Eigenvalue Problems for Generalized p-Laplacian Fractional Equations

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***Abstract*—This article mainly study under what conditions does the eigenvalue problem have positive solutions, in which the equation with generalized p-Laplacian operator involving both Caputo fractional derivatives and fractional derivatives of Riemann-Liouville. The novelty here consists of deriving some different intervals, when the λ is within it, there is at least one positive function satisfied the problem.**

***Index Terms*—Generalized p-Laplacian; Eigenvalue; Fractional differential equations; Fixed point theorem.**

I. Introduction

FOR studying turbulence problems in porous media in fundamental mechanics of engineering, Leibenson [1] proposed the following equation

$$(\varphi_p(v'(s)))' = f(s, v(s), v'(s)),$$

in which $\varphi_p(\tau) = |\tau|^{p-2}\tau, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ is called p-Laplacian operator.

Since then, many scholars have been interested in the p-Laplacian equation. Here we only briefly recall some remarkable results. D. Ji et al [2] discussed countably many solutions which is positive for the following multipoint problem which is singular

$$(\varphi_p(\rho'(t)))' + a(t)f(\rho(t)) = 0,\ t \in (0, 1),$$

$$\rho'(0) - \sum_{i=1}^{n-2} \alpha_i \rho(\xi_i) = 0,\ \rho'(1) + \sum_{i=1}^{n-2} \alpha_i \rho(\eta_i) = 0.$$

Recently, because of the importance of fractional differential equation in the modelling of many phenomena contained in engineering technology, many scholars began to study the differential equations which is fractional order with p-Laplacian operator [3-7,14,15].

In [3], Zhi wei Lv gave the existence results for the following relationship with fractional derivative involved p-Laplacian operator

$$D^\beta_{0^+}(\varphi_p(D^\alpha_{0^+}\theta(t))) + \varphi_p(\lambda)f(t, \theta(t)) = 0,\ t \in (0, 1),$$

$$\theta(0) = 0,\ D^\gamma_{0^+}\theta(1) = \sum_{i=1}^{n-2} \xi_i D^\gamma_{0^+}\theta(\eta_i),\ D^\alpha_{0^+}\theta(0) = 0.$$

The relationship in which involved the generalized p-Laplacian has received special attention in the last few years.

H. Wang [8] discussed the equation with generalized p-Laplacian operator

$$(\phi(v'))' + \lambda a(t)f(v) = 0, \quad 0 < t < 1,$$

$$v(0) = v(1) = 0.$$

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In [9], the authors investigated the solutions for the eigenvalue relationship, in which the equation involved generalized p-Laplacian operator

$$D^\beta_{0^+}(\phi(D^\alpha_{0^+}\gamma(t))) = \lambda f(\gamma(t)),\ t \in (0, 1),$$

$$\gamma(0) = \gamma'(0) = \gamma'(1) = 0,$$

$$\phi(D^\alpha_{0^+}\gamma(0)) = (\phi(D^\alpha_{0^+}\gamma(1)))' = 0.$$

Therefore, in order to enrich the research findings of fractional problems in which equations involved generalized p-Laplacian operator, here, we deal with such problem

$$D^\beta_{0^+}(\phi({}^cD^\alpha_{0^+}y(t))) + \lambda k(t, y(t)) = 0,\ t \in (0, 1), \tag{1}$$

$$y(0) + y'(0) = 0,\ y(1) + y'(1) = 0,\ {}^cD^\alpha_{0^+}y(0) = 0,$$

$$\phi({}^cD^\alpha_{0^+}y(1)) = \sum_{i=1}^{m-2} a_i \phi({}^cD^\alpha_{0^+}y(\xi_i)), \tag{2}$$

where λ is a constant which is positive, $1 < \alpha \leq 2,\ 1 < \beta \leq 2,\ 0 < a_i, \xi_i < 1,\ \sum_{i=1}^{m-2} a_i \xi_i^{\beta-1} < 1,\ D^\beta_{0^+}$ is the fractional derivative which is given by Riemann-Liouville, ${}^cD^\alpha_{0^+}$ is the Caputo fractional derivative. Furthermore, we need to introduce the following condition:

(H_1) The odd strictly increasing function $\phi \in C'(R, R)$ and the following relationship holds true

$$\omega_1(u)\phi(v) \leq \phi(u, v) \leq \omega_2(u)\phi(v),$$

for two increasing homeomorphisms $\omega_1, \omega_2 : (0, +\infty) \to (0, +\infty)$;

(H_2) $k \in C([0, 1] \times (0, +\infty), (0, +\infty))$.

