx_{n-k}}, \quad \mathrm{n} \in N,$$

and analyzed the dynamic behavior of its positive fuzzy solution, there, parameters $A, B \in \Re_f^+$, $x_i \in \Re_f^+(i = 0, -1, \cdots, -k), k \in N$.

Inspired with the previous works, we study the dynamical behaviors of k-order rational FDE

$$a_{n+1} = 1 + \frac{a_n}{T + a_{n-k}}, \; n \in N, \tag{1}$$

here the parameter $T \in \Re_f^+$, initial values $a_{-t} \in \Re_f^+(t = 0, 1, \cdots, k), k \in N$.

The research structure of the article: Section 2 provides a preliminary and partial definition for the reasoning of this paper. Section 3 studies the persistence, existence, boundedness and asymptotic behavior of the positive fuzzy solution of this kind of FDE. Section 4 gives an example to verify the applicability of the theoretical results. The last section draws a broad conclusion.

II. Preliminary and definitions

In order to prove convenience below, we provide some key definitions and lemmas that will be used in the following proofs, see ([9], [10], [26], [27]).

Lemma 2.1. If $v : C_a^{k+1} \times C_b^{k+1} \rightarrow C_a$, w: $C_a^{k+1} \times C_b^{k+1} \rightarrow C_b$ are continuously differentiable mappings for some intervals of real numbers C_a and C_b, then the system of difference equations is considered for any initial values $(a_t, b_t) \in C_a \times C_b$ $(t = -k, -k+1, \cdots, 0)$

$$\begin{cases} a_{n+1} = v\left(a_n, \cdots, a_{n-k}, b_n, \cdots, b_{n-k}\right), \\ b_{n+1} = w\left(a_n, \cdots, a_{n-k}, b_n, \cdots, b_{n-k}\right), \end{cases} \tag{2}$$

A unique solution exists for $n \in N$ in the system $\{(a_t, b_{\mathrm{t}})\}_{t=-k}^{+\infty}$.

Definition 2.1. Suppose

$$\bar{a} = v(\bar{a}, \cdots, \bar{a}, \bar{b}, \cdots, \bar{b}), \quad \bar{b} = w(\bar{a}, \cdots, \bar{a}, \bar{b}, \cdots, \bar{b}).$$

a point $(\bar{a}, \bar{b}) \in C_a \times C_b$ is referred to as the equilibrium point of system (2). In other words, which makes $(\bar{a}, \bar{b})$ the fixed point of vector mapping (v, w).

Definition 2.2. If such a function $C : R \rightarrow [0,1]$ that satisfies the following conditions, then we call it a fuzzy number.

(i) C is normal, meaning that $C(a) = 1$, for $a \in R$;

(ii) C is fuzzy convex, which means that for $a_t \in R(t = 1, 2)$, and $\lambda \in [0,1]$,

$$C(\lambda a_1 + (1-\lambda)a_2) \geq \min\{C(a_1), C(a_2)\};$$

(iii) C is upper semi-continuous;

(iv) The support of C is compact

$$\mathrm{supp}C = \overline{\{a : C(a) > 0\}} = \overline{\bigcup_{\vartheta \in [0,1]} [C]_\vartheta}$$

In this $\vartheta \in (0,1]$, the ϑ-cut of C on field of real numbers is defined by

$$[C]_\vartheta = \{a \in R : C(a) \geq \vartheta\}.$$

Especially, with $\vartheta = 0$, the support of C is defined by

$$\mathrm{supp}C = [C]_0 = \overline{\{a \in R : C(a) > 0\}}.$$

Obviously, a closed interval is denoted by $[C]_\vartheta$. If $\min(\mathrm{supp}C) > 0$, fuzzy number C is positive.

Apparently, if C is a positive real number here, afterwards $[C]_\vartheta = [C, C], \vartheta \in [0,1]$. Under this circumstance, we call C a trivial fuzzy number.

Definition 2.3. Suppose that C and D be fuzzy numbers represented by $[C]_\vartheta = [C_{l,\vartheta}, C_{r,\vartheta}]$, $[D]_\vartheta = [D_{l,\vartheta}, D_{r,\vartheta}]$, where $\vartheta \in (0,1]$. From this, the norm of fuzzy space is defined:

$$\|C\| = \sup_{\vartheta \in (0,1]} \max\{|C_{l,\vartheta}|, |C_{r,\vartheta}|\}.$$

Similarly, the metric between fuzzy numbers C and D can be defined as:

$$S(C, D) = \sup_{\vartheta \in (0,1]} \max\left\{|C_{l,\vartheta} - D_{l,\vartheta}|, |C_{r,\vartheta} - D_{r,\vartheta}|\right\}.$$

The set of all fuzzy numbers is represented by $\Re_f(\Re_f^+)$, which includes positive numbers. Therefore, the metric space $\Re_f(\Re_f^+)$ is complete.

