The Eigenvalue for a Class of p-Laplacian Equations with Integral Boundary Conditions

Yuejin Zhang

***Abstract*—We are interested in the eigenvalue for the differential equations with the conditions contain integrals**

$$(\phi_p(v'(t)))' + \lambda l(t)g(t,v(t)) = 0, \qquad 0 < t < 1,$$

$$v(0) - av'(0) = \int_0^1 h_1(s)v(s)ds,$$

$$v(1) + bv'(1) = \int_0^1 h_2(s)v(s)ds,$$

in which λ is a positive constant, ϕ_p is a classical p-Laplacian operator. We prove the following results: when $g_0 = g_\infty = 0$, there exists a positive number λ^0, at least two solutions to the problem are shown to exist for λ within the interval (λ, λ^0), at least one positive function satisfies the problem is shown to exist if λ equals to λ^0, we can also present nonexistence of positive solution if λ is greater than λ^0. We illustrate this fact by using the fixed point index theory.

***Index Terms*—p-Laplacian equations; Eigenvalue; Positive solutions; Fixed point index.**

I. Introduction

THE key aim of studying this eigenvalue problem is to understand the motion of equations with the boundary conditions contain integrals. Many phenomena in physics were expressed into nonlocal mathematical equations with the conditions contain integrals. The boundary conditions contain integrals It often appears in many fields of applied science and engineering, like underground water flow, thermoelasticity, population dynamics and chemical engineering, the researchers showed interest in [1-6,18,19].

W. T. Ang et al. [1] discussed the heat equation as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}, (x,y,z) \in V, t \geq 0,$$

with the boundary conditions contain integrals

$$\int\int\int_V u(x,y,z,t)dxdydz = m(t), \quad t \geq 0.$$

Bashir Ahmad [2]studied the equation with forced terms involved the conditions contain integrals

$$u''(t) + \sigma u'(t) + f(t,u) = 0,\ 0 < t < 1, \sigma \in R - \{0\},$$

$$u(0) - \mu_1 u'(0) = \int_0^1 q_1 u(s)ds,$$

$$u(1) + \mu_2 u'(1) = \int_0^1 q_2 u(s)ds.$$

In [3], the authors considered the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), (x,y) \in \Omega, t \in (0,T],$$

with the following conditions

$$u(0,y,t) = r_1 \int_0^1 u(x,y,t)dx + g_1(y,t),$$

$$u(1,y,t) = r_2 \int_0^1 u(x,y,t)dx + g_2(y,t),$$

$$u(x,0,t) = r_3 \int_0^1 u(x,y,t)dy + g_3(x,t),$$

$$u(x,1,t) = r_4 \int_0^1 u(x,y,t)dy + g_4(x,t).$$

The authors in [4] developed the singularly perturbed differential equations derived from engineering and applied sciences

$$-\varepsilon a(x)y''(x) = f(\mu y'(x), y(x), b(x)),\ x \in (0,1),$$

and

$$y(0) + \int_0^1 y(x)dx = G_1,\ y'(0) + \int_0^1 y(x)dx = G_2.$$

Due to the importance of the boundary conditions contain integrals, there is a lot of literature devoted to the mathematical theory of it. Some representative literature includes: numerical solution for parabolic equations with mixed boundary condition [7]; parabolic equation with three order involved conditions contain integrals [8]; positive solutions for the differential equation involved conditions contain integrals in Banach space [9]; approximate solution for the equation with forced terms involved the conditions contain integrals [10]; solutions to the classical p-Laplacian fourth order multipoint problem [11], nonlinear perturbed hammerstein integral boundary value problems [12] and so on.

In [13], You-Yan Yang and Qi-Ru Wang discussed the following problem

$$(\phi_p(u'(t)))' + h(t)f(t,u(t),u'(t)) = 0, \qquad 0 < t < 1,$$

$$u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds,$$

$$u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds,$$

by making use of Avery-Peterson's theorem.

At the same time, there exist some good results about existence of solutions for differential equation with a parameter, such as [14-16].

Based on the above facts, the eigenvalue for equations with the boundary conditions contain integrals is not taken

Manuscript received September 21, 2022; revised April 02, 2023.
* Y. J. Zhang is an Associate Professor of Basic Research Section of College of information & Business, Zhongyuan University of Technology, Zhengzhou, Henan, 450001, China (Corresponding author E-mail: zhangyuejinzy@163.com).

into account currently. In the following, applying fixed point index to obtain conclusions for the solution of p-Laplacian equations which the boundary conditions contain integrals.

In this article, we present existence of the model

$$(\phi_p(v'(t)))' + \lambda l(t)g(t,v(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$v(0) - av'(0) = \int_0^1 h_1(s)v(s)ds,$$

$$v(1) + bv'(1) = \int_0^1 h_2(s)v(s)ds, \tag{2}$$

here the parameter λ is positive, $\frac{1}{p}+\frac{1}{q}=1, \phi_p(\omega) = |\omega|^{p-2}\omega, p > 1, a, b \geq 0, (\phi_p)^{-1}(\omega) = \phi_q(\omega) = |\omega|^{q-2}\omega$. Throughout the article, the following assumptions are given.