In this work, under some natural assumptions, we obtain the solvability of the problem (1),(2). By scaling the value of the Green's function for (1),(2) and using some theorem, we obtain some different intervals, when the λ is within it, there is at least one positive function satisfied the relationship (1),(2).

II. The preliminary lemmas

This part gives some important definitions and useful Lemmas.

Definition 2.1 [10] Presume the function $h : (0, +\infty) \to R, \mu > 0$,

$$I^\mu h(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \theta)^{\mu-1} h(\theta) d\theta,$$

the above relationship refers to μ order fractional integration.

Definition 2.2 [10] Presume the function $h : (0, +\infty) \to R, \mu > 0$,

$$D^\mu h(t) = \frac{1}{\Gamma(n - \mu)} (\frac{d}{dt})^n \int_0^t (t - \theta)^{n-\mu-1} h(\theta) d\theta,$$

in this place $n = [\mu] + 1$, the above equation refers to μ order fractional derivative of Riemann-Liouville type.

Definition 2.3 [10] Let $h : (0, +\infty) \to R$ be a function, $\mu > 0$,

$$^{c}D^{\mu}h(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-\theta)^{n-\mu-1}h^n(\theta)d\theta,$$

in this place $n = [\mu] + 1$, the above equation refers to μ order fractional derivative of Caputo type.

Lemma 2.1 [10] Presume $\beta > 0$, $n = [\beta] + 1$, $h, ^{c}D_{0+}^{\beta}h, D_{0+}^{\beta}h \in L'(0,1)$, then there is

$$I^{\beta}\ ^{c}D^{\beta}h(t) = h(t) - c_1 - c_2 t - \cdots - c_n t^{n-1},$$

$$I^{\beta}D^{\beta}h(t) = h(t) - d_1 t^{\beta-1} - d_2 t^{\beta-2} - \cdots - d_n t^{\beta-n},$$

in this place c_i, d_i $i = 1, 2, \cdots, n$ are real numbers.

Lemma 2.2 [11] For continuous function h defined on interval $C[0,1]$,

$$\xi(t) = \int_0^1 G(t,s)h(s)ds, \tag{3}$$

in this place

$$G(t,s) = \begin{cases} \frac{(t-s)^{\mu-1}+(1-s)^{\mu-1}(1-t)}{\Gamma(\mu)} + \frac{(1-t)(1-s)^{\mu-2}}{\Gamma(\mu-1)}, \\ \quad 0 \le s \le t \le 1, \\ \frac{(1-t)(1-s)^{\mu-1}}{\Gamma(\mu)} + \frac{(1-t)(1-s)^{\mu-2}}{\Gamma(\mu-1)}, \\ \quad 0 \le t \le s \le 1. \end{cases} \tag{4}$$

meet the following equation

$$^{c}D_{0+}^{\mu}\xi(t) = h(t), \quad 0 < t < 1, \tag{5}$$

$$\xi(0) + \xi'(0) = 0, \quad \xi(1) + \xi'(1) = 0. \tag{6}$$

Lemma 2.3 [11] For continuous function h defined on interval $C[0,1]$, expression (4) satisfies:
(i) The function $G(t,s)$ is positive and continuous about t, s for s, t belongs to the interval $(0,1)$;
(ii) The following relationship holds true

$$\max_{0 \le t \le 1} G(t,s) \le N(s), \quad s \in (0,1),$$

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} G(t,s) \ge \delta(s)N(s), \quad s \in (0,1),$$

where $N(s) = \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$, $s \in (0,1)$. for some positive continuous function δ.