Let C , D be fuzzy numbers represented by $[C]_\vartheta = [C_{l,\vartheta}, C_{r,\vartheta}]$, $[D]_\vartheta = [D_{l,\vartheta}, D_{r,\vartheta}]$, and let λ be a real number. The operations of sum $C + D$, scalar product λC, multiplication CD, and division $\frac{C}{D}$ that can be defined as follows in the standard interval arithmetic setting:

$$[C + D]_\vartheta = [C]_\vartheta + [D]_\vartheta, [\lambda C] = \lambda[C]_\vartheta, \vartheta \in [0,1].$$

$$\begin{aligned}[CD]_\vartheta = \ & [\min\{C_{l,\eta}D_{l,\vartheta}, C_{l,\vartheta}D_{r,\vartheta}, C_{r,\vartheta}D_{l,\vartheta}, C_{r,\vartheta}D_{r,\vartheta}\}, \\ & \max\{C_{l,\vartheta}D_{l,\vartheta}, C_{l,\vartheta}D_{r,\vartheta}, C_{r,\vartheta}D_{l,\vartheta}, C_{r,\vartheta}D_{r,\vartheta}\}].\end{aligned}$$

$$\begin{aligned}\left[\frac{C}{D}\right]_\vartheta = \ & \Bigg[\min\{\frac{C_{l,\vartheta}}{D_{l,\vartheta}}, \frac{C_{l,\vartheta}}{D_{r,\vartheta}}, \frac{C_{r,\vartheta}}{D_{l,\vartheta}}, \frac{C_{r,\vartheta}}{D_{r,\vartheta}}\}, \\ & \max\{\frac{C_{l,\vartheta}}{D_{l,\vartheta}}, \frac{C_{l,\vartheta}}{D_{r,\vartheta}}, \frac{C_{r,\vartheta}}{D_{l,\vartheta}}, \frac{C_{r,\vartheta}}{D_{r,\vartheta}}\}\Bigg], 0 \notin [D]_\vartheta.\end{aligned}$$

Definition 2.4. If a_n is a series of positive fuzzy numbers satisfying FDE (1), it is considered a positive solution of FDE (1).

Definition 2.5. If there exists a $P_1 > 0$(resp. $P_2 > 0$) satisfying

$$\mathrm{supp}x_n \subset [P_1, \infty)(\mathrm{resp.supp}x_n \subset (0, P_2]), n \in N.$$

Sequence of positive fuzzy numbers $\{a_n\}$ is persistent or bounded.

Furthermore, assume $P_1, P_2 \in (0, +\infty)$ such that

$$\mathrm{supp}a_n \subset [P_1, P_2],\ n = 1, 2, \cdots,$$

then a_n is bounded and persistent.

Lemma 2.2. Suppose $v : R^+ \times R^+ \times R^+ \rightarrow R^+$ be continuous function, $\theta_1, \theta_2, \theta_3 \in \Re_f^+$. Then

$$[v(\theta_1, \theta_2, \theta_3)]_\vartheta = v([\theta_1]_\vartheta, [\theta_2]_\vartheta, [\theta_3]_\vartheta), \vartheta \in (0,1].$$

III. Main results

The present section discusses some dynamic characteristics of FDE (1). First, for all positive initial values, FDE (1) has a unique positive fuzzy solution, and we will prove its uniqueness.

Theorem 3.1. The unique positive solution a_n exists for FDE (1), where initial values $a_t \in \Re_f^+(t = -k, -k+1, \cdots, 0)$, and $k \in N$.

Proof. Assuming $a_t \in \Re_f^+(t = -k, -k+1, \cdots, 0)$, then we have a sequence of positive fuzzy numbers $\{a_n\}$, which satisfies equation (1).

Next, consider ϑ-cuts, where $\vartheta \in (0,1]$,

$$\begin{cases} [a_n]_\vartheta = [L_{n,\vartheta}, R_{n,\vartheta}], \\ [T]_\vartheta = [T_{l,\vartheta}, T_{r,\vartheta}]. \end{cases} \tag{3}$$

where, $n = 0, 1, 2, \cdots$. Form (1) and (3), by using Lemma 2.2, one has

$$\begin{aligned}[a_{n+1}]_\vartheta = [L_{n+1,\vartheta}, R_{n+1,\vartheta}] &= \left[1 + \tfrac{a_n}{T + a_{n-k}}\right]_\vartheta \\ &= 1 + \tfrac{[a_n]_\vartheta}{[T]_\vartheta + [a_{n-k}]_\vartheta} \\ &= 1 + \tfrac{[L_{n,\vartheta}, R_{n,\vartheta}]}{[T_{l,\vartheta}, T_{r,\vartheta}] + [L_{n-k,\vartheta}, R_{n-k,\vartheta}]} \\ &= \left[1 + \tfrac{L_{n,\vartheta}}{T_{r,\vartheta} + R_{\mathrm{n-k},\vartheta}}, 1 + \tfrac{R_{n,\vartheta}}{T_{l,\vartheta} + L_{n-k,\vartheta}}\right],\end{aligned} \tag{4}$$

for $n \in N, \vartheta \in (0,1]$. From this, we obtain the corresponding system of equations as follows.

$$\begin{cases} L_{n+1,\vartheta} = 1 + \frac{L_{n,\vartheta}}{T_{r,\vartheta}+R_{n-k,\vartheta}}, \\ R_{n+1,\vartheta} = 1 + \frac{R_{n,\vartheta}}{T_{l,\vartheta}+L_{n-k,\vartheta}}. \end{cases} \tag{5}$$

Clearly, for any initial values $L_{t,\vartheta}, R_{t,\vartheta}$ $(t = -k, -k+1, -k+2\cdots,0), \vartheta \in (0,1]$, then, there exists a unique positive solution $(L_{n,\vartheta}, R_{n,\vartheta})$ for $\vartheta \in (0,1]$.