(H_1) $g(s,v) \geq 0$ $\forall s \in [0,1], v \in [0,+\infty)$, $g(s,v) > 0$ for $v > 0, s \in [0,1]$ is nondecreasing about v, $l \geq 0$ is continuous for all $[0,1]$;

(H_2) For $i = 1,2, h_i \geq 0, h_i$ is integrable and $\int_0^1 h_i(s)ds \in [0,1)$.

II. Some lemmas

Under the standard norm $\|v\| := \max\limits_{0\leq t\leq 1}|v(t)|$, the space $B = C[0,1]$ is a Banach space. We mark

$$K = \{v \in B| \ v(s) \text{ is nonnegative and concave in the interval } [0,1]\},$$

$$P_r = \{v \in B: \ \|v\| < r\}, \partial P_r = \{v \in B: \ \|v\| = r\},$$

$$C^+[0,1] = \{v \in C[0,1]: \ v(s) \geq 0, \ s \in [0,1]\}.$$

Lemma 2.1 [13] Assume the condition (H_2) hold, let $x \in C^+[0,1]$, then the equation

$$\begin{aligned} v(t) &= \frac{-b\phi_q(A_x - \int_0^1 x(r)dr)}{1-\int_0^1 h_2(s)ds} \\ &\quad \frac{-\int_0^1 h_2(s)\int_s^1 \phi_q(A_x - \int_0^\tau x(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\ &\quad -\int_t^1 \phi_q(A_x - \int_0^s x(r)dr)ds, \end{aligned} \tag{3}$$

or

$$\begin{aligned} v(t) &= \frac{a\phi_q(A_x) + \int_0^1 h_1(s)\int_0^s \phi_q(A_x - \int_0^\tau x(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\ &\quad + \int_0^t \phi_q(A_x - \int_0^s x(r)dr)ds, \end{aligned} \tag{4}$$

meet the problem

$$(\phi_p(v'(t)))' + x(t) = 0, \quad 0 < t < 1, \tag{5}$$

$$v(0) - av'(0) = \int_0^1 h_1(s)v(s)ds,$$

$$v(1) + bv'(1) = \int_0^1 h_2(s)v(s)ds, \tag{6}$$

where A_x satisfies

$$\begin{aligned} &a\phi_q(A_x) = \int_0^1 h_1(s)\int_s^1 \phi_q(A_x - \int_0^\tau x(r)dr)d\tau ds \\ &- \int_0^1 \phi_q(A_x - \int_0^\tau x(r)dr)d\tau \\ &- \frac{[1-\int_0^1 h_1(s)ds][b\phi_q(A_x - \int_0^1 x(r)dr)}{1-\int_0^1 h_2(s)ds} \\ &+ \frac{\int_0^1 h_2(s)\int_s^1 \phi_q(A_x - \int_0^\tau x(r)dr)d\tau ds]}{1-\int_0^1 h_2(s)ds}, \end{aligned} \tag{7}$$

then there exists a unique constant $A_x \in (0, \int_0^1 x(r)dr)$ satisfying (7). It can be seen there exists a unique constant $\sigma_x \in (0,1)$ satisfying $A_x = \int_0^{\sigma_x} x(r)dr$.

Lemma 2.2 [13] Let (H_2) hold. Then for each $x \in C^+[0,1]$, the unique function $v(t)$ satisfied (5),(6) meet with

(i) the function $v(t)$ is nonnegative;

(ii) the function $v(t)$ is concave, $t \in (0,1)$;

(iii) there exists a unique $\sigma_x \in (0,1)$, st $v'(\sigma_x) = 0$;

(iv) $\max\limits_{0\leq t\leq 1} v(t) = v(\sigma_x), \quad \forall v \in K$.

Lemma 2.3 [13] Define the following map $S: K \to B$

$$(Sv)(t) = \begin{cases} \frac{a\phi_q(\int_0^{\sigma v} g(r,v(r))l(r)dr)}{1-\int_0^1 h_1(s)ds} \\ + \frac{\int_0^1 h_1(s)\int_0^s \phi_q(\int_\tau^{\sigma v} g(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\ + \int_0^t \phi_q(\int_s^{\sigma_v} g(r,v(r))l(r)dr)ds, 0 \leq t \leq \sigma_v, \\ \frac{b\phi_q(\int_{\sigma v}^1 g(r,v(r))l(r)dr)}{1-\int_0^1 h_2(s)ds} \\ + \frac{\int_0^1 h_2(s)\int_s^1 \phi_q(\int_{\sigma v}^\tau g(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\ + \int_t^1 \phi_q(\int_{\sigma_v}^s g(r,v(r))l(r)dr)ds, \sigma_v \leq t \leq 1. \end{cases} \tag{8}$$

Then S mapping K to K. Moreover, the S is continuous and compact. Moreover, if v satisfies $\lambda Sv = v$, the v is solutions for (1),(2).