Remark 2.1 [11] The expression of $\delta(s)$ is

$$\delta(s) = \frac{1}{4}\frac{(\alpha-1)(1-s)^{\alpha-2} + (1-s)^{\alpha-1}}{(\alpha-1)(1-s)^{\alpha-2} + 2(1-s)^{\alpha-1}}, \ s \in (0,1),$$

we see that $\delta(s) \ge \frac{1}{8}$.

Lemma 2.4 [12] Let

$$p(s) = 1 - \sum_{s \le \xi_i} a_i\left(\frac{\xi_i - s}{1-s}\right)^{\beta-1}, \tag{7}$$

then $p(s)$ is nondecreasing and positive on $[0,1]$.

Lemma 2.5 Choose $h \in C[0,1]$, then the following relationship $v(t)$

$$v(t) = \int_0^1 H(t,s)h(s)ds,$$

in this place

$$H(t,s) = \frac{1}{p(0)\Gamma(\beta)}\begin{cases} p(s)[(1-s)t]^{\beta-1} - (t-s)^{\beta-1}p(0), \\ \quad 0 \le s \le t \le 1, \\ p(s)[(1-s)t]^{\beta-1}, \\ \quad 0 \le t \le s \le 1, \end{cases} \tag{8}$$

meet the following equation

$$D_{0+}^{\beta}v(t) + h(t) = 0, \quad 0 < t < 1, \tag{9}$$

$$v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i), \quad v(0) = 0. \tag{10}$$

Proof: For the equation (9), in light of Lemma 2.1, one has

$$v(t) = -\frac{1}{\Gamma(\beta)}\int_0^t h(s)(t-s)^{\beta-1}ds + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

$v(0) = 0$ means $c_2 = 0$.

From

$$v(1) = -\frac{1}{\Gamma(\beta)}\int_0^1 h(s)(1-s)^{\beta-1}ds + c_1,$$

$$v(\xi_i) = -\frac{1}{\Gamma(\beta)}\int_0^{\xi_i} h(s)(\xi_i - s)^{\beta-1}ds + c_1\xi_i^{\beta-1},$$

and $v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i)$, one get

$$\begin{aligned} c_1 &= \tfrac{1}{p(0)\Gamma(\beta)}[\textstyle\int_0^1 h(s)(1-s)^{\beta-1}ds \\ &\quad - \textstyle\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(s)(\xi_i - s)^{\beta-1}ds] \\ &= \tfrac{1}{p(0)\Gamma(\beta)}\textstyle\int_0^1 h(s)p(s)(1-s)^{\beta-1}ds, \end{aligned}$$

so

$$\begin{aligned} v(t) &= -\tfrac{1}{\Gamma(\beta)}\textstyle\int_0^t h(s)(t-s)^{\beta-1}ds \\ &\quad + \tfrac{t^{\beta-1}}{p(0)\Gamma(\beta)}\textstyle\int_0^1 h(s)p(s)(1-s)^{\beta-1}ds \\ &= \textstyle\int_0^1 H(t,s)h(s)ds. \end{aligned}$$

■

Lemma 2.6 [12] The expression (8) satisfies:
(i) $H(t,s)$ is positive, $\forall\ t, s$ belong to $(0,1)$;
(ii) $H(t,s) \ge (1-t)m(1-s)^{\beta-1}t^{\beta-1}s$, $\forall\ s, t \in [0,1]$;
(iii) $H(t,s) \le (1-s)^{\beta-1}Ms$, $\forall\ s, t \in [0,1]$,
where

$$m_1 = \inf_{0<s\le 1}\frac{p(s) - p(0)}{s}, \quad M_1 = \sup_{0<s\le 1}\frac{p(s)-p(0)}{s},$$

$$m = \frac{m_1 + p(0)}{p(0)\Gamma(\beta)}, \quad M = \frac{M_1 + p(0)(\beta-1)}{p(0)\Gamma(\beta)}.$$

Lemma 2.7 Let $(H_1), (H_2)$ are true, the following function

$$y(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 \lambda H(s,r)k(r,y(r))dr\right)ds.$$

makes equation (1) and condition (2) hold.