Then, we demonstrate that $(L_{n,\vartheta}, R_{n,\vartheta})$ uniquely determines the solution a_n of (1), wich initial values $a_t, (t = -k, -k+1, \cdots, 0)$, which $(L_{n,\vartheta}, R_{n,\vartheta})$ is the positive solution of system (5) with initial values $(L_{t,\vartheta}, R_{t,\vartheta}), t = -k, -k+1, -k+2\cdots, 0$, for $\vartheta \in (0,1]$.

$$[a_n]_\vartheta = [L_{n,\vartheta}, R_{n,\vartheta}], \quad \vartheta \in (0,1], \quad n \in N. \tag{6}$$

For arbitrary $\vartheta \in (0,1], t = 1,2$, if $\vartheta_1 \leq \vartheta_2$, then we can use the following formula:

$$\begin{cases} 0 < T_{l,\vartheta_1} \leq T_{l,\vartheta_2} \leq T_{r,\vartheta_2} \leq T_{r,\vartheta_1} \\ 0 < L_{-k,\vartheta_1} \leq L_{-k,\vartheta_2} \leq R_{-k,\vartheta_2} \leq R_{-k,\vartheta_1}, \\ 0 < L_{-k+1,\vartheta_1} \leq L_{-k+1,\vartheta_2} \leq R_{-k+1,\vartheta_2} \leq R_{-k+1,\vartheta_1}, \\ \vdots \\ 0 < L_{0,\vartheta_1} \leq L_{0,\vartheta_2} \leq R_{0,\vartheta_2} \leq R_{0,\vartheta_1} \end{cases} \tag{7}$$

By induction, from (6) and (7), we will show that, for $n = 0,1,2,\cdots$,

$$L_{n,\vartheta_1} \leq L_{n,\vartheta_2} \leq R_{n,\vartheta_2} \leq R_{n,\vartheta_1}. \tag{8}$$

Inequality (8) is true for $n = 0$. For $n = 1$, one has

$$\begin{aligned} L_{1,\vartheta_1} &= 1 + \tfrac{L_{0,\vartheta_1}}{T_{r,\vartheta_1}+R_{-k,\vartheta_1}} \\ &\leq 1 + \tfrac{L_{0,\vartheta_2}}{T_{r,\vartheta_2}+R_{-k,\vartheta_2}} = L_{1,\vartheta_2} \\ &= 1 + \tfrac{L_{0,\vartheta_2}}{T_{r,\vartheta_2}+R_{-k,\vartheta_2}} \\ &\leq 1 + \tfrac{R_{0,\vartheta_2}}{T_{l,\vartheta_2}+L_{-k,\vartheta_2}} = R_{1,\vartheta_2} \\ &= 1 + \tfrac{R_{0,\vartheta_2}}{T_{l,\vartheta_2}+L_{-k,\vartheta_2}} \\ &\leq 1 + \tfrac{R_{0,\vartheta_1}}{T_{l,\vartheta_1}+L_{-k,\vartheta_1}} = R_{1,\vartheta_1}. \end{aligned} \tag{9}$$

So, when $n = 1$, (8) is true.

Suppose (8) holds for $n \leq m$, where $m \in 1,2,\ldots$, we can deduce from (6) and (7) the following

$$\begin{aligned} L_{m+1,\vartheta_1} &= 1 + \tfrac{L_{m,\vartheta_1}}{T_{r,\vartheta_1}+R_{m-k,\vartheta_1}} \\ &\leq 1 + \tfrac{L_{m,\vartheta_2}}{T_{r,\vartheta 2}+R_{m-k,\vartheta_2}} = L_{m+1,\vartheta_2} \\ &\leq 1 + \tfrac{R_{m,\vartheta_2}}{T_{l,\vartheta_2}+L_{m-k,\vartheta_2}} = R_{m+1,\vartheta_2} \\ &\leq 1 + \tfrac{R_{m,\vartheta_1}}{T_{l,\vartheta_1}+L_{m-k,\vartheta_1}} = R_{m+1,\vartheta_1}, \end{aligned} \tag{10}$$

So $L_{m+1,\vartheta_1} \leq L_{m+1,\vartheta_2} \leq R_{m+1,\vartheta_2} \leq R_{m+1,\vartheta_1}, m = 1,2,\cdots$. by virtue of induction, (8) is true.

On the other hand, from (5), we can get

$$\begin{cases} L_{1,\vartheta} = 1 + \frac{L_{0,\vartheta}}{T_{r,\vartheta}+R_{-k,\vartheta}}, \\ R_{1,\vartheta} = 1 + \frac{R_{0,\vartheta}}{T_{l,\vartheta}+L_{-k,\vartheta}}, \end{cases} \tag{11}$$

Since $T \in \Re_f^+, a_t \in \Re_f^+ (t = -k, -k+1, \cdots, 0), k \in N$. From Lemma 2.2, we get $T_{l,\vartheta}, T_{r,\vartheta}, \; L_{t,\vartheta}, R_{t,\vartheta} (t = -k, -k+1, -k+2, \cdots, 0)$, are left-continuous. Therefore, from equation (11), $L_{1,\vartheta}$ and $R_{1,\vartheta}$ are also left-continuous. Using induction method, we can conclude that $L_{1,\vartheta}$ and $R_{1,\vartheta}$ $(n \in N)$, are left-continuous.

Now, It will be proved that the set supp$a_n = \overline{\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}]}$ is compact. In fact, we need to prove the boundedness of $\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}]$.