Lemma 2.4 [13] For $\theta \in (0, \frac{1}{2})$ and $v \in K$, let

$$\theta\|v\| \leq v(t), \quad t \in [\theta, 1-\theta].$$

Lemma 2.5 [17] We denote K is a cone of Banach space B. Ω with θ elements is a bounded open set in B. Suppose S mapping $K \cap \bar{\Omega}$ to K is continuous and it is compact. Moreover, for $\mu \geq 1$ and $x \in K \cap \partial\Omega$, there have $Sx \neq \mu x$, then the fixed point index $i(S, K\cap\Omega, K) = 1$.

Lemma 2.6 [17]We denote K is a cone of Banach space B. Ω with θ elements is a bounded open set in B. If S mapping $K \cap \bar{\Omega}$ to K is continuous and it is compact. Moreover, for $\mu \geq 0$, $x \in K \cap \partial\Omega$ and $x_0 \in K\backslash\{\theta\}$, there have $x - Sx \neq \mu x_0$, then $i(S, K\cap\Omega, K) = 0$.

Lemma 2.7 [17] We denote K is a cone of Banach space B. Ω with θ elements is a bounded open set in B. S mapping $K \cap \bar{\Omega}$ to K is continuous and it is compact. Moreover, for $\mu \geq 1$, $x \in K \cap \partial\Omega$, there have $\inf\limits_{x\in K\cap\partial\Omega}\|Sx\| > 0$ and $\mu Sx \neq x$, then $i(S, K\cap\Omega, K) = 0$.

We provide the following markings

$$g_0 = \lim_{v\to 0^+}\min_{s\in[0,1]}\frac{g(s,v)}{v}, \ g_\infty = \lim_{v\to+\infty}\min_{s\in[0,1]}\frac{g(s,v)}{v},$$

$$B = \min\{\int_\theta^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}} l(r)dr)ds, \int_{\frac{1}{2}}^{1-\theta}\phi_q(\int_{\frac{1}{2}}^s l(r)dr)ds\},$$

$$\Psi = \{(\lambda, v): \lambda > 0, \ v \text{ is a function satisfying (1) and (2) in } K\},$$

$$\Lambda = \{\lambda > 0, \text{there has } v \in K \text{satisfing } (\lambda, v) \in \Psi\},$$

$$\lambda^0 = \sup\Lambda.$$

III. Main result

Lemma 3.1 Let $(H_1),(H_2)$ are true and $g_0=\infty$, we can conclude that $\Psi\neq\emptyset$.

Proof: Fix a positive number R, we can get $\lambda_* \sup\limits_{v\in K\cap\overline{P_R}}\|Sv\|<R$, if we choose $\lambda_*>0$ small enough. Obviously, the following relationship holds:

$$\lambda_* Sv\neq\mu v,\quad \forall v\in K\cap\partial P_R,\quad \mu\geq 1.$$

In the light of Lemma 2.5, one get

$$i(\lambda_* S, K\cap P_R, K)=1. \tag{9}$$

From $g_0=\infty$, we can find that the existence of number r belongs to $(0,R)$ satisfies

$$g(t,v)\geq\frac{1}{\lambda_*\theta^2 B}v,\ \forall t\in[0,1],\ v\in[0,r]. \tag{10}$$

Then for $K\cap\partial P_r$, $\lambda_* S$ has no fixed point. Let $e(t)$ is a constant function identical to 1 on $[0,1]$, thus $e\in\partial P_1$, therefore

$$v\neq\lambda_* Sv+\mu e,\quad \forall v\in K\cap\partial P_r,\quad \mu\geq 0. \tag{11}$$

Otherwise, there exists $v_*\in K\cap\partial P_r,\quad \mu_1\geq 0$ satisfying $v_*=\lambda_* Sv_*+\mu_1 e$, then $\mu_1>0$.

Next, we can prove it in two different situations.
(i) If $\sigma_v\in[\frac{1}{2},1)$. For $t\in[\theta,1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4 and (10), one get

$$\begin{aligned}&\|v_*\|\geq v_*(t)=\lambda_* Sv_*+\mu_1 e(t)\\&\geq\theta\lambda_*\|Sv_*(t)\|+\mu_1 e(t)\\&\geq\theta\lambda_*\textstyle\int_0^{\sigma_v}\phi_q(\int_s^{\sigma_v}g(r,v_*(r))l(r)dr)ds+\mu_1\\&\geq\lambda_*\theta\textstyle\int_\theta^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}}g(r,v_*(r))l(r)dr)ds+\mu_1\\&\geq\|v_*\|+\mu_1=r+\mu_1.\end{aligned}$$

(ii) If $\sigma_v\in(0,\frac{1}{2})$. For $t\in[\theta,1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4, and (10), one get

$$\begin{aligned}&\|v_*\|\geq v_*(t)=\lambda_* Sv_*+\mu_1 e(t)\\&\geq\theta\lambda_*\|Sv_*(t)\|+\mu_1 e(t)\\&\geq\theta\lambda_*\textstyle\int_{\sigma_v}^{1}\phi_q(\int_{\sigma_v}^{s}g(r,v_*(r))l(r)dr)ds+\mu_1\\&\geq\lambda_*\theta\textstyle\int_{\frac{1}{2}}^{1-\theta}\phi_q(\int_{\frac{1}{2}}^{s}g(r,v_*(r))l(r)dr)ds+\mu_1\\&\geq\|v_*\|+\mu_1=r+\mu_1.\end{aligned}$$