Proof: Choose $v(t) = \phi(^{c}D_{0+}^{\alpha}y(t))$, then the realtionship (1),(2) can be reduced to

$$D_{0+}^{\beta}v(t) + \lambda k(t, y(t)) = 0,$$

$$v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i).$$

From lemma 2.5, we have

$$v(t)=\int_0^1 k(s,y(s))\lambda H(t,s)ds,$$

this means

$$\phi({}^cD_{0+}^{\alpha}y(t))=\int_0^1 k(s,y(s))\lambda H(t,s)ds,$$

then

$${}^cD_{0+}^{\alpha}y(t)=\phi^{-1}(\int_0^1 k(s,y(s))\lambda H(t,s)ds)$$

and

$$y'(1)+y(1)=0,\quad y'(0)+y(0)=0.$$

Combining the above two formulas with Lemma 2.2, we can see

$$y(t)=\int_0^1 G(t,s)\phi^{-1}(\int_0^1 k(r,y(r))\lambda H(s,r)dr)ds.$$

■

Lemma 2.8 [9] Assuming condition (H_1) are met, then

$$\omega_1^{-1}(u)v\geq\phi^{-1}(u\phi(v))\geq\omega_2^{-1}(u)v,\ u,v\in(0,+\infty).$$

Lemma 2.9 [13] We denote K is a cone of Banach space B. Ω_1,Ω_2 with $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$ are bounded open set in B. Suppose S mapping K to K is continuous and it is compact. If one of the following holds true

(i) $\|Sv\|\leq\|v\|,\ \ \forall v\in K\cap\partial\Omega_1,$ $\|Sv\|\geq\|v\|,\ \ \forall v\in K\cap\partial\Omega_2;$

(ii) $\|Sv\|\geq\|v\|,\ \ \forall v\in K\cap\partial\Omega_1,$ $\|Sv\|\leq\|v\|,\ \ \forall v\in K\cap\partial\Omega_2.$

Then there is at least one function $v(t)\in K\cap(\overline{\Omega_2}\backslash\Omega_1)$ satisfies $Sv(t)=v(t)$.

III. Existence

Under the standard norm $\|v\|=\max_{0\leq t\leq 1}|v(t)|$, the space $C[0,1]$ defined as B is a space which is completed. We mark K as

$$K=\{v\in B|v(t)\geq\frac{1}{8}\|v\|,\quad t\in[0,1]\}.\tag{11}$$

Define the following map $T:K\rightarrow K$

$$(Ty)(t)=\int_0^1 G(t,s)\phi^{-1}(\int_0^1 k(r,y(r))\lambda H(s,r)dr)ds.\tag{12}$$

Moreover, if y satisfies $Ty=y$, the y is solutions for (1), (2).

Lemma 3.1 Let $(H_1),(H_2)$ are true, then T defined by (12) mapping K to K is compact, moreover, T is continuous.

Proof: In light of Lemma 2.3 and Remark 2.1, we have

$$\|Ty(t)\|\leq\int_0^1\phi^{-1}(\int_0^1 k(r,y(r))\lambda H(s,r)dr)N(s)ds,$$

$$\begin{aligned}Ty(t)&\geq\tfrac{1}{8}\textstyle\int_0^1\phi^{-1}(\int_0^1 k(r,y(r))\lambda H(s,r)dr)N(s)ds\\&\geq\tfrac{1}{8}\|Ty\|.\end{aligned}$$

Thus, $T(K)$ is included in K. The functions G,H and k is nonnegativeness and continuity, which give the result that T mapping K to K is continuous. According to the classical proof method, we can prove T mapping K to K is compact. In a word, we have T mapping K to K is continuous and compact. ■

We denote

$$k^0=\lim_{y\to0^+}\sup_{t\in[0,1]}\frac{k(t,y)}{\phi(y)},$$

$$k_0=\lim_{y\to0^+}\inf_{t\in[0,1]}\frac{k(t,y)}{\phi(y)},$$

$$k^\infty=\lim_{y\to+\infty}\sup_{t\in[0,1]}\frac{k(t,y)}{\phi(y)},$$

$$k_\infty=\lim_{y\to+\infty}\inf_{t\in[0,1]}\frac{k(t,y)}{\phi(y)},$$

$$B_1=\int_0^1\omega_1^{-1}(\int_0^1 M(1-r)^{\beta-1}rdr)N(s)ds,$$

$$\begin{aligned}B_2&=\tfrac{1}{8}\textstyle\int_{\frac14}^{\frac34}N(s)\omega_2^{-1}(s^{\beta-1}(1-s))\\&\quad\omega_2^{-1}(\textstyle\int_0^1(1-r)^{\beta-1}rm\omega_1(\frac18)dr)ds,\end{aligned}$$