Since $T \in \Re_f^+, a_t \in \Re_f^+ (t = -k, -k+1, \cdots, 0), k \in N$, for any $\vartheta \in (0,1]$, there exist the positive constants P_T, Q_T, P_t, Q_t satisfying,

$$[T_{l,\vartheta}, T_{r,\vartheta}] \subset [P_T, Q_T], [L_{t,\vartheta}, R_{t,\vartheta}] \subset [P_t, Q_t]. \tag{12}$$

Hence, from (11), (12), for $\vartheta \in (0,1]$, we obtain

$$[L_{1,\vartheta}, R_{1,\vartheta}] \subset \left[1 + \frac{P_0}{Q_T + Q_{-k}}, 1 + \frac{Q_0}{P_T + P_{-k}}\right]. \tag{13}$$

It is clear that

$$\bigcup_{\vartheta\in(0,1]} [L_{1,\vartheta_1}, R_{1,\vartheta_1}] \subset \left[1 + \frac{P_0}{Q_T + Q_{-k}}, 1 + \frac{Q_0}{P_T + P_{-k}}\right]. \tag{14}$$

Through observation, $\overline{\bigcup_{\vartheta\in(0,1]} [L_{1,\vartheta}, R_{1,\vartheta}]}$ is a compact set contained in $(0,+\infty)$. By using mathematical induction, we can conclude that $\overline{\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}]}$ is also compact for $n \in N$. Additionally, $\overline{\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}]} \subset (0,+\infty)$ holds for all $n \in N$.

Hence, using the left continuous of $L_{n,\vartheta}$ and $R_{n,\vartheta}$ for $n \in N$, and equation (8), we can conclude that the closed interval $[L_{n,\vartheta}, R_{n,\vartheta}]$ uniquely determines the sequence of positive fuzzy numbers $\{a_n\}$ satisfying equation (6).

Subsequently, we will prove that for any initial value $a_t \in \Re_f^+ (t = -k, k+1, \cdots, 0)$, $k \in N$, the sequence a_n is the solution of FDE (1).

$$\begin{aligned} [a_{n+1}]_\vartheta &= [L_{n+1,\vartheta}, R_{n+1,\vartheta}] \\ &= \left[1 + \tfrac{L_{n,\vartheta}}{T_{r,\vartheta}+L_{n-k,\vartheta}}, 1 + \tfrac{R_{n,\vartheta}}{T_{l,\vartheta}+L_{n-k,\vartheta}}\right] \\ &= \left[1 + \tfrac{a_n}{T+a_{n-k}}\right]_\vartheta \end{aligned} \tag{15}$$

Hence, a_n is the solution of FDE (1), which initial values $a_t \in \Re_f^+ (t = -k, k+1, \cdots, 0)$, and $k \in N$. This implies the existence of the positive solution a_n for FDE (1).

Then, we will prove the uniqueness of the positive solution of FDE (1).

Assume that for the initial values $a_t \in \Re_f^+ (t = -k, k+1, \cdots, 0)$, $k \in N$, FDE (1) has another positive solution $\overline{a}_n$. then, similar to the proof above, one can get, for $n \in N$,

$$\begin{aligned} [\overline{a}_{n+1}]_\vartheta &= [L_{n+1,\vartheta}, R_{n+1,\vartheta}] \\ &= \left[1 + \tfrac{L_{n,\vartheta}}{T_{r,\vartheta}+R_{n-k,\vartheta}}, 1 + \tfrac{R_{n,\vartheta}}{T_{l,\vartheta}+L_{n-k,\vartheta}}\right] \\ &= \left[1 + \tfrac{\overline{a}_n}{T+x_{n-k}}\right]_\vartheta. \end{aligned} \tag{16}$$

It implies that $[\overline{a}_n]_\vartheta = [L_{n,\vartheta}, R_{n,\vartheta}], \vartheta \in (0,1])$ for $n \in N$. Therefore, by using equation (6), we can obtain $[\overline{a}_n]_\vartheta = [a_n]_\vartheta$. So $\overline{a}_n = a_n$, i.e., we can conclude that FDE (1) has a unique positive solution.

To sum up, whatever the original values $a_t \in \Re_f^+ (t = -k, k+1, \cdots, 0)$, $k \in N$, there is just one positive fuzzy solution for FDE (1).

Next, we will further research the dynamical behaviors of FDE (1). The following lemma will be essential for our subsequent analysis.

Lemma 3.1. Let's consider the following difference equations.

$$\begin{cases} a_{n+1} = 1 + \frac{a_n}{T_1 + b_{n-k}}, \\ b_{n+1} = 1 + \frac{b_n}{T_2 + a_{n-k}}, \end{cases} \quad n = 0, 1, 2, \cdots, \tag{17}$$

where, initial values $a_t, b_t \in R^+ (t = -k, -k+1, -k+2, \cdots, 0)$. If $T_1 > 1, T_2 > 1$, then the conclusion below is correct.

(i) For each positive solution of system (17) in the form of (a_n, b_n) satisfies

$$\begin{cases} 1 \leq a_n \leq \frac{T_1}{T_1 - 1} + a_0, \\ 1 \leq b_n \leq \frac{T_2}{T_2 - 1} + b_0. \end{cases} \tag{18}$$

(ii) Difference equation system (17) has a single positive equilibrium $(\bar{a}, \bar{b})$ which can be expressed as follows:

$$\begin{cases} \bar{a} = 1 - \frac{T_2}{2} + \frac{\sqrt{T_1^2 T_2^2 + 4T_1 T_2}}{2T_1} \\ \bar{b} = 1 - \frac{T_1}{2} + \frac{\sqrt{T_1^2 T_2^2 + 4T_1 T_2}}{2T_2}, \end{cases} \tag{19}$$

and $\lim_{n\to\infty} a_n = \bar{a}, \lim_{n\to\infty} b_n = \bar{b}$ for $(a_t, b_t) \in (\psi_0, \Psi_0) \times (\gamma_0, \Gamma_0), t = -k, -k+1, \cdots, 0$, where

$$\begin{cases} \psi_0 \leq \min_{-k\leq t\leq 0}\{a_t\}, \quad \Psi_0 \geq \max_{-k\leq t\leq 0}\{a_t\}, \\ \gamma_0 \leq \min_{-k\leq t\leq 0}\{b_t\}, \quad \Gamma_0 \geq \max_{-k\leq t\leq 0}\{b_t\}. \end{cases}$$

(iii) The equilibrium point $(\bar{a}, \bar{b})$ shows local asymptotic stability.