Thus, in all cases, a contradiction $r\geq r+\mu_1$ has arisen. Lemma 2.6 implies that

$$i(\lambda_* S, K\cap P_r, K)=0. \tag{12}$$

Combining (9) and (12), one get

$$\begin{aligned}&i(\lambda_* S, K\cap(P_R\backslash\overline{P_r}),K)=i(\lambda_* S,K\cap P_R,K)\\&-i(\lambda_* S,K\cap P_r,K)=1,\end{aligned}$$

therefore, $v_*\in K\cap(P_R\backslash\overline{P_r})$ satisfies $\lambda_* Sv_*=v_*$, which means $(\lambda_*,v_*)\in\Psi$. ∎

Lemma 3.2 Let $(H_1),(H_2)$ are true and $g_0=g_\infty=\infty$, we can conclude that $0<\lambda^0<\infty$.

Proof: From Lemma 3.1, we can see that $\lambda^0>0$. Since $g_0=g_\infty=\infty$ and (H_2), for all $t\in[0,1]$ and $v\geq 0$, we can choose number $C>0$ satisfying $g(t,v)>Cv$. Let $(\lambda,v)\in\Psi$, in view of the cone K and Lemma 2.1, We can prove it in two different situations.

(i) If $\sigma_v\in[\frac{1}{2},1)$. For $t\in[\theta,1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4, one get

$$\begin{aligned}&v(t)=(\lambda Sv)(t)\geq\theta\lambda\|Sv(t)\|\\&\geq\theta\lambda\textstyle\int_0^{\sigma_v}\phi_q(\int_s^{\sigma_v}g(r,v(r))l(r)dr)ds\\&\geq\lambda\theta\textstyle\int_\theta^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}}g(r,v(r))l(r)dr)ds\\&\geq\theta^2\lambda C\|v\|\textstyle\int_\theta^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}}l(r)dr)ds\geq\theta^2\lambda C\|v\|B.\end{aligned}$$

(ii) If $\sigma_v\in(0,\frac{1}{2})$. For $t\in[\theta,1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4, one get

$$\begin{aligned}&v(t)=(\lambda Sv)(t)\geq\theta\lambda\|Sv(t)\|\\&\geq\theta\lambda\textstyle\int_{\sigma_v}^{1}\phi_q(\int_{\sigma_v}^{s}g(r,v(r))l(r)dr)ds\\&\geq\lambda\theta\textstyle\int_{\frac{1}{2}}^{1-\theta}\phi_q(\int_{\frac{1}{2}}^{s}g(r,v(r))l(r)dr)ds\\&\geq\theta^2\lambda C\|v\|\textstyle\int_{\frac{1}{2}}^{1-\theta}\phi_q(\int_{\frac{1}{2}}^{s}l(r)dr)ds\geq\theta^2\lambda C\|v\|B.\end{aligned}$$

Thus, in all cases this implies that $\|v\|\geq\theta^2\lambda C\|v\|B$, then $\lambda\leq\frac{1}{\theta^2 BC}$. ∎

Lemma 3.3 Let $(H_1),(H_2)$ are true and $g_0=g_\infty=\infty$, then $(0,\lambda^0)\subset\Lambda$. Furthermore, for $\lambda\in(0,\lambda^0)$, there are at least two solutions which is positive for problem (1), (2).

Proof: Given $\lambda\in(0,\lambda^0)$, we shall show $\lambda\in\Lambda$. In view of the notation of λ^0, we can obtain the existence of $\lambda_2\in\Lambda$ which satisfied $\lambda<\lambda_2\leq\lambda^0$ and $(\lambda_2,v_2)\in\Psi$. We choose $R<\min\limits_{t\in[0,1]}v_2(t)$. From Lemma 3.1, there exists $\lambda_1<\lambda, r<R$ and $v_1(t)\in K\cap(P_R\backslash\overline{P_r})$ that satisfied $(\lambda_1,v_1)\in\Psi$. Condition (H_1) implies that $0<v_1(t)<v_2(t), t\in[0,1]$. In addition

$$(\phi_p(v_1'(t)))'+\lambda_1 l(t)g(t,v_1(t))=0,\quad 0<t<1,$$

$$(\phi_p(v_2'(t)))'+\lambda_2 l(t)g(t,v_2(t))=0,\quad 0<t<1.$$

Let's discuss the modified problem:

$$(\phi_p(v'(t)))'+\lambda l(t)g_1(t,v(t))=0,\quad 0<t<1, \tag{13}$$

$$v(0)-av'(0)=\int_0^1 h_1(s)v(s)ds,$$

$$v(1)+bv'(1)=\int_0^1 h_2(s)v(s)ds, \tag{14}$$

where

$$g_1(t,v(t))=\begin{cases}g(t,v_2(t)),v(t)\geq v_2(t),\\ g(t,v(t)),v_1(t)<v(t)<v_2(t),\\ g(t,v_1(t)),v(t)\leq v_1(t).\end{cases}$$