$$\begin{aligned}B_3&=\tfrac{1}{8}\textstyle\int_{\frac14}^{\frac34}N(s)\omega_2^{-1}(s^{\beta-1}(1-s))\\&\quad\omega_2^{-1}(\textstyle\int_0^1(1-r)^{\beta-1}mrdr)ds.\end{aligned}$$

Theorem 3.1 Let $(H_1),(H_2)$ are true and $k_\infty\omega_1(B_1^{-1})>k^0\omega_2(B_2^{-1})$, then for

$$\lambda\in(\omega_2(B_2^{-1})k_\infty^{-1},\omega_1(B_1^{-1})(k^0)^{-1}),\tag{13}$$

there is at least one positive function satisfied problem (1), (2). In this place we write $k_\infty^{-1}=0$ if $k_\infty=+\infty$ and $(k^0)^{-1}=+\infty$ if $k^0=0$.

Proof: From (13), there exists $\varepsilon>0$ satisfying

$$(k^0+\varepsilon)^{-1}\omega_1(B_1^{-1})\geq\lambda\geq(k_\infty-\varepsilon)^{-1}\omega_2(B_2^{-1}).\tag{14}$$

First, by the notation of k^0,

$$k(t,y)\leq\phi(y)(k^0+\varepsilon),\quad 0<t<1,\ 0<y\leq r_1,\tag{15}$$

for some $r_1>0$. Select Ω_1 as $\{y\in B:\|y\|<r_1\}$. Then for any y belongs to $K\cap\partial\Omega_1$, from (14), (15), one get

$$\begin{aligned}\|Ty(t)\|&\leq\textstyle\int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}\\&\quad r(k^0+\varepsilon)\phi(r_1)dr)ds\\&\leq\omega_1^{-1}(\lambda(k^0+\varepsilon))B_1r_1\leq r_1=\|y\|,\end{aligned}$$

thus,

$$\|Ty\|\leq\|y\|,\quad\text{for } y \text{ belongs to } K\cap\partial\Omega_1.\tag{16}$$

Second, by the notation of k_∞,

$$k(t,y)\geq(k_\infty-\varepsilon)\phi(y),\quad 0<t<1,\ y\geq r_3,\tag{17}$$

for some $r_3>0$. Let $r_2=\max\{2r_1,r_3\}$, select Ω_2 as $\{y\in B:\|y\|<r_2\}$. Then for any y belongs to $K\cap\partial\Omega_2$, from (14), (17), one get

$$\begin{aligned}\|Ty(t)\|&\geq\textstyle\int_0^1 G(t,s)\phi^{-1}(\lambda\int_0^1 k(r,y(r))H(s,r)dr)ds\\&\geq\textstyle\int_{\frac14}^{\frac34}G(t,s)\phi^{-1}(\lambda\int_0^1 k(r,y(r))H(s,r)dr)ds\\&\geq\textstyle\int_{\frac14}^{\frac34}\frac18N(s)\phi^{-1}(\lambda\int_0^1 ms^{\beta-1}(1-s)(1-r)^{\beta-1}\\&\quad r(k_\infty-\varepsilon)\phi(\tfrac18\|y\|)dr)ds\\&\geq\textstyle\int_{\frac14}^{\frac34}\frac18N(s)\omega_2^{-1}(\lambda\int_0^1 ms^{\beta-1}(1-s)(1-r)^{\beta-1}\\&\quad r(k_\infty-\varepsilon)\tfrac18r_2dr)ds\\&=\omega_2^{-1}(\lambda(k_\infty-\varepsilon))B_2r_2\geq r_2=\|y\|,\end{aligned}$$

thus,

$$\|Ty\|\geq\|y\|,\quad\text{for } y \text{ belongs to } K\cap\partial\Omega_2.\tag{18}$$