Proof. (i) Apparently, the positive solution (a_n, b_n) of system (17) satisfies the inequality $a_n \geq 1, b_n \geq 1$. The following conclusion can be obtained recursively from system (17).

$$\begin{aligned} a_{n+1} &= 1 + \frac{a_n}{T_1 + b_{n-k}} \leq 1 + \frac{1}{T_1} a_n \\ &\leq 1 + \frac{1}{T_1} + \frac{1}{T_1^2} + \frac{1}{T_1^3} a_{n-2} \\ &\leq 1 + \frac{1}{T_1} + \frac{1}{T_1^2} + \frac{1}{T_1^3} + \frac{1}{T_1^4} a_{n-3} \\ &\leq 1 + \frac{1}{T_1} + \cdots + \frac{1}{T_1^n} + \frac{1}{T_1^{n+1}} a_0 \\ &\leq \frac{1 - \left(\frac{1}{T_1}\right)^n}{1 - \frac{1}{T_1}} + \frac{1}{T_1^{n+1}} a_0 \\ &\leq \frac{T_1}{T_1 - 1} + a_0. \end{aligned} \tag{20}$$

Proved by the same method $b_{n+1} \leq \frac{T_2}{T_2 - 1} + b_0$. Therefore (18) holds true.

(ii) If a, b satisfy

$$a = 1 + \frac{a}{T_1 + b}, \quad b = 1 + \frac{b}{T_2 + a}. \tag{21}$$

Thus, based on equation (21), the positive equilibrium point $(\bar{a}, \bar{b})$ can be expressed as (19).

To construct the corresponding linearized mapping for the nonlinear system (17), let

$$\begin{aligned} &(a_n, a_{n-1}, \cdots, a_{n-k}, b_n, b_{n-1}, \cdots, b_{n-k}) \\ &\mapsto (v, v_1, \cdots, v_k, w, w_1, \cdots w_k), \end{aligned} \tag{22}$$

where

$$\begin{cases} v = 1 + \frac{a_n}{T_1 + b_{n-k}}, v_t = a_{n-t+1}, \\ w = 1 + \frac{b_n}{T_2 + a_{n-k}}, w_t = b_{n-t+1}, \end{cases} \quad t = 1, 2, \cdots, k+1. \tag{23}$$

Using ψ_0, Ψ_0, γ_0 and Γ_0 as two couples of initial iterations, namely,

$$\begin{cases} \psi_0 \leq \min_{-k\leq t\leq 0}\{a_t\} \\ \quad \leq \max_{-k\leq t\leq 0}\{a_t\} \\ \quad \leq \Psi_0, \\ \gamma_0 \leq \min_{-k\leq t\leq 0}\{b_t\} \\ \quad \leq \max_{-k\leq t\leq 0}\{b_t\} \\ \quad \leq \Gamma_0. \end{cases} \tag{24}$$

We create four sequences $\{\psi_t\}, \{\Psi_t\}, \{\gamma_t\}, \{\Gamma_t\}$ for $t = 1, 2, \cdots$, as follows

$$\begin{cases} \psi_t = v\left([\psi_{t-1}]_j, [\Psi_{t-1}]_k, [\gamma_{t-1}]_f, [\Gamma_{t-1}]_g\right) \\ \Psi_t = v\left([\Psi_{t-1}]_j, [\psi_{t-1}]_k, [\Gamma_{t-1}]_f, [\gamma_{t-1}]_g\right) \\ \gamma_t = w\left([\psi_{t-1}]_{j_1}, [\Psi_{t-1}]_{k_1}, [\gamma_{t-1}]_{f_1}, [\Gamma_{t-1}]_{g_1}\right) \\ \Gamma_t = w\left([\Psi_{t-1}]_{j_1}, [\psi_{t-1}]_{k_1}, [\Gamma_{t-1}]_{f_1}, [\gamma_{t-1}]_{g_1}\right) \end{cases} \tag{25}$$

The mixed monotonicity of v and w implies that the sequences $\{\psi_t\}, \{\Psi_t\}, \{\gamma_t\}$, and $\{\Gamma_t\}$ possess the following properties

$$\begin{cases} \psi_0 \leq \psi_1 \leq \cdots \leq \psi_t \leq \cdots \leq \Psi_t \leq \cdots \leq \Psi_1 \leq \Psi_0, \\ \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_t \leq \cdots \leq \Gamma_t \leq \cdots \leq \Gamma_1 \leq \Gamma_0, \end{cases} \tag{26}$$

where $t = 1, 2, \cdots$.