Obviously, for $v\in K$ and $t\in[0,1]$, λg_1 is a bounded function. Moreover, $\lambda g_1(t,v)$ is continuous about v. Let

$$(S_1v)(t)=\begin{cases}\frac{a\phi_q(\int_0^{\sigma v}g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_1(s)ds}\\ +\frac{\int_0^1 h_1(s)\int_0^s\phi_q(\int_\tau^{\sigma v}g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds}\\ +\int_0^t\phi_q(\int_s^{\sigma_v}g_1(r,v(r))l(r)dr)ds,\\ \qquad\qquad 0\leq t\leq\sigma_v,\\ \frac{b\phi_q(\int_{\sigma v}^1 g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_2(s)ds}\\ +\frac{\int_0^1 h_2(s)\int_s^1\phi_q(\int_{\sigma v}^\tau g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds}\\ +\int_t^1\phi_q(\int_{\sigma_v}^s g_1(r,v(r))l(r)dr)ds,\\ \qquad\qquad \sigma_v\leq t\leq 1.\end{cases}$$

Clearly, $S_1 : K \to K$ is continuous and it is compact. Moreover, the solution v for the problem (13), (14) satisfies the equation $\lambda S_1 v = v$. Obviously, for $\forall v \in K$, $\exists\, r_0 > \|v_2\|$ satisfied $\|\lambda S_1 v\| < r_0$, in the light of Lemma 2.5,

$$i(\lambda S_1, K \cap P_{r_0}, K) = 1. \tag{15}$$

Denote

$$W = \{v \in K : v_1(t) < v(t) < v_2(t),\ \forall t \in [0,1]\}.$$

Now, we prove the fixed point $v \in K$ of λS_1, must belong to W. Noting that if $v = \lambda S_1 v$, then

$$\begin{aligned}
v(t) &= (\lambda S_1 v)(t)\\
&= \begin{cases}
\frac{\lambda a \phi_q(\int_0^{\sigma v} g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_1(s)ds} \\
+\frac{\lambda \int_0^1 h_1(s)\int_0^s \phi_q(\int_\tau^{\sigma v} g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\
+\lambda \int_0^t \phi_q(\int_s^{\sigma_v} g_1(r,v(r))l(r)dr)ds, 0 \le t \le \sigma_v,\\
\frac{\lambda b \phi_q(\int_{\sigma v}^1 g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_2(s)ds} \\
+\frac{\lambda \int_0^1 h_2(s)\int_s^1 \phi_q(\int_{\sigma v}^{\tau} g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\
+\lambda \int_t^1 \phi_q(\int_{\sigma_v}^{s} g_1(r,v(r))l(r)dr)ds, \sigma_v \le t \le 1.
\end{cases}\\
&< \begin{cases}
\frac{\lambda_2 a \phi_q(\int_0^{\sigma v} g(r,v_2(r))l(r)dr)}{1-\int_0^1 h_1(s)ds} \\
+\frac{\lambda_2 \int_0^1 h_1(s)\int_0^s \phi_q(\int_\tau^{\sigma v} g(r,v_2(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\
+\lambda_2 \int_0^t \phi_q(\int_s^{\sigma_v} g(r,v_2(r))l(r)dr)ds, 0 \le t \le \sigma_v,\\
\frac{\lambda_2 b \phi_q(\int_{\sigma v}^1 g(r,v_2(r))l(r)dr)}{1-\int_0^1 h_2(s)ds} \\
+\frac{\lambda_2 \int_0^1 h_2(s)\int_s^1 \phi_q(\int_{\sigma v}^{\tau} g(r,v_2(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\
+\lambda_2 \int_t^1 \phi_q(\int_{\sigma_v}^{s} g(r,v_2(r))l(r)dr)ds, \sigma_v \le t \le 1.
\end{cases}\\
&= (\lambda_2 S v_2)(t) = v_2(t)
\end{aligned}$$