Therefore, in view of (16), (18) and Lemma 2.9, there exists y belongs to $K\cap(\overline{\Omega_2}\backslash\Omega_1)$ and $Ty=y$ with $r_1\le\|y\|\le r_2$. Obviously, y is a positive function satisfied problem (1), (2). ■

Theorem 3.2 Let $(H_1),(H_2)$ are true and $k_0\omega_1(B_1^{-1})>k^\infty\omega_2(B_2^{-1})$, then for

$$\lambda\in(\omega_2(B_2^{-1})k_0^{-1},\omega_1(B_1^{-1})(k^\infty)^{-1}),\tag{19}$$

there is at least one positive function satisfied problem (1), (2). In this place we write $k_0^{-1}=0$ if $k_0=+\infty$ and $(k^\infty)^{-1}=+\infty$ if $k^\infty=0$.

Proof: From (19), there exists $\varepsilon>0$ satisfying

$$(k^\infty+\varepsilon)^{-1}\omega_1(B_1^{-1})\ge\lambda\ge(k_0-\varepsilon)^{-1}\omega_2(B_2^{-1}).\tag{20}$$

First, by the notation of k_0,

$$k(t,y)\ge(k_0-\varepsilon)\phi(y),\quad 0<t<1,\ 0<y\le r_1.\tag{21}$$

for some $r_1>0$. Select Ω_1 as $\{y\in B:\|y\|<r_1\}$. Then for any y belongs to $K\cap\partial\Omega_1$, from (20), (21), one get

$$\begin{aligned}\|Ty(t)\| &\ge \int_0^1 G(t,s)\phi^{-1}(\lambda\int_0^1 k(r,y(r))H(s,r)dr)ds\\ &\ge \int_{\frac14}^{\frac34} G(t,s)\phi^{-1}(\lambda\int_0^1 k(r,y(r))H(s,r)dr)ds\\ &\ge \int_{\frac14}^{\frac34}\tfrac18 N(s)\phi^{-1}(\lambda\int_0^1 ms^{\beta-1}(1-s)(1-r)^{\beta-1}\\ &\quad r(k_0-\varepsilon)\phi(\tfrac18\|y\|)dr)ds\\ &\ge \int_{\frac14}^{\frac34}\tfrac18 N(s)\phi^{-1}(\lambda\int_0^1 ms^{\beta-1}(1-s)(1-r)^{\beta-1}\\ &\quad r(k_0-\varepsilon)\omega_1(\tfrac18)\phi(r_1)dr)ds\\ &\ge \int_{\frac14}^{\frac34}\tfrac18 N(s)\omega_2^{-1}(s^{\beta-1}(1-s))\omega_2^{-1}(m\omega_1(\tfrac18))\\ &\quad \omega_2^{-1}(\int_0^1(1-r)^{\beta-1}rdr)ds\cdot\omega_2^{-1}(\lambda(k_0-\varepsilon))r_1\\ &= \omega_2^{-1}(\lambda(k_0-\varepsilon))B_2r_1\ge r_1=\|y\|,\end{aligned}$$

thus,

$$\|Ty\|\ge\|y\|,\quad\text{for } y \text{ belongs to } K\cap\partial\Omega_1.\tag{22}$$

Second, we choose a positive constant R_1 which satisfy

$$k(t,y)\le\phi(y)(k^\infty+\varepsilon),\quad 0<t<1,\ y\ge R_1.\tag{23}$$

In the following, we will prove it by dividing it into k is bounded and k is unbounded.
(i) k is bounded,

$$k(t,y)\le D,\quad\text{for } 0<t<1,\quad 0<y<+\infty,$$

for some positive constant D. Let $r_3=\max\{2r_1,\phi^{-1}(\lambda D)B_1\}$, select Ω_3 as $\{y\in B:\|y\|<r_3\}$. Then for any y belongs to $K\cap\partial\Omega_3$, from (20), (23), one get

$$\begin{aligned}\|Ty(t)\| &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}\\ &\quad rk(r,y(r))dr)ds\\ &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}rDdr)ds\\ &\le \int_0^1 N(s)\phi^{-1}(\lambda D)\omega_1^{-1}(\int_0^1 M(1-r)^{\beta-1}rdr)ds\\ &= \phi^{-1}(\lambda D)B_1\le r_3=\|y\|,\end{aligned}$$