One can choose two new sequences $\{a_l\}, \{b_l\}$ for $l \geq (k+1)t + 1$, satisfying

$$\psi_t \leq y_l \leq \Psi_t, \quad \gamma_t \leq z_l \leq \Gamma_t, \tag{27}$$

where $t \in N$. Suppose that

$$\begin{cases} \psi = \lim_{t\to\infty} \psi_t, \quad \Psi = \lim_{t\to\infty} \Psi_t, \\ \gamma = \lim_{t\to\infty} \gamma_t, \quad \Gamma = \lim_{t\to\infty} \Gamma_t. \end{cases} \tag{28}$$

Then

$$\begin{cases} \psi \leq \lim_{t\to\infty} \inf a_t \leq \lim_{t\to\infty} \sup a_t \leq \Psi, \\ \gamma \leq \lim_{t\to\infty} \inf b_t \leq \lim_{t\to\infty} \sup b_t \leq \Gamma. \end{cases} \tag{29}$$

Since v and w are continuous, so

$$\begin{cases} \Psi = v\left([\Psi]_j, [\psi]_k, [\Gamma]_f, [\gamma]_g\right), \\ \psi = v\left([\psi]_j, [\Psi]_k, [\gamma]_f, [\Gamma]_g\right), \\ \Gamma = w\left([\Psi]_{j_1}, [\psi]_{k_1}, [\Gamma]_{f_1}, [\gamma]_{g_1}\right), \\ \gamma = w\left([\psi]_{j_1}, [\Psi]_{k_1}, [\gamma]_{f_1}, [\Gamma]_{g_1}\right). \end{cases} \tag{30}$$

And then, if $\psi = \Psi, \gamma = \Gamma$, then $\psi = \Psi = \lim_{t\to\infty} a_t = \bar{a}, \gamma = \Gamma = \lim_{t\to\infty} b_t = \bar{b}$, and then, the proof is completed.

(iii) Using statement (ii), we can derive the linearized equation of (17) at the equilibrium point $(\bar{a}, \bar{b})$ as follows

$$\varphi_{n+1} = D\varphi_n, \tag{31}$$

here $\varphi_n = (a_n, a_{n-1}, \cdots, a_{n-k}, b_n, b_{n-1}, \cdots, b_{n-k})^T$, and

$$D = \begin{pmatrix} \Upsilon & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \Theta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where

$$\begin{cases} \Upsilon = \frac{2T_2}{T_1T_2+2T_2+\sqrt{T_1^2T_2^2+4T_1T_2}}, \\ \Theta = \frac{2T_1}{T_1T_2+2T_1+\sqrt{T_1^2T_2^2+4T_1T_2}}. \end{cases} \tag{32}$$

The characteristic equation of (31) is

$$P(\lambda) = (-\lambda)^{k-2}(\Upsilon - \lambda)(\Theta - \lambda) = 0.$$

Obviously, the characteristic roots of the equations lie within the unit disk. Thus the equilibrium point $(\bar{a}, \bar{b})$ is locally asymptotically stable.

Theorem 3.2. Consider FDE (1), where $T \in \Re_f^+$, and the initial values $a_t \in \Re_f^+ (t = -k, -k+1, -k+2, \cdots, 0)$, $k \in N$. If

$$T_{l,\vartheta} > 1, \vartheta \in (0,1], \tag{33}$$

the conclusions can be drawn from this.

(i) Positive fuzzy solutions a_n of FDE (1) are both bounded and persistent.

(ii) All positive fuzzy solutions a_n of FDE (1) converge to the unique positive equilibrium $\bar{a}$ as $n \to \infty$..

Proof. (i) Assume x_n is a positive fuzzy solution of FDE (1). Since (3) and (12) holds, from (7) and using Lemma 3.1, we have, for $n = 0, 1, 2, \cdots$,

$$\begin{cases} 1 \le L_{n,\vartheta} \le \frac{T_{r,\vartheta}}{T_{r,\vartheta}-1} + L_{0,\vartheta} \\ 1 \le R_{n,\vartheta} \le \frac{T_{l,\vartheta}}{T_{l,\vartheta}-1} + R_{0,\vartheta}. \end{cases} \tag{34}$$

From (3), (12) and (34), we have

$$[L_{n,\vartheta}, R_{n,\vartheta}] \subset [1, \iota], \tag{35}$$

where

$$\iota = \frac{P_T}{P_T - 1} + Q_0.$$

From (35), it follows that $\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}] \subset [1, \iota]$, so

$$\overline{\bigcup_{\vartheta\in(0,1]} [L_{n,\vartheta}, R_{n,\vartheta}]} \subset [1, \iota].$$

In this way, (i) has been proven.

(ii) The following system is considered with positive equilibrium $\bar{a}, [\bar{a}]_\vartheta = [L_\vartheta, R_\vartheta]$, which is a solution of FDE (1).

$$L_\vartheta = 1 + \frac{L_\vartheta}{T_{r,\vartheta} + R_\vartheta}, \qquad R_\vartheta = 1 + \frac{R_\vartheta}{T_{l,\vartheta} + L_\vartheta}. \tag{36}$$

Then the positive solution $(L_\vartheta, R_\vartheta)$ of (36) is given by

$$L_\vartheta = 1 - \frac{T_{l,\vartheta}}{2} + \frac{\sqrt{\phi}}{2T_{r,\vartheta}}, \quad R_\vartheta = 1 - \frac{T_{r,\vartheta}}{2} + \frac{\sqrt{\phi}}{2T_{l,\vartheta}}, \tag{37}$$

where

$$\phi = T_{r,\vartheta}^2 T_{l,\vartheta}^2 + 4T_{r,\vartheta}T_{l,\vartheta}.$$

From (37), we have, for $0 < \vartheta 1 \le \vartheta 2 \le 1$,

$$0 < L_{\vartheta 1} \le L_{\vartheta 2} \le R_{\vartheta 2} \le R_{\vartheta 1} \tag{38}$$

Since $T_{l,\vartheta}, T_{r,\vartheta}$ are left continuous, from (37), we can obtain that L_ϑ, R_ϑ are also left continuous. Combining (12) and (37), it can be concluded that

$$R_\vartheta \le d = 1 - \frac{Q_T}{2} + \frac{\sqrt{Q_T^2 P_T^2 + 4Q_T P_T}}{2P_T}. \tag{39}$$