and

$$\begin{aligned}
v(t) &= (\lambda S_1 v)(t)\\
&= \begin{cases}
\frac{\lambda a \phi_q(\int_0^{\sigma v} g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_1(s)ds} \\
+\frac{\lambda \int_0^1 h_1(s)\int_0^s \phi_q(\int_\tau^{\sigma v} g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\
+\lambda \int_0^t \phi_q(\int_s^{\sigma_v} g_1(r,v(r))l(r)dr)ds, 0 \le t \le \sigma_v,\\
\frac{\lambda b \phi_q(\int_{\sigma v}^1 g_1(r,v(r))l(r)dr)}{1-\int_0^1 h_2(s)ds} \\
+\frac{\lambda \int_0^1 h_2(s)\int_s^1 \phi_q(\int_{\sigma v}^{\tau} g_1(r,v(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\
+\lambda \int_t^1 \phi_q(\int_{\sigma_v}^{s} g_1(r,v(r))l(r)dr)ds, \sigma_v \le t \le 1.
\end{cases}\\
&> \begin{cases}
\frac{\lambda_1 a \phi_q(\int_0^{\sigma v} g(r,v_1(r))l(r)dr)}{1-\int_0^1 h_1(s)ds} \\
+\frac{\lambda_1 \int_0^1 h_1(s)\int_0^s \phi_q(\int_\tau^{\sigma v} g(r,v_1(r))l(r)dr)d\tau ds}{1-\int_0^1 h_1(s)ds} \\
+\lambda_1 \int_0^t \phi_q(\int_s^{\sigma_v} g(r,v_1(r))l(r)dr)ds, 0 \le t \le \sigma_v,\\
\frac{\lambda_1 b \phi_q(\int_{\sigma v}^1 g(r,v_1(r))l(r)dr)}{1-\int_0^1 h_2(s)ds} \\
+\frac{\lambda_1 \int_0^1 h_2(s)\int_s^1 \phi_q(\int_{\sigma v}^{\tau} g(r,v_1(r))l(r)dr)d\tau ds}{1-\int_0^1 h_2(s)ds} \\
+\lambda_1 \int_t^1 \phi_q(\int_{\sigma_v}^{s} g(r,v_1(r))l(r)dr)ds, \sigma_v \le t \le 1.
\end{cases}\\
&= (\lambda_1 S v_1)(t) = v_1(t)
\end{aligned}$$

Hence, from (15), we have $i(\lambda S_1, W, K) = i(\lambda S_1, K \cap P_{r_0}, K) = 1$, which implies $S_1 = S$ on $\bar{W}$, and thus

$$i(\lambda S, W, K) = 1. \tag{16}$$

Now we have $w_1 \in W$ is a fixed point of λS. Which means w_1 is a positive solution for (1), (2), hence $\lambda \in \Lambda, (\lambda, w_1) \in \Psi$ and $(0, \lambda) \subset \Lambda$. In the following, we shall provide the second positive function satisfies the problem (1), (2). $g_\infty = \infty$ and $g(t,v)$ is continuous about v implies that

$$g(t,v) \ge \frac{2v}{\lambda\theta^2 B} - \frac{c}{\theta B}, \forall t \in [0,1], v \ge 0. \tag{17}$$

for some positive constant c. For constant function $e(t)$ corresponding to 1, denote

$$\begin{aligned}\Omega = \{&v \in K : \text{for a nonnegative constant } \mu\\ &\text{satisfying } v = \lambda S v + \mu e\}.\end{aligned}$$

We can prove that Ω is a bounded subset of B. We will divide into two cases to complete the proof.
(i) If $\sigma_v \in [\frac{1}{2}, 1)$. For $t \in [\theta, 1-\theta]$, in the light of Lemma 2.2, 2.3, 2.4 and (17), one get

$$\begin{aligned}
&v(t) = (\lambda S v)(t) + \mu e(t) = (\lambda S v)(t) + \mu\\
&\ge \theta\lambda \|Sv(t)\| + \mu\\
&\ge \theta\lambda \textstyle\int_0^{\sigma_v} \phi_q(\int_s^{\sigma_v} g(r,v(r))l(r)dr)ds\\
&\ge \lambda\theta \textstyle\int_\theta^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} l(r)\big[\frac{2v(t)}{\lambda\theta^2 B} - \frac{c}{\theta B}\big]dr)ds\\
&\ge \lambda\theta \textstyle\int_\theta^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} l(r)\big[\frac{2\theta\|v\|}{\lambda\theta^2 B} - \frac{c}{\theta B}\big]dr)ds = 2\|v\| - \lambda c.
\end{aligned}$$

(ii) If $\sigma_v \in (0, \frac{1}{2})$. For $t \in [\theta, 1-\theta]$, in the light of Lemma 2.2, 2.3, 2.4 and (17), one get

$$\begin{aligned}
&v(t) = (\lambda S v)(t) + \mu e(t) = (\lambda S v)(t) + \mu\\
&\ge \theta\lambda \|Sv(t)\| + \mu\\
&\ge \theta\lambda \textstyle\int_{\sigma_v}^{1} \phi_q(\int_{\sigma_v}^{s} g(r,v(r))l(r)dr)ds\\
&\ge \lambda\theta \textstyle\int_{\frac{1}{2}}^{1-\theta} \phi_q(\int_{\frac{1}{2}}^{s} l(r)\big[\frac{2v(t)}{\lambda\theta^2 B} - \frac{c}{\theta B}\big]dr)ds\\
&\ge \lambda\theta \textstyle\int_{\frac{1}{2}}^{1-\theta} \phi_q(\int_{\frac{1}{2}}^{s} l(r)\big[\frac{2\theta\|v\|}{\lambda\theta^2 B} - \frac{c}{\theta B}\big]dr)ds = 2\|v\| - \lambda c.
\end{aligned}$$

In conclusion, we have $\|v\| \le \lambda c$. It shows that as a subset of B, Ω is bounded. Hence, there exists $R_1 > \|v_2\|$ such that

$$v \ne \lambda S v + \mu e, \quad \forall \mu \ge 0, \quad v \in K \cap \partial P_{R_1}.$$