thus,

$$\|Ty\|\le\|y\|,\quad\text{for } y \text{ belongs to} K\cap\partial\Omega_3.$$

(ii) k is unbounded,

$$k(t,y)\le k(t,r_4),\quad\text{for } 0<t<1,\quad 0<y\le r_4,$$

for some positive constant $r_4>\max\{2r_1,R_1\}$. Select Ω_4 as $\{y\in B:\|y\|<r_4\}$. Then for any y belongs to$K\cap\partial\Omega_4$, from (20), (23), we get

$$\begin{aligned}\|Ty(t)\| &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}\\ &\quad rk(r,u(r))dr)ds\\ &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}r\\ &\quad (k^\infty+\varepsilon)\phi(r_4)dr)ds\\ &= \omega_1^{-1}(\lambda(k^\infty+\varepsilon))B_1r_4\le r_4=\|y\|,\end{aligned}$$

thus,

$$\|Ty\|\le\|y\|,\quad\text{for } y \text{ belongs to} K\cap\partial\Omega_4.$$

Considering the above two cases, select Ω_2 as $\{y\in B:\|y\|<r_2=\max\{r_3,r_4\}\}$, we get

$$\|Ty\|\le\|y\|,\quad\text{for } y \text{ belongs to} K\cap\partial\Omega_2.$$

Therefore, Lemma 2.9 implies that there exists y belongs to $K\cap(\overline{\Omega_2}\backslash\Omega_1)$ and $Ty=y$ with $r_1\le\|y\|\le r_2$. Obviously, y is a positive function satisfied problem (1), (2). ■

Theorem 3.3 The following inequality holds

$$\lambda\min_{0\le t\le1,\frac18 r_1\le y\le r_1}k(t,y)\ge\phi(\frac{r_1}{B_3}),$$

$$\lambda\max_{0\le t\le1,0\le y\le r_2}k(t,y)\le\phi(\frac{r_2}{B_1}),$$

for some $r_2>r_1>0$, then there exists a positive function y satisfying problem (1), (2), moreover $r_1\le\|y\|\le r_2$.

Proof: First, let Ω_1 as $\{y\in B:\|y\|<r_1\}$. Thus for any $y\in K\cap\partial\Omega_1$, we get

$$\begin{aligned}\|Ty\| &\ge \int_0^1 G(t,s)\phi^{-1}(\lambda\int_0^1 H(s,r)k(r,y(r))dr)ds\\ &\ge \int_{\frac14}^{\frac34} G(t,s)\phi^{-1}(\lambda\int_0^1 H(s,r)k(r,y(r))dr)ds\\ &\ge \int_{\frac14}^{\frac34}\tfrac18 N(s)\phi^{-1}(\lambda\int_0^1 ms^{\beta-1}(1-s)(1-r)^{\beta-1}\\ &\quad r\min\nolimits_{0\le r\le1,\frac18 r_1\le y\le r_1}k(r,y(r))dr)ds\\ &\ge \tfrac{r_1}{B_3}B_3\\ &\ge r_1=\|y\|,\end{aligned}$$

Second, let Ω_2 as $\{y\in B:\|y\|<r_2\}$. Then for any y belongs to $K\cap\partial\Omega_2$, we get

$$\begin{aligned}\|Ty\| &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}\\ &\quad rk(r,y(r))dr)ds\\ &\le \int_0^1 N(s)\phi^{-1}(\lambda\int_0^1 M(1-r)^{\beta-1}r\\ &\quad \max\nolimits_{0\le r\le1,0\le y\le r_2}k(r,y(r))dr)ds\\ &\le \tfrac{r_2}{B_1}B_1\\ &= r_2=\|y\|,\end{aligned}$$

Therefore, in light of Lemma 2.9, there exists a function y satisfying $Ty=y$, moreover, $y\in K\cap(\overline{\Omega_2}\backslash\Omega_1)$ and $r_1\le\|y\|\le r_2$. Obviously, y is a solution which is positive for the relationship (1), (2). ■

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