Then from (37), it is easy to get

$$L_\vartheta \ge 1. \tag{40}$$

Therefore, from (39) and (40), it implies $[L_\vartheta, R_\vartheta] \subset [1, d]$, and so

$$\bigcup_{\vartheta\in(0,1]} [L_\vartheta, R_\vartheta] \subset [1, d]. \tag{41}$$

Clearly, $\bigcup_{\vartheta\in(0,1]} [L_\vartheta, R_\vartheta]$ is compact set. and

$$\bigcup_{\vartheta\in(0,1]} [L_\vartheta, R_\vartheta] \subset [0, \infty]. \tag{42}$$

From (38) and (42), so $L_\vartheta, R_\vartheta, \vartheta \in (0,1]$, determine a fuzzy number a satisfying

$$a = 1 + \frac{a}{T + a}, \quad [a]_\vartheta = [L_\vartheta, R_\vartheta], \quad \vartheta \in (0,1], \tag{43}$$

This means that $a = \bar{a}$ is a positive equilibrium of FDE (1).

Assume $\bar{a}'$ is another positive equilibrium of FDE (1). Then, it can be defined that there is such a function $\bar{L}_\vartheta, \bar{R}_\vartheta : (0,1] \to (0, \infty)$ satisfied

$$\bar{a}' = \frac{\bar{a}'}{T + \bar{a}'}, \quad [\bar{a}']_\vartheta = [\bar{L}_\vartheta, \bar{R}_\vartheta], \quad \vartheta \in (0,1]. \tag{44}$$

One can get that

$$\bar{L}_\vartheta = 1 + \frac{\bar{L}_\vartheta}{T_{r,\vartheta} + \bar{R}_\vartheta}, \bar{R}_\vartheta = 1 + \frac{\bar{R}_\vartheta}{T_{l,\vartheta} + \bar{L}_\vartheta}, \tag{45}$$

thus $L_\vartheta = \bar{L}_\vartheta, R_\vartheta = \bar{R}_\vartheta, \vartheta \in (0,1]$. Thus $\bar{a} = \bar{a}'$.

On the other hand, suppose that a_n is the positive fuzzy solution of FDE (1), and $[a_n]_\vartheta = [L_{n,\vartheta}, R_{n,\vartheta}], \vartheta \in (0,1], n \in N$. Next, by using Lemma 3.1 to the system (5), we can easily get

$$\lim_{n\to\infty} L_{n,\vartheta} = L_\vartheta, \quad \lim_{n\to\infty} R_{n,\vartheta} = R_\vartheta. \tag{46}$$

So $\lim_{n\to\infty} D(a_n, \bar{a}) = 0$. The proof of (ii) is completed.

IV. Numerical Example

It is helpful to better understand the dynamic behavior of FDE (1) by using some numerical values to simulate and analyze. In this section, to illustrate effectiveness of our theoretical analysis, we provide a numerical example.

Example 4.1. If $k = 2$ in FDE (1). Consider

$$a_{n+1} = 1 + \frac{a_n}{T + a_{n-2}}, n = 0, 1, 2, \cdots, \tag{47}$$

in which $T \in \Re_f^+$, initial values $a_t \in \Re_f^+ (t = -2, -1, 0)$.

Taking $\vartheta-$cuts, from (47) , a parameterized difference equation system exists as follows.

$$\begin{cases} L_{n+1,\vartheta} = 1 + \frac{L_{n,\vartheta}}{T_{r,\vartheta} + R_{n-2,\vartheta}}, \\ R_{n+1,\vartheta} = 1 + \frac{R_{n,\vartheta}}{T_{l,\vartheta} + L_{n-2,\vartheta}}, \end{cases} \vartheta \in (0, 1]. \tag{48}$$

We take T such that

$$T(\tau) = \begin{cases} 2\tau - 3, & 1.5 \le \tau \le 2, \\ -2\tau + 5, & 2 \le \tau \le 2.5, \end{cases} \tag{49}$$

With the initial values of a_{-2}, a_{-1} and a_0, there are

$$a_{-2}(\tau) = \begin{cases} \frac{2}{3}\tau - \frac{4}{3}, & 2 \le \tau \le 3.5, \\ -\frac{2}{3}\tau + \frac{10}{3}, & 3.5 \le \tau \le 5, \end{cases} \tag{50}$$

$$a_{-1}(\tau) = \begin{cases} \frac{1}{2}\tau - 3, & 6 \le \tau \le 8, \\ -\frac{1}{2}\tau + 5, & 8 \le \tau \le 10, \end{cases} \tag{51}$$

$$a_0(\tau) = \begin{cases} \frac{1}{3}\tau - \frac{7}{3}, & 7 \le \tau \le 10, \\ -\frac{1}{3}\tau + \frac{13}{3}, & 10 \le \tau \le 13, \end{cases} \tag{52}$$

From (49), we have

$$[T]_\vartheta = \left[\frac{3}{2} + \frac{1}{2}\vartheta, \frac{5}{2} - \frac{1}{2}\vartheta\right], \vartheta \in (0, 1]. \tag{53}$$

And so

$$\overline{\bigcup_{\vartheta \in (0,1]} [T]_\vartheta} = \left[\frac{3}{2}, \frac{5}{2}\right]. \tag{54}$$