In the light of Lemma 2.6, we get

$$i(\lambda S, K \cap \partial P_{R_1}, K) = 0. \tag{18}$$

Using the same derivation method as equation (12), one get

$$i(\lambda S, K \cap \partial P_{r_1}, K) = 0, \tag{19}$$

here $0 < r_1 < \min\limits_{t\in[0,1]} v_1(t)$. Hence, by (16), (18), (19), so that

$$\begin{aligned}
&i(\lambda S, K \cap (P_{R_1} \backslash (\overline{W} \cup \overline{P_r}), K) = i(\lambda S, K \cap P_{R_1}, K)\\
&-i(\lambda S, W, K) - i(\lambda S, K \cap P_{r_1}, K) = -1,
\end{aligned}$$

therefore, there exists $w_2 \in K \cap (P_{R_1} \backslash (\overline{W} \cup \overline{P_r})$ satisfying the equation $\lambda S w_2 = w_2$. Thus, we get another positive function which satisfies the problem (1), (2). ■

Lemma 3.4 If $(H_1), (H_2)$ are true and $g_0 = g_\infty = \infty$, then $\Lambda = (0, \lambda^0)$.

Proof: From Lemma 3.3, we only need to prove $\lambda^0 \in \Lambda$. From $\lambda^0 = \sup \Lambda$, there exists $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \ge \frac{\lambda^0}{2} (n = 1, 2, \cdots)$ satisfying $\lambda_n \to \lambda^0$ as $n \to \infty$. Considering the definition of Λ, there is a non-zero sequence $\{v_n\} \subset K \backslash \{\theta\}$

satisfying $(\lambda_n, v_n) \in \Psi$. We now prove that the sequence $\{v_n\}$ is bounded. In fact, if not, we can find a subsequence of $\{v_n\}$ that satisfied $\|v_n\| \to \infty$ as $n \to \infty$. Lemma 2.4 implies $v_n \geq \theta\|v_n\|$, $\{v_n\} \subset K\backslash\{\theta\}$. Choose sufficiently large μ such that

$$\frac{\lambda^0\theta^2 B\mu}{2} > 1.$$

$g_\infty = \infty$, implies $g(t, v) \geq \mu v$ for $v > \theta R, t \in [0, 1]$, where R is a positive constant. Considering $\|v_n\| \to \infty$ as $n \to \infty$. We can find a sufficiently large n_0 such that $\|v_{n_0}\| \geq R$. Let's prove in two situations below.

(i) If $\sigma_v \in [\frac{1}{2}, 1)$. For $t \in [\theta, 1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4, one get

$$\begin{aligned}
&\|v_{n_0}\| \geq v_{n_0}(t) = (\lambda_{n_0} S v_{n_0})(t) \geq \theta\lambda_{n_0}\|Sv_{n_0}(t)\| \\
&\geq \theta\lambda_{n_0}\int_0^{\sigma_v}\phi_q(\int_s^{\sigma_v} g(r, v_{n_0}(r))l(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta\int_0^{\sigma_v}\phi_q(\int_s^{\sigma_v} l(r)v_{n_0}(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta^2\|v_{n_0}\|\int_\theta^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}} l(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta^2\|v_{n_0}\|B.
\end{aligned}$$

(ii) If $\sigma_v \in (0, \frac{1}{2})$. For $t \in [\theta, 1-\theta]$, in the light of Lemma 2.2, 2.3 and 2.4, one get

$$\begin{aligned}
&\|v_{n_0}\| \geq v_{n_0}(t) = (\lambda_{n_0} S v_{n_0})(t) \geq \theta\lambda_{n_0}\|Sv_{n_0}(t)\| \\
&\geq \theta\lambda_{n_0}\int_{\sigma_v}^1\phi_q(\int_{\sigma_v}^s g(r, v_{n_0}(r))l(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta\int_{\sigma_v}^1\phi_q(\int_{\sigma_v}^s l(r)v_{n_0}(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta^2\|v_{n_0}\|\int_{\frac{1}{2}}^{1-\theta}\phi_q(\int_{\frac{1}{2}}^s l(r)dr)ds \\
&\geq \frac{\lambda^0}{2}\mu\theta^2\|v_{n_0}\|B.
\end{aligned}$$

In conclusion, we have

$$\frac{\lambda^0\mu\theta^2 B}{2} \leq 1, \tag{20}$$

this is a contradiction. Thus, $\{v_n\}$ is bounded, $\{Sv_n\}$ is equicontinuous imply that for each positive ε, there $\exists$ positive δ that satisfied

$$|v_n(t_1) - v_n(t_2)| = \lambda_n|(Sv_n)(t_1) - (Sv_n)(t_2)| < \lambda_n\varepsilon \leq \lambda^0\varepsilon,$$

where t_1, t_2 are any two numbers on an interval $[0, 1]$ and $|t_1 - t_2| < \delta$, $n = 1, 2, \cdots$, which means $\{v_n\}$ is equicontinuous. Hence, $\{v_n\}$ is relatively compact by use of the Ascoli-Arzela theorem. Therefore, we can find a subsequence of $\{v_n\}$ and $v^* \in K$ that satisfied $v_n \to v^*$ as $n \to \infty$. Letting $n \to \infty$ in $v_n = \lambda_n S v_n$, we get

$$v^* = \lambda^0 S v^*.$$

If $v^* = \theta$, turning to $g_0 = \infty$, we can also get the contradictions. Thus $v^* \in K\backslash\{\theta\}$ and so $\lambda^0 \in \Lambda$. ■

Theorem 3.1 Let $(H_1), (H_2)$ are true. Moreover, $g_0 = g_\infty = \infty$, then we can find a positive constant λ^0, if $\lambda \in (0, \lambda^0)$, there are at least two positive functions which solve the equation (1), (2); if $\lambda = \lambda^0$, there is at least one positive function which solves the equation (1), (2); if $\lambda > \lambda^0$, there is no positive function which solves the relation (1), (2).

Proof: In the light of Lemma 3.1, 3.2, 3.3 and 3.4, we can easily prove Theorem 3.1. ■

References

[1] W.T. Ang , A.B. Gumel, "A boundary integral method for the three-dimensional heat equation subject to specification of energy," *Journal of Computational and Applied Mathematics*, vol. 135, pp.303-311, 2001.

[2] B. Ahmad, A. Alsaedi, B.S. Alghamdi, "Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions," *Nonlinear Analysis:Real World Applications*, vol. 9, pp.1727-1740, 2008.

[3] M. Sapagovas, J. Novickij, "Alternating direction method for the wave equation with integral boundary conditions," *Applied Numerical Mathematics*, vol. 182, pp.1-13, 2022.

[4] M. Ahsan , M. Bohner , A. Ullah, A.A. Khan, S. Ahmad, "A Haar wavelet multi-resolution collocation method for singularly perturbed differential equations with integral boundary conditions," *Mathematics and Computers in Simulation*, vol. 204, pp.166-180, 2023.

[5] X.M. Zhang, M.Q. Feng, W.G. Ge, "Existence result of second-order differential equations with integral boundary conditions at resonance," *J. Math. Anal. Appl.*, vol. 35, pp.311-319, 2009.

[6] F. Nicoud, T. Schonfeld, "Integral boundary conditions for unsteady biomedical CFD applications," *Int. J. Numer. Meth. Fluids*, vol. 40, pp.457-465, 2002.

[7] G.W. Batten Jr., "Second-order correct boundary conditions for the numerical solution of the mixed boundary problem for parabolic equations," *Math. Comp.*, vol. 17, pp.405-413, 1963.

[8] A. Bouziani, N.E. Benouar, "Mixed problem with integral conditions for a third order parabolic equation," *Kobe J. Math.*, vol. 15, pp.47-58, 1998.

[9] M. Feng, D. Ji, W. Ge, "Positive solutions for a class of boundary value problem with integral boundary conditions in Banach spaces," *J. Comput. Appl. Math.*, vol. 222, pp.351-363, 2008.

[10] J.M. Gallardo, "Second order differential operators with integral boundary conditions and generation of semigroups," *Rocky Mountain J. Math.*, vol. 30, pp.1265-1292, 2000.

[11] X. Zhang, L. Liu, "A necessary and sufficient condition of positive solutions for fourth order multi-point boundary value problem with p-Laplacian," *Nonlinear Anal.*, vol. 68, pp. 3127-3137, 2008.

[12] G. Infante, J.R.L. Webb, "Nonlinear non-local boundary-value problems and perturbed hammerstein integral equations," *Proc. Edinb. Math. Soc.*, vol. 49, pp.637-656, 2006.

[13] Y.Y. Yang, Q.R. Wang, "Multiple positive solutions for p-Laplacian equations with integral boundary conditions," *J. Math. Anal. Appl.*, vol. 453, pp.558-571, 2017.

[14] D.Y. Bai, Y.M. Chen, "Three positive solutions for a generalized Laplacian boundary value problem with a parameter," *Applied Mathematics and Computation*, vol. 219, pp.4782-4788, 2013.

[15] T.S. He, F.J. Yang, C.Y. Chen, S.G. Peng, " Existence and multiplicity of positive solutions for nonlinear boundary value problems with a parameter," *Computers and Mathematics with Applications*, vol. 61, pp.3355-3363, 2011.

[16] Y. Yang, J. H. Zhang, "Existence of solutions for some discrete boundary value problems with a parameter," *Applied Mathematics and Computation*, vol. 211, no.2, pp.293-302, 2009.

[17] D. J. Guo, V. Lakskmikantham, (1988), Nonlinear Problems in Abstract Cones.

[18] Jinping Xu, Baiyan Xu, "Iteration and Existence of Positive Solutions for Fractional Integral Problems involved p(t)-Laplacian Operator," *Engineering Letters*, vol. 30, no.2, pp.782-787, 2022.

[19] Qinghua Feng, "A New Approach for Seeking Exact Solutions of Fractional Partial Differential Equations in the Sense of Conformable Fractional Derivative," *IAENG International Journal of Computer Science*, vol. 49, no.4, pp.1242-1248, 2022.