From (50), (51) and (52), we get

$$\begin{aligned} [a_{-2}]_\vartheta &= \left[2 + \tfrac{3}{2}\vartheta, 5 - \tfrac{3}{2}\vartheta\right], \\ [a_{-1}]_\vartheta &= [6 + 2\vartheta, 10 - 2\vartheta], \\ [a_0]_\vartheta &= [7 + 3\vartheta, 13 - 3\vartheta], \end{aligned} \tag{55}$$

for $\vartheta \in (0, 1]$. For this reason

$$\begin{cases} \overline{\bigcup_{\vartheta \in (0,1]} [a_{-2}]_\vartheta} = [2, 5], \\ \overline{\bigcup_{\vartheta \in (0,1]} [a_{-1}]_\vartheta} = [6, 10], \\ \overline{\bigcup_{\vartheta \in (0,1]} [a_0]_\vartheta} = [7, 13]. \end{cases} \tag{56}$$

Apparently, we know $T_{l,\vartheta} > 1$, for any $\vartheta \in (0, 1]$. Applying Theorem 3.2, the conclusion can be drawn that every positive solution a_n of (47) is bounded and persistent. Besides, as demonstrated the Fig.1-Fig.3, the positive equilibrium point $\overline{a} = (1.3282, 1, 4142, 1.5471)$ is globally asymptotically stable for all positive solutions a_n of (47).

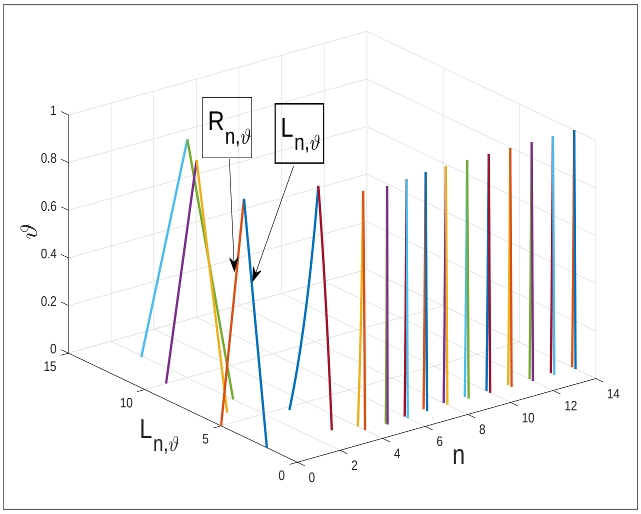

Fig. 1. Behavior of system (47).

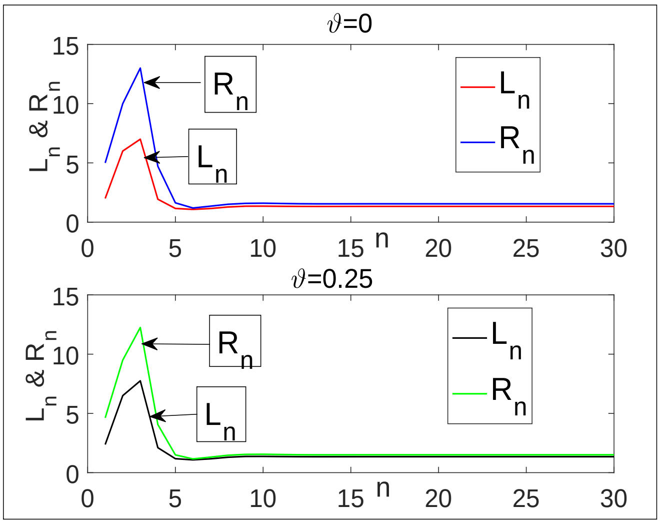

Fig. 2. Results for system (47) at $\vartheta = 0$ and $\vartheta = 0.25$

V. Conclusion

This paper explores the dynamic behavior of positive fuzzy solutions in FDE (1). Through the demonstration, we conclude the following main results.

(i) With any initial value $a_t \in \Re_f^+ (t = -k, -k + 1, \cdots, 0), k \in N$, there exist a unique positive fuzzy solution a_n for FDE (1) of k-order. A unique positive fuzzy solution a_n to k-order FDE (1) exists for any initial values $a_t \in \Re_f^+ (t = -k, -k + 1, \cdots, 0), k \in N$.

(ii) Applying Zadeh extension principle, FDE can be transformed into two parameter ordinary difference equations. We demonstrate that the positive fuzzy solution to FDE (1) is bounded and persistent using the theory of difference equations. Additionally, there is a unique locally asymptotically stable positive equilibrium point for this system.

(iii) Under condition $T_{l,\vartheta} > 1, \vartheta \in (0, 1]$, the k-order FDE (1) possesses a unique positive equilibrium point $\overline{a}$, and every positive fuzzy solution converges to $\overline{a}$ as $n \to \infty$.

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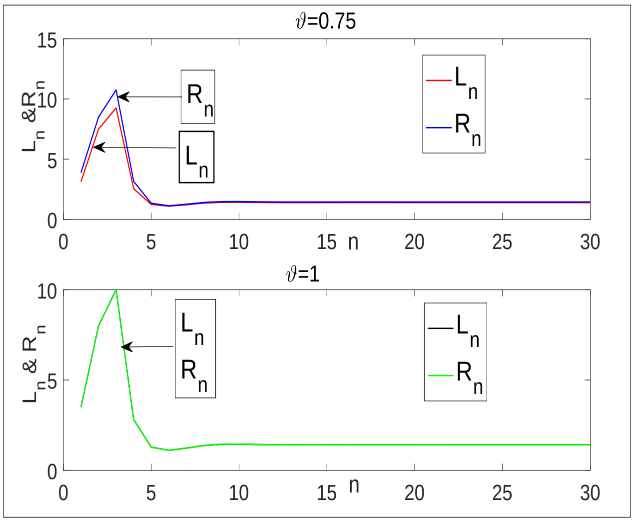

Fig. 3. Results for system (47) at $\vartheta = 0.75$ and $\vartheta = 1$.

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